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# COMPLEX ANALYSIS

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## CONTENTS

## 1. ANALYTIC FUNCTIONS

A *domain* in the complex plane $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$. For example, every open disc:

$$\mathbb{D}(w,r) = \{z \in \mathbb{C} : |z-w| < r\}$$

is a domain. Throughout this course we will consider functions defined on domains.

Suppose that $D$ is a domain and $f : D \to \mathbb{C}$ a function. This function is *complex differentiable* at a point $z \in D$ if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. The value of the limit is the derivative $f'(z)$. The function $f : D \to \mathbb{C}$ is *analytic* if it is complex differentiable at each point $z$ of the domain $D$. (The terms *holomorphic* and *regular* are more commonly used in place of *analytic*.)

For example, $f : z \mapsto z^n$ is analytic on all of $\mathbb{C}$ with $f'(z) = nz^{n-1}$ but $g : z \mapsto \overline{z}$ is not complex differentiable at any point and so $g$ is not analytic.

It is important to observe that asking for a function to be complex differentiable is much stronger than asking for it to be real differentiable. To see this, first recall the definition of real differentiability. Let $D$ be a domain in $\mathbb{R}^2$ and write the points in $D$ as $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let $f : D \to \mathbb{R}^2$ be a function. Then we can write

$$f(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix}$$

with $f_1, f_2 : D \to \mathbb{R}$ as the two components of $f$. The function $f$ is *real differentiable* at a point $\boldsymbol{a} \in D$ if there is a real linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$||f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - T(\boldsymbol{h})|| = o(||\boldsymbol{h}||) \quad \text{as } \boldsymbol{h} \to \boldsymbol{0} \, .$$

This means that

$$\frac{||f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - T(\boldsymbol{h})||}{||\boldsymbol{h}||} \to 0 \quad \text{as } \boldsymbol{h} \to \boldsymbol{0} \, .$$

We can write this out in terms of the components. Let $T$ be given by the $2\times 2$ real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\left|\left| \begin{pmatrix} f_1(\boldsymbol{a}+\boldsymbol{h}) \\ f_2(\boldsymbol{a}+\boldsymbol{h}) \end{pmatrix} - \begin{pmatrix} f_1(\boldsymbol{a}) \\ f_2(\boldsymbol{a}) \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right|\right| = o(||\boldsymbol{h}||) \quad \text{as } \boldsymbol{h} \to \boldsymbol{0} \, .$$

This means that

$$|f_1(\boldsymbol{a}+\boldsymbol{h}) - f_1(\boldsymbol{a}) - (ah_1 + bh_2)| = o(||\boldsymbol{h}||) \quad \text{and}$$
$$|f_2(\boldsymbol{a}+\boldsymbol{h}) - f_2(\boldsymbol{a}) - (ch_1 + dh_2)| = o(||\boldsymbol{h}||)$$

as $\boldsymbol{h} \to \boldsymbol{0}$. By taking one of the components of $\boldsymbol{h}$ to be 0 in this formula, we see that the matrix for $T$ must be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{a}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{a}) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{a}) \end{pmatrix} .$$

We can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by letting $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ correspond to $x_1 + ix_2$. Then $f$ gives a map $f : D \to \mathbb{C}$. This is **complex** differentiable if it is **real** differentiable and the map $T$ is linear over the complex numbers. The complex linear maps $T : \mathbb{C} \to \mathbb{C}$ are just multiplication by a complex number $w = w_1 + iw_2$, so $T$ must be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix} .$$

In particular, this shows that a complex differentiable function must satisfy the *Cauchy – Riemann equations*:

$$\frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) = \frac{\partial f_2}{\partial x_2}(\boldsymbol{a}) \quad \text{and} \quad \frac{\partial f_1}{\partial x_2}(\boldsymbol{a}) = -\frac{\partial f_2}{\partial x_1}(\boldsymbol{a}) \ .$$

There are also more direct ways to obtain the Cauchy – Riemann equations. For example, if $f : D \to \mathbb{C}$ is complex differentiable at a point $a$ with derivative $f'(a)$, then we can consider the functions

$$x_1 \mapsto f(a + x_1) \quad \text{and} \quad x_2 \mapsto f(a + ix_2)$$

for real values of $x_1$ and $x_2$. These must also be differentiable and so

$$f'(a) = \frac{\partial f}{\partial x_1}(a) = \frac{\partial f_1}{\partial x_1}(a) + i\frac{\partial f_2}{\partial x_1}(a) \quad \text{and} \quad f'(a) = \frac{1}{i}\frac{\partial f}{\partial x_2}(a) = -i\frac{\partial f_1}{\partial x_2}(a) + \frac{\partial f_2}{\partial x_2}(a) \ .$$

## 2. POWER SERIES

A *power series* is an infinite sum of the form $\sum_{n=0}^{\infty} a_n(z-z_o)^n$. Recall that a power series converges on a disc.

**Proposition 2.1** Radius of convergence
*For the sequence of complex numbers $(a_n)$ define $R = \sup\{r : a_n r^n \to 0 \text{ as } n \to \infty\}$. Then the power series $\sum a_n z^n$ converges absolutely on the open disc $\mathbb{D}(z_o, R)$ and diverges outside the corresponding closed disc $\overline{\mathbb{D}(z_o, R)}$. Indeed, the power series converges uniformly on each disc $B(z_o, r)$ with $r$ strictly less than $R$.*

We call $R$ the *radius of convergence of the power series* $\sum a_n(z-z_o)^n$. *It can take any value from* 0 *to* $+\infty$ *including the extreme values. The series may converge or diverge on the circle* $\partial\mathbb{D}(z_o, R)$.

*Proof:*
It is clear that if $\sum a_n(z-z_o)^n$ converges then the terms $a_n(z-z_o)^n$ must tend to 0 as $n \to \infty$. Therefore, $a_n r^n \to 0$ as $n \to \infty$ for each $r \leqslant |z-z_o|$. Hence $R \geqslant |z-z_o|$ and we see that the power series diverges for $|z-z_o| > R$.

Suppose that $|z-z_o| < R$. Then we can find $r$ with $|z-z_o| < r < R$ and $a_n r^n \to 0$ as $n \to \infty$. This means that there is a constant $K$ with $|a_n|r^n \leqslant K$ for each $n \in \mathbb{N}$. Hence

$$\sum |a_n||z-z_o|^n \leqslant \sum K \left(\frac{|z-z_o|}{r}\right)^n .$$

The series on the right is a convergent geometric series, and $\sum a_n z^n$ converges, absolutely, by comparison with it. Also, this convergence is uniform on $\mathbb{D}(z_o, r)$. □

We wish to prove that a power series can be differentiated term-by-term within its disc of convergence.

**Proposition 2.2** Power series are differentiable.
*Let $R$ be the radius of convergence of the power series $\sum a_n(z-z_o)^n$. The sum $s(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$ is complex differentiable on the disc $\mathbb{D}(z_o, R)$ and has derivative $t(z) = \sum_{n=1}^{\infty} n a_n (z-z_o)^{n-1}$.*

*Proof:*
We may assume that $z_o = 0$. For a fixed point $w$ with $|w| < R$, we can choose $r$ with $|w| < r < R$. We will consider $h$ satisfying $|h| < r - |w|$ so that $|w+h| < r$.

Consider the function (curve):

$$\gamma : [0,1] \to \mathbb{C} \; ; \quad t \mapsto n(n-1)(w+th)^{n-2}h^2 \; .$$

Straightforward integration shows that

$$\int_0^s \gamma(t)\, dt = n(w+th)^{n-1}h \Big|_0^s = n(w+sh)^{n-1}h - nw^{n-1}h$$

and

$$\int_0^1 \int_0^s \gamma(t)\, dt\, ds = (w+sh)^n - nw^{n-1}sh \Big|_0^1 = (w+h)^n - w^n - nw^{n-1}h \; .$$

For each $t \in [0, 1]$ we have $|w + th| < r$, so $|\gamma(t)| \leqslant n(n-1)r^{n-2}|h|^2$. This implies that

$$|(w+h)^n - w^n - nw^{n-1}| \leqslant \int_0^1 \int_0^s n(n-1)r^{n-2}|h|^2 \, dt \, ds = \tfrac{1}{2}n(n-1)r^{n-2}|h|^2$$

Hence,

$$\begin{aligned} |s(w+h) - s(w) - t(w)h| &= \left| \sum_{n=0}^{\infty} a_n \left( (w+h)^n - w^n - nw^{n-1}h \right) \right| \\ &\leqslant \sum_{n=0}^{\infty} |a_n| |(w+h)^n - w^n - nw^{n-1}h| \\ &\leqslant \frac{1}{2} \left( \sum_{n=0}^{\infty} n(n-1)|a_n|r^{n-2} \right) |h|^2 \ . \end{aligned}$$

The series $\sum n(n-1)|a_n|r^{n-2}$ converges by comparison with $\sum |a_n|s^n$ for any $s$ with $r < s < R$. Therefore, $s$ is differentiable at $w$ and $s'(w) = t(w)$. □

The derivative of the power series $s$ is itself a power series, so $s$ is twice differentiable. Repeating this shows that $s$ is infinitely differentiable, that is we can differentiate it as many times as we wish.

**Corollary 2.3** Power series are infinitely differentiable
*Let $R$ be the radius of convergence of the power series $\sum a_n(z - z_o)^n$. Then the sum*

$$s(z) = \sum_{n=0}^{\infty} a_n(z - z_o)^n$$

*is infinitely differentiable on $B(z_o, R)$ with*

$$s^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n(z - z_o)^{n-k} \ .$$

*In particular, $s^{(k)}(z_o) = k!a_k$, so the power series is the Taylor series for $s$.*

□

**The Exponential Function**

One of the most important applications of power series is to the exponential function. This is defined as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \ .$$

The ratio test shows that the series converges for all complex numbers $z$. Hence, it defines a function

$$\exp : \mathbb{C} \to \mathbb{C} \ .$$

We know, from Proposition 2.1 , that the exponential function is differentiable with

$$\exp'(z) = \sum_{n=0}^{\infty} n \frac{1}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \exp(z) \ .$$

This is the key property of the function and we will use it to establish the other properties.

**Proposition 2.4** Products of exponentials
*For any complex numbers $w, z$ we have*

$$\exp(z + w) = \exp(z) \exp(w) \ .$$

*Proof:*

Let $a$ be a fixed complex number and consider the function

$$g(z) = \exp(z)\exp(a-z) \ .$$

This is differentiable and its derivative is

$$g'(z) = \exp(z)\exp(a-z) - \exp(z)\exp(z-a) = 0 \ .$$

This implies that $g$ is constant. (For consider the function $\gamma : t \mapsto g(tz)$ defined on the unit interval $[0,1] \subset \mathbb{R}$. This has derivative 0 and so the mean value theorem shows that it is constant. Therefore, $g(z) = g(0)$ for each $z$.) The value of $g$ at 0 is $\exp(0)\exp(a) = \exp(a)$, so we see that

$$\exp(z)\exp(a-z) = \exp(a) \ .$$

□

This Proposition allows us to establish many of the properties of the exponential function very easily.

**Corollary 2.5** Properties of the exponential

*(a) The exponential function has no zeros.*

*(b) For any complex number $z$ we have $\exp\overline{z} = \overline{\exp z}$.*

*(c) $e : x \mapsto \exp x$ is a strictly increasing function from $\mathbb{R}$ onto $(0,\infty)$.*

*(d) For real numbers $y$, the map $f : y \mapsto \exp iy$ traces out the unit circle, at unit speed, in the positive direction.*

*Proof:*

(a) For $\exp(z)\exp(-z) = \exp(0) = 1$.

(b) Is an immediate consequence of the power series.

(c) For $x \in \mathbb{R}$ it is clear from the power series that $\exp x$ is real. Moreover $\exp x = \left(\exp \tfrac{1}{2}x\right)^2 > 0$. This shows that $e'(x) = e(x) > 0$ and so $e$ is a strictly increasing positive function. The power series also shows that

$$e(x) = \exp x > x \qquad \text{for } x > 1$$

so $e(x) \nearrow +\infty$ as $x \nearrow +\infty$. Finally,

$$e(x) = \frac{1}{e(-x)} \searrow 0 \qquad \text{as } x \searrow -\infty \ .$$

(d) Part (b) shows that $|\exp iy|^2 = \exp iy \ \exp -iy = 1$, so $f$ maps into the unit circle. Moreover, $f$ is differentiable with $f'(y) = i\exp iy$, so $f$ traces out the unit circle at unit speed in the positive direction. □

Any complex number $w$ can be written as $r(\cos\theta + i\sin\theta)$ for some modulus $r \geqslant 0$ and some argument $\theta \in \mathbb{R}$. The modulus $r = |z|$ is unique but the argument is only determined up to adding an integer multiple of $2\pi$ (and is completely arbitrary when $w = 0$).

Part (a) of the Corollary shows that $\exp z$ is never 0. Suppose that $w \neq 0$. Then (c) shows that we can find a unique real number $x$ with $\exp x = |w|$. Part (d) shows that $\exp i\theta = \cos\theta + i\sin\theta$. Hence

$$w = |w| \exp i\theta = \exp x \exp i\theta = \exp(x + i\theta) \ .$$

So there is a complex number $z_o = x + i\theta$ with $\exp z_o = w$. Furthermore, parts (c) and (d) show that the only solutions of $\exp z = w$ are $z = z_o + 2n\pi i$ for an integer $n \in \mathbb{Z}$.

**Logarithms**

Corollary 2.5(c) shows that the exponential function on the real line gives a strictly increasing map $e : \mathbb{R} \to (0, \infty)$ from $\mathbb{R}$ onto $(0, \infty)$. This map must then be invertible and we call its inverse the *natural logarithm* and denote it by $\ln : (0, \infty) \to \mathbb{R}$. We want to consider analogous complex logarithms that are inverse to the complex exponential function.

We know that $\exp z$ is never 0, so we can not hope to define a complex logarithm of 0. For any non-zero complex number $w$ we have seen that there are infinitely many complex numbers $z$ with $\exp z = w$ and any two differ by an integer multiple of $2\pi i$. Therefore, the exponential function can not be invertible.

However, if we restrict our attention to a suitable domain $D$ in $\mathbb{C} \setminus \{0\}$, then we can try to find a continuous function $\lambda : D \to \mathbb{C}$ with $\exp \lambda(z) = z$ for each $z \in D$. Such a map is called a *branch of the logarithm* on $D$. If one branch $\lambda$ exists, then $z \mapsto \lambda(z) + 2n\pi i$ is another branch of the logarithm.

Consider, for example, the domain

$$D = \{z = r \exp i\theta : 0 < r \text{ and } \alpha < \theta < \alpha + 2\pi\}$$

that is obtained by removing a half-line from $\mathbb{C}$. The map

$$\lambda : D \to \mathbb{C} \ ; \quad r \exp i\theta \mapsto \ln r + i\theta$$

for $r > 0$ and $\alpha < \theta < \alpha + 2\pi$ is certainly continuous and satisfies $\exp \lambda(z) = z$ for each $z \in D$. Hence it is one of the branches of the logarithm on $D$.

As remarked above, the point 0 is special and there is no branch of the logarithm defined at 0. We call 0 a *logarithmic singularity.* Many authors abuse the notation by writing $\log z$ for $\lambda(z)$. However, it is important to remember that there are many branches of the logarithm and that there is none defined on all of $\mathbb{C} \setminus \{0\}$.

The branches of the logarithm are important and we will use them throughout this course. Note that, for any branch $\lambda$ of the logarithm, we have

$$\lambda(z) = \ln |z| + i\theta$$

where $\theta$ is an argument of $z$. The real part is unique and clearly continuous. However, the imaginary part is only determined up to an additive integer multiple of $2\pi$. The choice of a branch of the logarithm on $D$ corresponds to a continuous choice of the argument $\theta : D \to \mathbb{R}$.

Since the branch $\lambda : D \to \mathbb{C}$ is inverse to the exponential function, the inverse function theorem shows that $\lambda$ is differentiable with

$$\lambda'(w) = \frac{1}{\exp' \lambda(w)} = \frac{1}{\exp \lambda(w)} = \frac{1}{w} \ .$$

To do this more carefully, let $w$ be a point of $D$. Choose $k \neq 0$ so small that $w + k \in D$. Then set $z = \lambda(w)$ and $z + h = \lambda(w + k)$. Since $\lambda$ is continuous, $h \to 0$ as $k \to 0$. Hence

$$\frac{\lambda(w+k) - \lambda(w)}{k} = \frac{(z+h) - z}{\exp\lambda(w+k) - \exp\lambda(w)} = \frac{h}{\exp(z+h) - \exp z}$$

tends to $\frac{1}{\exp' z}$ as $K \to 0$. This shows that $\lambda$ is complex differentiable at $w$ with

$$\lambda'(w) = \frac{1}{\exp z} = \frac{1}{w} \ .$$

Thus every branch of the logarithm is analytic.

Let $\Omega$ be the complex plane cut along the negative real axis: $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Every $z \in \Omega$ can be written uniquely as

$$r \exp i\theta \qquad \text{with } r > 0 \ \text{ and } \ -\pi < \theta < \pi.$$

We call this $\theta$ the *principal branch of the argument of* $z$ and denote it by $\mathrm{Arg}(z)$. In a similar way, the *principal branch of the logarithm* is:

$$\mathrm{Log} : \Omega \to \mathbb{C} \ ; \quad z \mapsto \ln|z| + i\,\mathrm{Arg}(z) \ .$$

**Powers**

We can also define branches of powers of complex numbers. Suppose that $n \in \mathbb{Z}$, $z$ a complex number and $\lambda : D \to \mathbb{C}$ any branch of the complex logarithm defined at $z$. Then

$$z^n = (\exp\lambda(z))^n = \exp(n\lambda(z))$$

and the value of the right side does not depend on which branch $\lambda$ we choose. When $\alpha$ is a complex number but not an integer, we may define a *branch* of the $\alpha$th power on $D$ by

$$p_\alpha : D \to \mathbb{C} \ ; \quad z \mapsto (\exp\alpha\lambda(z)) \ .$$

This behaves as we would expect an $\alpha$th power to, for example,

$$p_\alpha(z)p_\beta(z) = p_{\alpha+\beta}(z)$$

analogously to $z^\alpha z^\beta = z^{\alpha+\beta}$ for integers $\alpha$ and $\beta$. Moreover, $p_\alpha$ is analytic on $D$ since exp and $\lambda$ are both analytic with

$$p'_\alpha(z) = \exp'(\alpha\lambda(z))\alpha\lambda'(z) = (\exp\alpha\lambda(z))\,\frac{\alpha}{z} = \alpha\exp((\alpha-1)\lambda(z)) = \alpha p_{\alpha-1}(z) \ .$$

However, there are many different branches of the $\alpha$th power coming from different branches of the logarithm.

For example, on the cut plane $\Omega = \mathbb{C} \setminus (-\infty, 0]$ the principal branch of the $\alpha$th power is given by

$$z \mapsto \exp(\alpha\,\mathrm{Log}\,z) = \exp(\alpha(\ln|z| + i\,\mathrm{Arg}(z))) \ .$$

When $\alpha = \frac{1}{2}$ this is

$$r\exp i\theta \mapsto r^{1/2}\exp\tfrac{1}{2}i\theta \qquad \text{for } r > 0 \text{ and } -\pi < \theta < \pi.$$

Note that none of these branches of powers is defined at 0 since no branch of the logarithm is defined there. The point 0 is called a *branch point* for the power. The only powers that can be defined to be analytic at 0 are the non-negative integer powers.

If we set $e = \exp 1 = 2.71828\ldots$, then $\exp z$ is one of the values for the $z$th power of $e$. Despite the fact that there are other values (unless $z \in \mathbb{Z}$) we often write this as $e^z$. In particular, it is very common to write $e^{i\theta}$ for $\exp i\theta$.

**Conformal Maps**

A *conformal map* is an analytic map $f : D \to \Omega$ between two domains $D, \Omega$ that has an analytic inverse $g : \Omega \to D$. This certainly implies that $f$ is a bijection and that $f'(z)$ is never 0, since the chain rule gives $g'(f(z))f'(z) = 1$. When there is a conformal map $f : D \to \Omega$ then the complex analysis on $D$ and $\Omega$ are the same, for we can transform any analytic map $h : D \to \mathbb{C}$ into a map $h \circ g : \Omega \to \mathbb{C}$ and *vice versa*.

You have already met Möbius transformations as examples of conformal maps. For instance, $z \mapsto \dfrac{1+z}{1-z}$ is a conformal map from the unit disc $\mathbb{D}$ onto the right half-plane $H = \{x + iy : x > 0\}$. Its inverse is $w \mapsto \dfrac{w-1}{w+1}$. Powers also give useful examples, for instance:

$$\{x + iy : x, y > 0\} \to \{u + iv : v > 0\} \ ; \quad z \mapsto z^2$$

is a conformal map. Its inverse is a branch of the square root. Similarly, the exponential map gives us examples. The map

$$\{x + iy : -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi\} \to \{u + iv : u > 0\} \ ; \quad z \mapsto \exp z$$

is conformal. Its inverse is the principal branch of the logarithm.

Conformal maps preserve the angles between curves. For consider the straight line $\beta : t \mapsto z_o + t\omega$ where $|\omega| = 1$. The analytic map $f$ sends this to the curve

$$f \circ \beta : t \mapsto f(z_o + t\omega) \ .$$

The tangent to this curve at $t = 0$ is in the direction of

$$\lim_{t \to 0} \frac{f(z_o + t\omega) - f(z_o)}{|f(z_o + t\omega) - f(z_o)|} = \frac{f'(z_o)\omega}{|f'(z_o)\omega|} \ .$$

Provided that $f'(z_o) \neq 0$, this shows that $f \circ \beta$ is a curve through $f(z_o)$ in the direction of $f'(z_o)\omega$. Consequently, such a function $f$ preserves the angle between two curves, in both magnitude and orientation. This shows that conformal maps preserve the angles between any two curves.

## 3. INTEGRATION ALONG CURVES

We have seen that it is a much stronger condition on a function to be complex differentiable than to be real differentiable. The reason for this is that we can apply the fundamental theorem of calculus when we integrate $f$ along a curve in $D$ that starts and ends at the same point. This will show that, for suitable curves, the integral is 0 — a result we call Cauchy's theorem. This theorem has many important consequences and is the key to the rest of the course.

We therefore wish to integrate functions along curves in $D$. First recall some of the properties of integrals along intervals of the real line. If $\phi : [a, b] \to \mathbb{C}$ is a continuous function, then the Riemann integral

$$I = \int_a^b \phi(t)\ dt$$

exists. For any angle $\theta$, we have

$$Ie^{i\theta} = \Re\left(\int_a^b \phi(t)e^{i\theta}\ dt\right) = \int_a^b \Re\left(\phi(t)e^{i\theta}\right)\ dt \leqslant \int_a^b |\phi(t)|\ dt$$

so we have the inequality

$$\left|\int_a^b \phi(t)\ dt\right| \leqslant \int_a^b |\phi(t)|\ dt\ .$$

A *continuously differentiable curve* in $D$ is a map $\gamma : [a, b] \to D$ defined on a compact interval $[a, b] \subset \mathbb{R}$ that is continuously differentiable at each point of $[a, b]$. (At the endpoints $a, b$ we demand a one-sided derivative.) The image $\gamma([a, b])$ will be denoted by $[\gamma]$. We think of the parameter $t$ as time and the point $z = \gamma(t)$ traces out the curve as time increases. The direction that we move along the curve is important and is often denoted by an arrow.

As the time increases by a small amount $\delta t$, so the point $z = \gamma(t)$ on the curve moves by $\delta z = \gamma(t + \delta t) - \gamma(t) \approx \gamma'(t)\ \delta t$. Hence, it is natural to define the integral of a continuous function $f : D \to \mathbb{C}$ along $\gamma$ to be

$$\int_\gamma f(z)\ dz\ =\ \int_a^b f(\gamma(t))\gamma'(t)\ dt\ .$$

We can also define integrals with respect to the arc-length s along $\gamma$ where $\frac{ds}{dt} = |\gamma'(t)|$. This is usually denoted by:

$$\int_\gamma f(z)\ |dz| = \int_a^b f(\gamma(t))\ |\gamma'(t)|\ dt\ .$$

In particular, the *length* of $\gamma$ is:

$$L(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\ dt\ .$$

Then we have the important inequality:

**Proposition 3.1**
*Let $\gamma : [a, b] \to D$ be a continuously differentiable curve in the domain $D$ and let $f : D \to \mathbb{C}$ be a continuous function. Then*

$$\left|\int_\gamma f(z)\ dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\ dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|\ dt \leqslant L(\gamma). \sup\{|f(z)| : z \in [\gamma]\}\ .$$

□

**Example:** The straight-line curve $[w_0, w_1]$ between two points of $\mathbb{C}$ is given by

$$[0,1] \to \mathbb{C} \; ; \quad t \mapsto (1-t)w_o + tw_1 \; .$$

This has length $|w_1 - w_0|$. The unit circle $c$ is given by

$$c : [0, 2\pi] \to \mathbb{C} \; ; \quad t \mapsto z_o + r \exp it$$

and has length $2\pi$. For any integer $n$ we have

$$\int_c z^n \, dz = \int_0^{2\pi} \exp int \; i \exp it \; dt = \begin{cases} 0 & \text{if } n \neq -1; \\ 2\pi i & \text{if } n = -1. \end{cases}$$

It is possible to re-parametrise a curve $\gamma : [a,b] \to D$. Suppose that $h : [c,d] \to [a,b]$ is a continuously differentiable, strictly increasing function with a continuously differentiable inverse $h^{-1} : [a,b] \to [c,d]$. Then $\gamma \circ h : [c,d] \to D$ is a curve and the substitution rule for integrals shows that

$$\int_{\gamma \circ h} f(z) \, dz = \int_c^d f(\gamma(h(s))\gamma'(h(s))h'(s) \, ds = \int_a^b f(\gamma(t))\gamma'(t) \, dt = \int_\gamma f(z) \, dz$$

and similarly that $L(\gamma \circ h) = L(\gamma)$. Sometimes it is useful to reverse the orientation of the curve. For any curve $\gamma : [a,b] \to D$, the *reversed curve* $-\gamma$ is given by

$$-\gamma : [-b, -a] \to D \; ; \quad t \mapsto \gamma(-t) \; .$$

This traces out the same image as $\gamma$ but in the reverse direction.

It is useful to generalise the definition of a curve slightly. A *piecewise continuously differentiable curve* is a map $\gamma : [a,b] \to D$ for which there is a subdivision

$$a = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = b$$

with each of the restrictions $\gamma| : [t_n, t_{n+1}] \to D$ $(n = 0, 1, \ldots, N)$ being a continuously differentiable curve. The integral along $\gamma$ is then

$$\int_\gamma f(z) \, dz \;=\; \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f(\gamma(t))\gamma'(t) \, dt$$

and

$$\int_\gamma f(z) \, |dz| = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f(\gamma(t))|\gamma'(t)| \, dt \; .$$

We clearly have

$$\left| \int_\gamma f(z) \, dz \right| \leqslant \int_\gamma |f(z)| \, |dz| \leqslant L(\gamma). \sup\{|f(z)| : z \in [\gamma]\} \; .$$

From now on, we will suppose, tacitly, that all the curves we consider are piecewise continuously differentiable.

**Proposition 3.2** Fundamental Theorem of Calculus
*Let $f : D \to \mathbb{C}$ be an analytic function. If $f$ is the derivative of another analytic function $F : D \to \mathbb{C}$, then*

$$\int_\gamma f(z) \, dz = F(\gamma(b)) - F(\gamma(a))$$

*for any piecewise continuously differentiable curve $\gamma : [a,b] \to D$.*

We call $F : D \to \mathbb{C}$ an *antiderivative* of $f$ if $F'(z) = f(z)$ for all $z \in D$.

*Proof:*

The fundamental theorem of calculus show that

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_a^b F'(\gamma(t))\gamma'(t)\ dt = \int_a^b (F \circ \gamma)'(t)\ dt = F(\gamma(b)) - F(\gamma(a))$$

for any continuously differentiable curve $\gamma$. The result follows for piecewise continuously differentiable curves by adding the results for each continuously differentiable section. □

A curve $\gamma : [a, b] \to D$ is *closed* if $\gamma(b) = \gamma(a)$. In this case, the Proposition shows that

$$\int_\gamma f(z)\ dz = 0$$

provided that $f$ is the derivative of a function $F : D \to \mathbb{C}$. This is our first form of Cauchy's theorem.

For the sake of variety, we use many different names for curves, such as paths or routes. Closed curves are sometimes called contours.

**Example:** Let $A$ be the domain $\mathbb{C} \setminus \{0\}$ and $\gamma$ the closed curve

$$\gamma : [0, 2\pi] \to A\ ;\quad t \mapsto \exp it$$

that traces out the unit circle in a positive direction. Let $f(z) = z^n$ for $n \in \mathbb{Z}$. Then

$$\int_\gamma z^n\ dz = \int_0^{2\pi} \exp int\ (2\pi i \exp it)\ dt = \begin{cases} 2\pi i & \text{when } n = -1; \\ 0 & \text{otherwise.} \end{cases}$$

For each function $f(z) = z^n$ with $n \neq -1$ there is a function $F(z) = z^{n+1}/(n + 1)$ with $F'(z) = f(z)$ on $A$, so the integral around $\gamma$ should be 0. However, for $n = -1$ the Proposition shows that there can be no such function $F : A \to \mathbb{C}$ with $F'(z) = \frac{1}{z}$. This means that there is no branch of the logarithm $f$ defined on all of $A$.

## Winding Numbers

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve that does not pass through 0. A *continuous choice of the argument on* $\gamma$ is a continuous map $\theta : [a, b] \to \mathbb{R}$ with $\gamma(t) = |\gamma(t)| \exp i\theta(t)$ for each $t \in [a, b]$. The change $\theta(b) - \theta(a)$ measures the angle about 0 turned through by $\gamma$. We call $(\theta(b) - \theta(a))/2\pi$ the *winding number* $n(\gamma, 0)$ *of* $\gamma$ *about* 0. Suppose that $\phi$ is another continuous choice of the argument on $\gamma$. Then $\theta(t) - \phi(t)$ must be an integer multiple of $2\pi$. Since $\theta - \phi$ is continuous on the connected interval $[a, b]$, we see that there is an integer $k$ with $\phi(t) - \theta(t) = 2k\pi$ for all $t \in [a, b]$. Hence $\theta(b) - \theta(a) = \phi(b) - \phi(a)$ and the winding number is well defined.

When $\gamma$ is a piecewise continuously differentiable curve, we can give a continuous choice of $\theta(t)$ explicitly and hence find an expression for the winding number. Let

$$h(t) = \int_{\gamma|[a,t]} \frac{1}{z}\ dz = \int_a^t \frac{\gamma'(t)}{\gamma(t)}\ dt$$

for $t \in [a, b]$. The chain rule shows that

$$\frac{d}{dt}\left(\gamma(t) \exp -h(t)\right) = \gamma'(t)(\exp -h(t)) - \gamma(t)h'(t)(\exp -h(t)) = \left(\gamma'(t) - \gamma(t)\frac{\gamma'(t)}{\gamma(t)}\right) \exp -h(t) = 0\ .$$

Hence $\gamma(t) \exp -h(t)$ is constant. Therefore,

$$\gamma(t) = \gamma(a) \exp h(t) = \gamma(a) \exp \Re h(t) \exp i \Im h(t) \ .$$

This means that $\theta(t) = \arg \gamma(a) + \Im h(t)$ gives a continuous choice of the argument of $\gamma(t)$. Consequently, the total angle turned through by $\gamma$ is

$$\Im \left( \int_\gamma \frac{1}{z} \, dz \right) \ .$$

If $\gamma$ is piecewise continuously differentiable, we can apply this argument to each section of $\gamma$ and so find that the final formula still holds.

The formula is particularly important when $\gamma$ is a closed curve. Then $\gamma(b) = \gamma(a)$, so $\exp h(b) = 1$ and we must have $h(b) = 2N\pi i$ for some integer $N$. The number $N$ counts the number of times $\gamma$ winds positively around 0. We have the formula:

$$N = \frac{h(b)}{2\pi i} = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, dz \ .$$

We can also consider how many times a closed curve $\gamma$ winds around any point $w_o$ that does not lie on $\gamma$. By translating $w_o$ to 0 we see that this is

$$n(\gamma; w_o) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w_o} \, dz \ ,$$

which is called the *winding number of $\gamma$ about $w_o$*.

**Example:** The curve $\gamma : [0, 1] \to \mathbb{C}; \ t \mapsto z_o + re^{2\pi i t}$ has winding number

$$n(\gamma; w_o) = \begin{cases} 1 & \text{when } |w_o - z_o| < r; \\ 0 & \text{when } |w_o - z_o| > r. \end{cases}$$

It is not defined when $|w_o - z_o| = r$.

**Lemma 3.3**
*Let $\gamma$ be a piecewise continuously differentiable closed curve taking values in the disc $B(z_o, R)$. Then $n(\gamma; w_o) = 0$ for all points $w_o \notin B(z_o, R)$.*

*Proof:*
By translating and rotating the curve, we may assume that $w_o = 0$ and $z_o$ is a positive real number no smaller than $R$. For $z$ in the disc $B(z_o, R)$, we can find an unique real number $\phi(z) \in (-\pi, \pi)$ with $z = |z|e^{i\phi(z)}$. (This is the principal branch of the argument of $z$.) The map $\phi : B(1, 1) \to \mathbb{R}$ is then continuous. Hence, $t \mapsto \phi(\gamma(t))$ is a continuous choice of the argument on $\gamma$. So

$$n(\gamma; 0) = \frac{\phi(\gamma(b)) - \phi(\gamma(a))}{2\pi} \ .$$

Since $\gamma(b) = \gamma(a)$, this winding number must be 0. □

The winding number $n(\gamma; w)$ is unchanged if we perturb $\gamma$ by a sufficiently small amount.

**Proposition 3.4** Winding numbers under perturbation
*Let $\alpha, \beta : [a, b] \to \mathbb{C}$ be two closed curve and $w$ a point not on $[\alpha]$. If*

$$|\beta(t) - \alpha(t)| < |\alpha(t) - w| \qquad \textit{for each } t \in [a, b]$$

*then $n(\beta; w) = n(\alpha; w)$.*

*Proof:*

By translating the curves, we may assume that $w = 0$. Then $|\beta(t) - \alpha(t)| < |\alpha(t)|$ for $t \in [a, b]$. This certainly implies that $\beta(t) \neq 0$, so the winding number $n(\beta; 0)$ exists. Write

$$\beta(t) = \alpha(t)\left(1 + \frac{\beta(t) - \alpha(t)}{\alpha(t)}\right) = \alpha(t)\gamma(t) \ .$$

Since the argument of a product is the sum of the arguments, this implies that

$$n(\beta; 0) = n(\alpha; 0) + n(\gamma; 0) \ .$$

However the inequality in the proposition shows that $\gamma$ takes values in the disc $B(1, 1)$ so the lemma proves that $n(\gamma; 0) = 0$. □

**Proposition 3.5** Winding number constant on each component
*Let $\gamma$ be a piecewise continuously differentiable closed curve in $\mathbb{C}$. The winding number $n(\gamma; w)$ is constant for $w$ in each component of $\mathbb{C} \setminus [\gamma]$ and is 0 on the unbounded component.*

*Proof:*

The image $[\gamma]$ is a compact subset of $\mathbb{C}$, so it is bounded, say $[\gamma] \subset B(0, R)$. The complement $U = \mathbb{C} \setminus [\gamma]$ is open, so each component of the complement is also open. One component contains $\mathbb{C} \setminus B(0, R)$, so it is the unique unbounded component that contains all points of sufficiently large modulus.

Let $w_o \in U = \mathbb{C} \setminus [\gamma]$. Then there is a disc $B(w_o, r) \subset U$. For $w$ with $|w - w_o| < r$ we have

$$|(\gamma(t) - w) - (\gamma(t) - w_o)| = |w - w_o| < r \leqslant |\gamma(t) - w_o| \ .$$

Proposition 3.4 then shows that $n(\gamma; w) = n(\gamma; w_o)$. So the function $w \mapsto n(\gamma; w)$ is continuous (indeed constant) at $w_o$. It follows that $w \mapsto n(\gamma; w)$ is a continuous integer-valued function on $U$. It must therefore be constant on each component of $U$.

Lemma 3.3 shows that $n(\gamma; w) = 0$ for $w$ outside the disc $B(0, R)$. So the winding number must be 0 on the unbounded component of $U$. □

**Homotopy**

Let $\gamma_0, \gamma_1 : [a, b] \to D$ be two piecewise continuously differentiable closed curves in the domain $D$. A homotopy from $\gamma_0$ to $\gamma_1$ is a family of piecewise continuously differentiable closed curves $\gamma_s$ for $s \in [0, 1]$ that vary continuously from $\gamma_0$ to $\gamma_1$. This means that the map

$$h : [0, 1] \times [a, b] \to D \ ; \quad (s, t) \mapsto \gamma_s(t)$$

is continuous. More formally, we define a *homotopy* to be a continuous map $h : [0, 1] \times [a, b] \to D$ with

$$h_s : [a, b] \to D \ ; \quad t \mapsto h(s, t)$$

being a piecewise continuously differentiable closed curve in $D$ for each $s \in [0, 1]$. We then say that the curves $h_0$ and $h_1$ are homotopic and write $h_0 \simeq h_1$. This gives an equivalence relation between closed curves in $D$.

**Example:** Suppose that $\gamma_0, \gamma_1 : [0, 1] \to D$ are closed paths in the domain $D$ and that, for each $t \in [0, 1]$, the line segment $[\gamma_0(t), \gamma_1(t)]$ lies within $D$. Then the map

$$h : [0, 1] \times [0, 1] \to D \ ; \quad (s, t) \mapsto (1 - s)\gamma_0(t) + s\gamma_1(t)$$

is continuous and defines a homotopy from $\gamma_0$ to $\gamma_1$. We sometimes call such a homotopy a *linear* homotopy.

A closed curve $\gamma$ in $D$ is *null-homotopic* if it is homotopic in $D$ to a constant curve. The domain $D$ is *simply-connected* if every closed curve in $D$ is null-homotopic. For example, a disc $B(z_o, r)$ is simply-connected since there is a linear homotopy from any curve $\gamma$ in the disc to $z_o$.

A domain $D \subset \mathbb{C}$ is called a *star with centre* $z_o$ if, for each point $w \in D$ the entire line segment $[z_o, w]$ lies within $D$. A domain $D$ is a *star domain* if it is a star with some centre $z_o$. Clearly every disc is a star domain but such domains as $\mathbb{C} \setminus \{0\}$ are not. Every star domain is simply-connected because a curve is linearly homotopic to the constant curve at the centre.

**Proposition 3.6** Winding number and homotopy
*If two closed curves $\gamma_0$ and $\gamma_1$ are homotopic in a domain $D$ and $w \in \mathbb{C} \setminus D$, then $n(\gamma_0; w) = n(\gamma_1; w)$.*

*Proof:*
By translating the curves and the domain, we may assume that $w = 0$.

Let $h : [0,1] \times [a,b] \to D$ be the homotopy with $\gamma_0 = h_0$ and $\gamma_1 = h_1$. Since $[0,1] \times [a,b]$ is a compact subset of $D$, there is an $\varepsilon > 0$ with $|h_s(t)| > \varepsilon$ for each $(s,t) \in [0,1] \times [a,b]$. The homotopy $h$ is uniformly continuous. Hence there is a $\delta > 0$ with

$$|h_s(t) - h_u(t)| < \varepsilon \qquad \text{whenever} \quad |s - u| < \delta \ .$$

This means that

$$|h_s(t) - h_u(t)| < |h_u(t)| \qquad \text{whenever} \quad |s - u| < \delta \ .$$

Hence Proposition 3.4 shows that

$$n(h_s; 0) = n(h_u; 0) \qquad \text{whenever} \quad |s - u| < \delta \ .$$

This clearly establishes the result. □

**Chains and Cycles**

Let $D$ be a domain in $\mathbb{C}$. A *chain* in $D$ is a finite collection $\gamma_n : [a_n, b_n] \to D$ (for $n = 1, 2, 3, \ldots, N$) of piecewise continuously differentiable curves in $D$. We will write $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ for this collection. The empty chain will be written as 0. We can add two chains and obtain another chain. The integral of a continuous function $f : D \to \mathbb{C}$ around $\Gamma$ is then defined to be the sum

$$\int_\Gamma f(z)\, dz = \sum_{n=1}^{N} \int_{\gamma_n} f(z)\, dz \ .$$

In particular, the *winding number* $n(\Gamma; w)$ of a chain $\Gamma$ about any point $w \notin [\Gamma]$ is

$$n(\Gamma; w) = \sum_{n=1}^{N} n(\gamma_n; w) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - w}\, dz \ .$$

A *cycle* in $D$ is a chain $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ where each point $w \in \mathbb{C}$ occurs the same number of times as an initial point $\gamma_n(a_n)$ as it does as an final point $\gamma_n(b_n)$. This means that a cycle consists of a finite number of closed curves, each of which may be made up from a number of the curves $\gamma_n$. The winding number $n(\Gamma; w)$ of a cycle $\Gamma$ is therefore an integer.

Proposition 3.2 shows that any analytic function $f : D \to \mathbb{C}$ that has an antiderivative on $D$ must satisfy

$$\int_\Gamma f(z)\, dz = 0$$

for every cycle $\Gamma$ in the domain $D$.

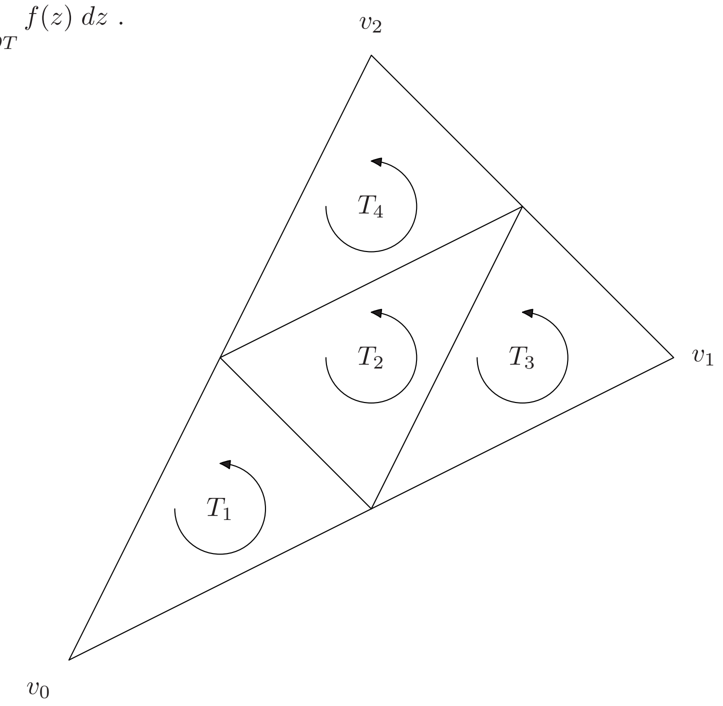
Let $T$ be a closed triangle that lies inside the domain $D$. Let $v_0, v_1, v_2$ be the vertices labelled in anti-clockwise order around $T$. Then the edges $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_0]$ are straight-line paths in $D$. The three sides taken in order give a closed curve $[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$in $D$ that we denote by $\partial T$.

**Proposition 4.1** Cauchy's theorem for triangles
*Let $f : D \to \mathbb{C}$ be an analytic function and $T$ a closed triangle that lies within $D$. Then*

$$\int_{\partial T} f(z)\, dz = 0\ .$$

This proof is due to Goursat and relies on repeated bisection. It underlies all the stronger versions of Cauchy's theorem that we will prove later.

*Proof:*
Set $I = \int_{\partial T} f(z)\, dz$ .




Subdivide $T$ into four similar triangles $T_1, T_2, T_3, T_4$ as shown. Then we have

$$\sum_{k=1}^{4} \int_{\partial T_k} f(z)\, dz = \int_{\partial T} f(z)\, dz$$

because the integrals along the sides of $T_k$ in the interior of $T$ cancel. At least one the integrals

$$\int_{\partial T_k} f(z)\, dz$$

must have modulus at least $\frac{1}{4}|I|$. Choose one of the triangles with this property and call it $T'$. Repeating this procedure we obtain sequence of triangles $(T^{(n)})$ nested inside one another with

$$\left| \int_{\partial T^{(n)}} f(z)\, dz \right| \geqslant \frac{|I|}{4^n}\ .$$

Let $L(\gamma)$ denote the length of a path $\gamma$ and set $L = L(\partial T)$. Then each $T_k$ has $L(\partial T_k) = \frac{1}{2}L$. Therefore, $L(\partial T^{(n)}) = L/2^n$.

The triangle $T$ is a compact subset of $\mathbb{C}$ with $T^{(n)}$ closed subsets. If the intersection $\bigcap_{n\in\mathbb{N}} T^{(n)}$ of these sets were empty, then the complements $T \setminus T^{(n)}$ would form an open cover of $T$ with no finite subcover. Therefore, we must have $\bigcap_{n\in\mathbb{N}} T^{(n)}$ non-empty. Choose a point $z_o \in \bigcap_{n\in\mathbb{N}} T^{(n)}$.

The function $f$ is differentiable at $z_o$. So, for each $\varepsilon > 0$, there is a $\delta > 0$ with

$$\left| \frac{f(z) - f(z_o)}{z - z_o} - f'(z_o) \right| < \varepsilon$$

whenever $z \in B(z_o, \delta)$. This means that

$$f(z) = f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)$$

with $|\eta(z)| < \varepsilon$ for $z \in B(z_o, \delta)$. For $n$ sufficiently large, we have $T^{(n)} \subset B(z_o, \delta)$, so

$$\left| \int_{\partial T^{(n)}} f(z)\ dz \right| = \left| \int_{\partial T^{(n)}} f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)\ dz \right| .$$

The integrals

$$\int_{\partial T^{(n)}} f(z_o)\ dz \quad \text{and} \quad \int_{\partial T^{(n)}} f'(z_o)(z - z_o)\ dz$$

can be evaluated explicitly and are both zero, so

$$\left| \int_{\partial T^{(n)}} f(z)\ dz \right| \leqslant \int_{\partial T^{(n)}} \varepsilon |z - z_o|\ dz \leqslant \varepsilon L(\partial T^{(n)}) \sup\{|z - z_o| : z \in \partial T^{(n)}\} \leqslant \varepsilon L(\partial T^{(n)})^2 = \varepsilon \frac{L^2}{4^n} .$$

This gives

$$|I| = \left| \int_{\partial T} f(z)\ dz \right| \leqslant 4^n \left| \int_{\partial T^{(n)}} f(z)\ dz \right| \leqslant \varepsilon L^2 .$$

This is true for all $\varepsilon > 0$, so we must have $I = 0$. □

We can use this proposition to prove Cauchy's theorem for discs. The proof actually works for any star domain.

**Theorem 4.2** Cauchy's theorem for a star domain
*Let $f : D \to \mathbb{C}$ be an analytic function on a star domain $D \subset \mathbb{C}$ and let $\gamma$ be a piecewise continuously differentiable closed curve in $D$. Then*

$$\int_\gamma f(z)\ dz = 0 .$$

*Proof:*
Let $D$ be the star domain with centre $z_o$ then each line segment $[z_o, z]$ to a point $z \in D$ lies within $D$. By Proposition 3.1 we need only show that there is an antiderivative $F$ of $f$, that is a function with $F'(z) = f(z)$ for $z \in D$. Define $F : D \to \mathbb{C}$ by

$$F(w) = \int_{[z_o, w]} f(z)\ dz .$$

Since $D$ is open, each $w \in D$ is contained in a disc $\mathbb{D}(w, r)$ that lies within $D$. This implies that the triangle with vertices $z_o, w, w + h$ lies within the star domain $D$ provided that $|h| < r$. Then Cauchy's theorem for this triangle gives

$$F(w + h) - F(w) = \int_{[w, w+h]} f(z)\ dz .$$

Consequently,

$$|F(w + h) - F(w) - f(w)h| = \left| \int_{[w, w+h]} f(z) - f(w)\ dz \right| \leqslant |h|. \sup\{|f(z) - f(w)| : z \in [w, w + h]\} .$$

The continuity of $f$ at $w$ shows that $\sup\{|f(z) - f(w)| : z \in [w, w + h]\}$ tends to 0 as $h$ tends to 0. Hence $F$ is differentiable at $w$ and $F'(w) = f(w)$. □

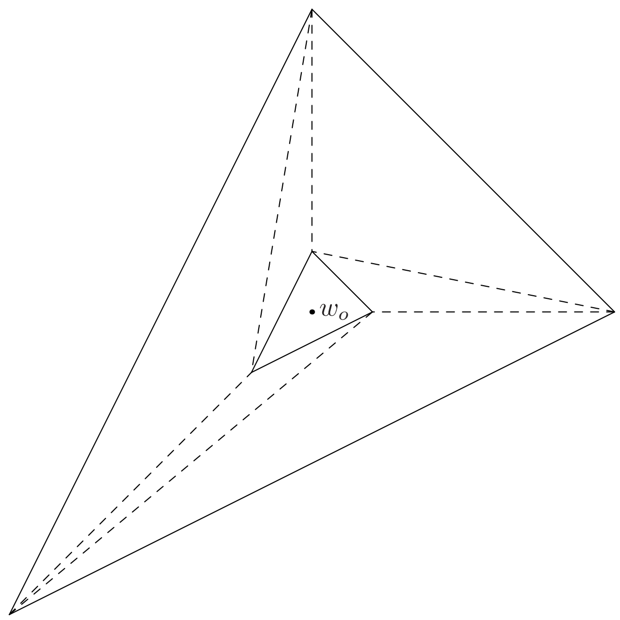
We wish to apply Theorem 4.2 under slightly weaker conditions on $f$. We want to allow there to be a finite number of exceptional points in $D$ where $f$ is not necessarily differentiable but is continuous. Later we will see that such a function must, in fact, be differentiable at each exceptional point.

**Proposition 4.1′** Cauchy's theorem for triangles
*Let $f : D \to \mathbb{C}$ be a continuous function that is complex differentiable at every point except $w_o \in D$. Let $T$ be a closed triangle that lies within $D$. Then*

$$\int_{\partial T} f(z)\, dz = 0 \ .$$

*Proof:*
If $w_o \notin T$, then this result is simply Proposition 4.1 . Hence, we may assume that $w_o \in T$. Let $T^\varepsilon$ be the triangle obtained by enlarging $T$ with centre $w_o$ by a factor $\varepsilon < 1$. Then we can divide $T \setminus T^\varepsilon$ into triangles that lie entirely within $T \setminus \{w_o\}$. The integral around each of these triangles is 0 by Proposition 4.1. Adding these results we see that

$$\int_{\partial T} f(z)\, dz = \int_{\partial T^\varepsilon} f(z)\, dz \ .$$




Since $f$ is continuous on $D$, there is a constant $K$ with $|f(z)| \leqslant K$ for every $z \in T$. Therefore,

$$\left| \int_{\partial T} f(z)\, dz \right| = \left| \int_{\partial T^\varepsilon} f(z)\, dz \right| \leqslant L(\partial T^\varepsilon) K = \varepsilon L(\partial T) K \ .$$

This is true for every $\varepsilon > 0$, so we must have $\displaystyle\int_{\partial T} f(z)\, dz = 0$ as required. □

This proposition allows us to extend Cauchy's Theorem 4.2 to functions that fail to be differentiable at one point (or, indeed, at a finite number of points).

**Theorem 4.2′** Cauchy's theorem for a star domain
*Let $f : D \to \mathbb{C}$ be a continuous function on a star domain $D \subset \mathbb{C}$ that is complex differentiable at every point except $w_o \in D$. Let $\gamma$ be a piecewise continuously differentiable closed curve in $D$. Then*

$$\int_{\gamma} f(z)\, dz = 0 \ .$$

*Proof:*

We argue exactly as in the proof of Theorem 4.2 . Let $z_o$ be a centre for the star domain $D$ and define $F(z)$ to be the integral of $f$ along the straight line path $[z_o, z]$ from $z_o$ to $z$. The previous proposition shows that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(z)\, dz \ .$$

So $F$ is differentiable with $F'(z) = f(z)$ for each $z \in D$. Now Proposition 3.1 gives the result. □

The crucial application of this corollary is the following. Suppose that $f : D \to \mathbb{C}$ is an analytic function on a disc $D = B(z_o, R) \subset \mathbb{C}$ and $w_o \in D$. Then we can define a new function $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(w_o)}{z - w_o} & \text{for } z \neq w_o; \\ f'(w_o) & \text{for } z = w_o. \end{cases}$$

This is certainly complex differentiable at each point of $D$ except $w_o$. At $w_o$ we know that $f$ is differentiable, so $g$ is continuous. We can now apply Theorem 4.2′ to $g$ and obtain

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz$$

for any closed curve $\gamma$ in $D$ that does not pass through $w_o$. Now

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz = \int_\gamma \frac{f(z)}{z - w_o}\, dz - f(w_o) \int_\gamma \frac{1}{z - w_o}\, dz \ .$$

So we obtain

$$f(w_o) n(\gamma; w_o) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w_o}\, dz \ . \qquad (*)$$

This applies, in particular, when $\gamma$ is the boundary of a circle contained in $D$.

**Theorem 4.3** Cauchy's Representation Formula
*Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$ and let $\overline{B(z_o, R)}$ be a closed disc in $D$. Then*

$$f(w) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - w}\, dz \qquad \textit{for } w \in D(z_o, R)$$

*when $C(z_0, R)$ is the circular path $C(z_0, R) : [0, 2\pi] \to \mathbb{C}$ ; $t \mapsto z_o + Re^{it}$.*

*Proof:*

This follows immediately from formula $(*)$ above since the winding number of $C(z_o, R)$ about any $w \in B(z_o, R)$ is 1. □

Cauchy's representation formula is immensely useful for proving the local properties of analytic functions. These are the properties that hold on small discs rather then the global properties that require we study a function on its entire domain. The next chapter will use the representation formula frequently but, as a first example:

**Example:** Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D$. For $z_o \in D$ there is a closed disc $\overline{B(z_o, R)}$ within $D$ and Cauchy's representation formula gives

$$f(z_o) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - z_o}\, dz = \int_0^{2\pi} f(z_o + Re^{i\theta})\, \frac{d\theta}{2\pi} \ .$$

So the value of $f$ at the centre of the circle is the average of the values on the circle $C$.

**Theorem 4.4** Liouville's theorem
*Any bounded analytic function $f : \mathbb{C} \to \mathbb{C}$ defined on the entire complex plane is constant.*

*Proof:*

Let $w, w'$ be any two points of $\mathbb{C}$ and let $M$ be an upper bound for $|f(z)|$ for $z \in \mathbb{C}$. Then Cauchy's representation formula gives

$$f(w) = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w}\, dz \qquad \text{for each } r > |w| \ .$$

Hence,

$$f(w) - f(w') = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w} - \frac{f(z)}{z-w'}\, dz = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)(w-w')}{(z-w)(z-w')}$$

for $r > \max\{|w|, |w'|\}$. Consequently,

$$|f(w) - f(w')| \leqslant \frac{L(C(0,r))}{2\pi} \sup \left\{ \frac{|f(z)||w-w'|}{|z-w||z-w'|} : |z| = r \right\} \leqslant r \left( \frac{M|w-w'|}{(r-|w|)(r-|w'|)} \right) \ .$$

The right side tends to 0 as $r \nearrow +\infty$, so the left side must be 0. Thus $f(w) = f(w')$. □

**Exercise:** Show that an analytic function $f : \mathbb{C} \to \mathbb{C}$ that never takes values in the disc $D(w_o, R)$ is constant.

*For the function*

$$g : \mathbb{C} \to \mathbb{C} \ ; \quad z \mapsto \frac{1}{f(z) - w_o}$$

*is bounded by* $1/R$ *and so is constant by Liouville's theorem.*

**Corollary 4.5** The Fundamental Theorem of Algebra
*Every non-constant polynomial has a zero in* $\mathbb{C}$.

*Proof:*

Suppose that $p(z) = z^N + a_{N-1}z^{N-1} + \ldots a_1 z + a_0$ is a polynomial that has no zero in $\mathbb{C}$. Then $f(z) = 1/p(z)$ is an analytic function. As $z \to \infty$ so $f(z) \to 0$. Hence $f$ is bounded. By Liouville's theorem, $p$ must be constant. □

By dividing a polynomial by $z - z_o$ for each zero $z_o$ we see that the total number of zeros of $p$, counting multiplicity, is equal to the degree of $p$.

## Homotopy form of Cauchy's Theorem.

Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D$. We wish to study how the integral

$$\int_\gamma f(z)\, dz$$

varies as we vary the closed curve $\gamma$ in $D$. Recall that two closed curves $\beta, \gamma : [a, b] \to D$ are *linearly homotopic in* $D$ if, for each $t \in [a, b]$ the line segment $[\beta(t), \gamma(t)]$ is a subset of $D$.

**Theorem 4.6** Homotopy form of Cauchy's Theorem.
*Let* $f : D \to \mathbb{C}$ *be an analytic map on a domain* $D \subset \mathbb{C}$. *If the two piecewise continuously differentiable closed curves* $\alpha$, $\beta$ *are homotopic in* $D$, *then*

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz \ .$$

*Proof:*

Let $h : [0,1] \times [a,b] \to D$ be the homotopy. So each map $h_s : [a,b] \to D$ ; $t \mapsto h(s,t)$ is a piecewise continuously differentiable closed curve in $D$, $h_0 = \alpha$ and $h_1 = \beta$. This means that $h$ is piecewise continuously differentiable on each "vertical" line $\{s\} \times [a,b]$. Initially we will assume that $h$ is also continuously differentiable on each "horizontal" line $[0,1] \times \{t\}$. For any rectangle

$$Q = \{(s,t) \in [0,1] \times [a,b] : s_1 \leqslant s \leqslant s_2 \text{ and } t_1 \leqslant t \leqslant t_2\}$$

let $\partial Q$ denote the boundary of $Q$ positively oriented. Then $h$ is piecewise continuously differentiable on each segment of the boundary, so $h(\partial Q)$ is a piecewise continuously differentiable closed curve in $D$. If we divide $Q$ into two smaller rectangles $Q_1, Q_2$ by drawing a horizontal or vertical line $\ell$ then the segments of the integrals $\int_{h(\partial Q_1)} f(z)\,dz$ and $\int_{h(\partial Q_2)} f(z)\,dz$ along $\ell$ cancel, so

$$\int_{h(\partial Q)} f(z)\,dz = \int_{h(\partial Q_1)} f(z)\,dz + \int_{h(\partial Q_2)} f(z)\,dz\ .$$

For the original rectangle $R = [0,1] \times [a,b]$ the image of the horizontal sides $[0,1] \times \{a\}$ and $[0,1] \times \{b\}$ are the same since each $h_s$ is closed. Hence

$$\int_{h(\partial R)} f(z)\,dz = \int_\beta f(z)\,dz - \int_\alpha f(z)\,dz\ .$$

We need to show that this is 0.

Define $\rho(z) = \inf\{|z-w| : w \in \mathbb{C} \setminus D\}$ to be the distance from $z \in D$ to the complement of $D$. Since $D$ is open, $\rho(z) > 0$ for each $z \in D$. Moreover, $\rho$ is continuous since $|\rho(z) - \rho(z')| \leqslant |z - z'|$. Hence, $\rho$ attains a minimum value on the compact set $h(R)$, say

$$\rho(h(s,t)) \geqslant r > 0 \qquad \text{for every } s \in [0,1],\ t \in [a,b]\ .$$

This means that each disc $B(h(s,t), r)$ is contained in $D$.

Furthermore, we know that $h$ is uniformly continuous on the compact set $[0,1] \times [a,b]$. So there is a $\delta > 0$ with

$$|h(u,v) - h(s,t)| \leqslant r \qquad \text{whenever} \qquad ||(u,v) - (s,t)|| < \delta\ . \tag{$*$}$$

Suppose that $Q$ is a rectangle in $R$ with diameter less than $\delta$ and $P_o$ a point in $Q$. Then $h(Q) \subset B(h(P_o), r)$ and the disc $B(h(P_o), r)$ is a subset of $D$. Cauchy's theorem for star domains (4.2) can now be applied to this disc to see that

$$\int_{h(\partial Q)} f(z)\,dz = 0\ .$$

We can divide $R$ into rectangles $(Q_n)_{n=1}^N$ each with diameter less than $\delta$. So

$$\int_{h(\partial R)} f(z)\,dz = \sum_{n=1}^{N} \int_{h(\partial Q_n)} f(z)\,dz = 0$$

as required.

It remains to deal with the case where the homotopy $h$ is not continuously differentiable on each horizontal line. Choose a subdivision

$$0 = s(0) < s(1) < \ldots < s(N-1) < s(N) = 1$$

of $[0,1]$ with $|s(k+1) - s(k)| < \delta$ for $k = 0, 1, \ldots, N-1$. Then equation $(*)$ above shows that $|h(s(k),t) - h(s(k+1),t)| < r$ for each $t \in [a,b]$. Hence the entire line segment $[h(s(k),t), h(s(k+1),t)]$ lies in the disc $B(h(s(k),t), r)$ and hence in $D$. So $h_{s(k)}$ and $h_{s(k+1)}$ are **LINEARLY** homotopic in $D$. We can certainly apply the above argument to linear homotopies, so we see that

$$\int_{h_{s(k)}} f(z)\,dz = \int_{h_{s(k+1)}} f(z)\,dz\ .$$

Adding these results gives

$$\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz\ .$$

□

**Corollary 4.7** Cauchy's Theorem for null-homotopic curves
*Let $f : D \to \mathbb{C}$ be an analytic map on a domain $D$ and $\gamma$ a piecewise continuously differentiable closed curve in $D$ that is null-homotopic in $D$. Then*

$$\int_\gamma f(z)\, dz = 0 \ .$$

□

If the domain $D$ is simply connected, then any closed curve in $D$ is null-homotopic, so Cauchy's theorem will apply.

# 5. CONSEQUENCES OF CAUCHY'S THEOREM

## Cauchy Transforms

Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise continuously differentiable path in $\mathbb{C}$ and $\phi : [\gamma] \to \mathbb{C}$ a continuous function on $[\gamma]$. Then the integral

$$\Phi(w) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{z-w}\, dz$$

exists for each $w \in \mathbb{C}\setminus[\gamma]$. This is the *Cauchy transform of* $\phi$. We will show that it defines a function analytic everywhere except on $[\gamma]$.

**Proposition 5.1** Cauchy transforms have power series
*Let $\Phi$ be the Cauchy transform of a continuous function $\phi : [\gamma] \to \mathbb{C}$. For $z_o \in \mathbb{C}\setminus[\gamma]$ let $R$ be the radius of the largest disc $B(z_o, R)$ that lies within $\mathbb{C}\setminus[\gamma]$. Then*

$$\Phi(w) = \sum_{n=0}^{\infty} a_n (w-z_o)^n \qquad \text{for} \quad |w-z_o| < R$$

*where the coefficients $a_n$ are given by*

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}}\, dz\ .$$

*Proof:*

We may assume, by translating $\gamma$, that $z_o = 0$. The formula for the sum of a geometric series shows that

$$\frac{1}{z-w} = \frac{1}{z} + \frac{w}{z^2} + \ldots + \frac{w^{N-1}}{z^N} + \frac{w^N}{z^N(z-w)}\ .$$

Integrating this gives

$$\Phi(w) = a_0 + a_1 w + \ldots + a_{N-1}w^{N-1} + E_N(w)$$

where

$$a_k = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)}{z^{k+1}}\, dz \qquad \text{and} \qquad E_N(w) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(z)w^N}{z^N(z-w)}\, dz\ .$$

Let $||\phi||_\infty = \sup\{|\phi(z)| : z \in [\gamma]\}$. For $z \in [\gamma]$ we have $|z| \geqslant R$ and $|z-w| \geqslant R - |w|$, so

$$|E_N(w)| \leqslant \frac{L(\gamma)}{2\pi}\frac{||\phi||_\infty}{(R-|w|)}\left(\frac{|w|}{R}\right)^N\ .$$

This shows that, for $|w| < R$,

$$\left|\Phi(w) - \sum_{n=0}^{N-1} a_n w^n\right| = |E_N(w)| \to 0 \qquad \text{as } N \to \infty\ .$$

Therefore the power series $\sum a_n w^n$ converges on $B(0,R)$ to $\Phi$. □

**Corollary 5.2** Cauchy transforms are infinitely differentiable
*The Cauchy transform $\Phi$ of a continuous function $\phi : [\gamma] \to \mathbb{C}$ is infinitely differentiable on $\mathbb{C}\setminus[\gamma]$ with*

$$\Phi^{(n)}(z_o) = \frac{n!}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}}\, dz\ .$$

*Proof:*

We know that $\Phi$ is given by a power series $\Phi(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$ on the disc $B(z_o, R)$. By Corollary 2.3 this power series is infinitely differentiable. Moreover,

$$\Phi^{(n)}(z_o) = n! a_n = \frac{n!}{2\pi i}\int_\gamma \frac{\phi(z)}{(z-z_o)^{n+1}}\,dz$$

as required. □

If we apply these results to the Cauchy representation formula we obtain the following theorem.

**Theorem 5.3** Analytic functions have power series
*Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. For each point $z_o \in D$, let $R$ be the radius of the largest disc $B(z_o, R)$ that lies within $D$. Then*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n \qquad \text{for} \quad |z-z_o| < R$$

*where the coefficients $a_n$ are given by*

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_o)^{n+1}}\,dz$$

*for $C_r$ the circle of radius $r$ $(0 < r < R)$ about $z_o$. Therefore, $f$ is infinitely differentiable on $D$ and we have representation formulae*

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(z)}{(z-w)^{n+1}}\,dz$$

*for $w$ with $|w-z_o| < r$.*

*Proof:*

For $0 < r < R$, let $C_r$ be the circle of radius $r$ with centre $z_o$. The Cauchy representation formula ( Theorem 4.3 ) shows that $f$ is the Cauchy transform

$$f(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz$$

for $w \in B(z_o, r)$. Hence, $f$ must be given by a power series $\sum_{n=0}^{\infty} a_n(w-z_o)^n$ on this disc $B(z_o, r)$. The coefficients $a_n$ must be

$$a_n = \frac{f^{(n)}(z_o)}{n!} \ ,$$

which is independent of $r$. This holds for all $r < R$, so the series $\sum_{n=0}^{\infty} a_n(w-z_o)^n$ must converge on all of $B(z_o, R)$.

Also Corollary 5.2 shows that the Cauchy transform satisfies

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(z)}{(z-w)^{n+1}}\,dz \ .$$

□

Note that the expression for the $n$th derivative clearly implies that

$$|f^{(n)}(w)| \leqslant \frac{n! r}{(r-|w|)^{n+1}} \sup\{|f(z)| : |z| = r\}$$

for each $w \in \mathbb{D}(0, r)$. These are *Cauchy's inequalities.*

This theorem has many useful consequences. Our first will be a partial converse of Cauchy's theorem.

**Proposition 5.4** Morera's theorem
*Let $f : D \to \mathbb{C}$ be a continuous function on a domain $D \subset \mathbb{C}$. If, for every closed triangle $T \subset D$, the integral $\int_{\partial T} f(z)\,dz$ is 0, then $f$ is analytic.*

*Proof:*

Let $z_o \in D$ and choose $R > 0$ so that $B(z_o, R) \subset D$. Then we can define a function $F : B(z_o, R) \to \mathbb{C}$ by

$$F(z) = \int_{[z_o,z]} f(z)\, dz \ .$$

Since $f$ is continuous, the fundamental theorem of calculus shows that $F$ is complex differentiable at each point of $B(z_o, R)$ with $F'(z) = f(z)$ (compare Theorem 4.2 ). Now $F$ is analytic on the disc $B(z_o, R)$ and so the previous theorem shows that it is twice continuously differentiable. Thus $f'(z) = F''(z)$ exists. □

Note that the result fails if we do not insist that $f$ is continuous. For example the function $f : \mathbb{C} \to \mathbb{C}$ that is 0 except at at a single point is not analytic.

**The Local Behaviour of Analytic Functions**

The power series expansion for an analytic function is very useful for describing the local behaviour of analytic functions. A key result is that the zeros of an non-constant analytic function are isolated. This means that if $f : D \to \mathbb{C}$ is a non-constant analytic function and $f(z_o) = 0$, then there is a neighbourhood $V$ of $z_o$ on which $f$ has no other zeros.

**Theorem 5.5** Isolated Zeros
*The zeros of a non-constant analytic function are isolated.*

*Proof:*

Let $f : D \to \mathbb{C}$ be an analytic function. For each $z \in D$ we know that there is a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n$$

that converges to $f(w)$ on some disc $B(z, R)$. The coefficients $a_n$ are given by $f^{(n)}(z)/n!$. If all the coefficients $a_n$ are 0, then $f$ is zero on the entire disc $B(z, R)$. Conversely, if $f$ is zero on some neighbourhood $V$ of $z$, then each derivative $f^{(n)}(z)$ is 0 and so each coefficient $a_n$ is 0.

Let $A$ be the set: $\{z \in D :$ there is a neighbourhood $V$ of $z$ with $f(w) = 0$ for all $w \in V\}$. This is clearly open. However, we have shown that $A = \{z \in D : f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \dots\}$. If $z \in B = D \setminus A$, then there is a natural number $n$ with $f^{(n)}(z) \neq 0$. Since $f^{(n)}$ is continuous, $f^{(n)}(w) \neq 0$ on some neighbourhood of $z$. Therefore, $B$ is also open. Since $D$ is connected, one of the two sets $A, B$ must be empty. If $B$ is empty, then $f$ is constantly 0 on $D$. If $A$ is empty, we will show that the zeros of $f$ are isolated.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function with $f(z) = 0$ for some $z \in D$. Since $f$ is not constant, the set $B$ can not be all of $D$ and must therefore be empty. This means that at least one of the coefficients of the power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n \qquad \text{for} \quad w \in B(z, r)$$

is non-zero. Let $a_N$ be the first such coefficient. Then

$$f(w) = (w - z)^N \left( \sum_{n=N}^{\infty} a_n (w - z)^{n-N} \right) .$$

Since the power series $\sum a_n (w - z)^n$ converges on $B(z, r)$, so does $\sum a_n (w - z)^{n-N}$ and it gives an analytic function $F : B(z, r) \to \mathbb{C}$. Note that $F(z) = a_N \neq 0$. Since $F$ is continuous, there is an $r$ with $0 < r < R$ and $F(w) \neq 0$ for $w \in B(z_o, r)$. This means that $f(w) = (w - z_o)^N F(w)$ is not 0 on $B(z_o, r)$ except at $z_o$. Thus $z_o$ is an isolated zero. □

**Corollary 5.6** Identity Theorem
*Let $f, g : D \to \mathbb{C}$ be two analytic functions on a domain $D$. If the set $E = \{z \in D : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ everywhere on $D$.*

*Proof:*

$E$ is the set of zeros of the analytic function $f - g$. □

This corollary gives us the *principle of analytic continuation*: If $f : D \to \mathbb{C}$ is an analytic function on a (non-empty) domain $D$ and $f$ extends to an analytic function $F : \Omega \to \mathbb{C}$ on some larger domain $\Omega$, then $F$ is unique. For, if $\widetilde{F} : \Omega \to \mathbb{C}$ were another extension of $f$, then $F$ and $\widetilde{F}$ would agree on $D$ and hence on all of $\Omega$. However, there may not be any extension of $f$ to a larger domain.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function on a domain $D \subset \mathbb{C}$. For any point $z_o \in D$, we know that $f(z)$ is represented by a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z_o)^n$$

on some disc $B(z_o, R)$. Clearly $a_0 = f(z_o)$. Since the zeros of $f - f(z_o)$ are isolated, there must be a first coefficient (after $a_0$) that is non-zero, say $a_N$. We call $N$ the *degree of $f$ at $z_o$* and write it $\deg(f; z_o)$. We can write $f$ as

$$f(w) = f(z_o) + (w - z_o)^N g(w)$$

for $w \in B(z_o, R)$ and some analytic function $g : B(z_o, R) \to \mathbb{C}$ with $g(z_o) \neq 0$. Indeed, we can define a function $F$ on all of $D$ by

$$F(w) = \begin{cases} \frac{f(w) - f(z_o)}{(w - z_o)^N} & \text{when } w \in D \setminus \{z_o\}; \\ g(w) & \text{when } w \in B(z_o, R). \end{cases}$$

These definitions agree on $B(z_o, R) \setminus \{z_o\}$ and so do define an analytic function $F : D \to \mathbb{C}$ with $f(w) = f(z_o) + (w - z_o)^N F(w)$ on all of $D$.

## Locally Uniform Convergence

Let $f_n$ and $f$ be functions from a domain $D$ into $\mathbb{C}$. We say that $f_n \to f$ *locally uniformly* on $D$ if, for each $z_o \in D$, there is a neighbourhood $V$ of $z_o$ in $D$ with $f_n(z) \to f(z)$ uniformly for $z \in V$.

**Example:** Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

converge locally uniformly on $B(0, R)$ to $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This was proven in Proposition 2.1 .

Suppose that $f_n \to f$ on the domain $D$. Then, for each $z_o \in D$, there is an open disc $\Delta(z_o)$ in $D$, centred on $z_o$, with $f_n(z) \to f(z)$ uniformly on $\Delta(z_o)$. If $K$ is any compact subset of $D$, then $K$ is covered by these sets $\Delta(z_o)$ for $z_o \in K$. Hence, there is a finite subcover. This shows that $f_n \to f$ uniformly on the compact set $K$. We will use this particularly when $K$ is the image $[\gamma]$ of a curve $\gamma$.

Suppose that each of the functions $f_n$ is continuous on $D$. The uniform limit of continuous functions is continuous, so $f$ is continuous on each $\Delta(z_o)$ and hence on all of $D$. We will now prove the the locally uniform limit of analytic functions is analytic.

**Proposition 5.7** Locally uniform convergence of analytic functions
*Let $f_n : D \to \mathbb{C}$ be a sequence of analytic functions on a domain $D$ that converges locally uniformly to a function $f$. Then $f$ is analytic on $D$. Moreover, the derivatives $f_n^{(k)}$ converge locally uniformly on $D$ to $f^{(k)}$.*

*Proof:*

Let $z_o \in D$. Then there is a disc $\Delta = B(z_o, r)$ on which $f_n$ converge uniformly to $f$. The functions $f_n$ are continuous so the uniform limit $f$ is also continuous on $\Delta$. Also, the uniform convergence implies that

$$\int_\gamma f_n(z)\, dz \to \int_\gamma f(z)\, dz$$

for any closed curve $\gamma$ in $\Delta$. Since $f_n$ is analytic, Cauchy's theorem for the disc $\Delta$ implies that $\int_\gamma f_n(z)\, dz = 0$. Therefore, $\int_\gamma f(z)\, dz = 0$. Morera's theorem now shows that $f$ is analytic on $\Delta$. Since $z_o$ is arbitrary, $f$ is analytic on all of $D$.

Now let $C(z_o, s)$ be the circle of radius $s < r$ about $z_o$. For $|w| < s$ Cauchy's representation formula ( 4.3 ) gives

$$f_n^{(k)}(w) = \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z)}{(z-w)^{k+1}}\, dz$$

and a similar formula for $f$, which we now know is analytic. Therefore,

$$\begin{aligned}
|f_n^{(k)}(w) - f^{(k)}(w)| &= \left| \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z) - f(z)}{(z-w)^{k+1}}\, dz \right| \\
&\leqslant \frac{k!}{2\pi} L(C(z_o, s)) \sup \left\{ \left| \frac{f_n(z) - f(z)}{(z-w)^{k+1}} \right| : |z - z_o| = s \right\} \\
&\leqslant \frac{k! s}{(s - |w - z_o|)^k} \sup\{|f_n(z) - f(z)| : |z - z_o| = s\}
\end{aligned}$$

and we see that $f_n^{(k)}(w) \to f^{(k)}(w)$ uniformly on any disc $D(z_o, t)$ with $t < s$. □

This theorem gives us an alternative proof of Proposition 2.2 , which showed that a power series could be differentiated term by term inside its radius of convergence. For suppose that $s(z) = \sum a_n (z - z_o)^n$ is a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n (z - z_o)^n$$

converge locally uniformly to $s$ on $B(z_o, R)$. Each $S_N$ is a polynomial and so is certainly analytic. Therefore $s$ is analytic on $B(z_o, R)$. Moreover,

$$s'(z) = \lim_{N\to\infty} S_N'(z) = \lim_{N\to\infty} \sum_{n=0}^{N} n a_n (z - z_o)^{n-1} = \sum_{n=0}^{\infty} n a_n (z - z_o)^{n-1} \ .$$

**Isolated Singularities**

Let $D$ be a domain and $z_o$ a point of $D$. We are concerned about an analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ that is not defined at the point $z_o$. We call $z_o$ an *isolated singularity of* $f$. It is defined and analytic at every point of some disc $B(z_o, R)$ except the centre $z_o$. We will study the behaviour of $f$ as we approach the singular point.

The simplest possibility for $f$ is that we can extend it to a function analytic on all of $D$, even at the point $z_o$. If this is the case, we say that $f$ has a *removable singularity at* $z_o$. Usually we replace $f$ by the analytic extension:

$$F(z) = \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

Since $F$ is to be continuous, the value $w_o$ it takes at $z_o$ must be $\lim_{z\to z_o} f(z)$ and $F$ is unique. We will now show that $f$ has a removable singularity at $z_o$ if and only if the limit $\lim_{z\to z_o} f(z)$ exists.

**Example:** The function

$$s : \mathbb{C} \setminus \{0\} \to \mathbb{C} \; ; \quad z \mapsto \frac{\sin z}{z}$$

has a removable singularity at 0. For the power series for the sine function shows that

$$s(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} \; .$$

So we can extend $s$ to 0 by sending 0 to 1. This extension is given by a power series and so is analytic on all of $\mathbb{C}$.

**Proposition 5.8** Removable singularities
*The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a removable singularity at $z_o \in D$ if and only if there is a finite limit $w_o \in \mathbb{C}$ with $f(z) \to w_o$ as $z \to z_o$.*

*Proof:*
If $f$ has a removable singularity at $z_o$, then there is an analytic extension $F : D \to \mathbb{C}$. This extension is continuous, so $f(z) = F(z) \to F(z_o)$ as $z \to z_o$.

For the converse, suppose that $f(z) \to w_o$ as $z \to z_o$. Then we can define

$$F : D \to \mathbb{C} \; ; \quad z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

This is certainly continuous at $z_o$ and analytic elsewhere on $D$. Therefore, we can apply Cauchy's theorem to any triangle $T$ within $D$ using Proposition 4.1′ and obtain $\int_{\partial T} f(z)\, dz = 0$. Morera's theorem now shows that $F$ is analytic on all of $D$. □

When we proved Cauchy's theorem we considered a function $f : D \to \mathbb{C}$ that was analytic except at one point $z_o$ where it was continuous. The last proposition shows that such a function is actually analytic even at $z_o$. So the exceptional point is no different from any other.

It is useful to strengthen the last proposition a little.

**Corollary 5.9** Riemann's Removable Singularity Criterion
*The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a removable singularity at $z_o \in D$ if and only if $\lim_{z\to z_o} (z - z_o) f(z) = 0$.*

Note that when $f$ is bounded in a neighbourhood of $z_o$, then the limit $\lim (z - z_o) f(z)$ certainly exists and is 0 and so there must be a removable singularity at $z_o$.

*Proof:*
The function $g(z) = (z - z_o) f(z)$ is analytic on $D \setminus \{z_o\}$ and tends to 0 as $z \to z_o$. Hence the previous proposition tells us that $g$ has a removable singularity at $z_o$. Let $G : D \to \mathbb{C}$ be the analytic extension of $g$. We certainly have $G(z_o) = \lim_{z\to z_o} g(z) = 0$. Hence

$$f(z) = \frac{G(z) - G(z_o)}{z - z_o} \to G'(z_o) \qquad \text{as} \qquad z \to z_o \; .$$

Therefore, the previous proposition shows that $f$ has a removable singularity at $z_o$. □

So far we have only considered functions $f : D \to \mathbb{C}$ taking values in the finite complex plane $\mathbb{C}$. However, in the Algebra and Geometry course you considered functions taking values in the Riemann sphere (or extended complex plane) $\mathbb{C}_\infty$. The Riemann sphere consists of the complex plane $\mathbb{C}$ and one extra point $\infty$. You saw that the extra point $\infty$ behaved in the same way as the finite points in $\mathbb{C}$ and that the Möbius transformations $z \mapsto (az+b)/(cz+d)$ permuted the points of $\mathbb{C}_\infty$. We now wish to explain what it means for a function $f : D \to \mathbb{C}_\infty$ that takes values in the Riemann sphere to be analytic.

Let $f : D \to \mathbb{C}_\infty$ be a function defined on a domain $D \subset \mathbb{C}$ and $z_o \in D$. If $f(z_o) \in \mathbb{C}$, then $f$ is complex differentiable at $z_o$ if the limit $\displaystyle\lim_{z\to z_o} \frac{f(z)-f(z_o)}{z-z_o}$ exists and is a point of $\mathbb{C}$. If $f(z_o) = \infty$, we use the Möbius transformation $J : w \mapsto 1/w$ to send $\infty$ to a finite point and then ask if $J \circ f$ is complex differentiable at $z_o$. Thus we say that $f$ *is complex differentiable at the point* $z_o$ *with* $f(z_o) = \infty$ if $z \mapsto 1/f(z)$ is complex differentiable at $z_o$. (It is not useful to define a value for $f'(z_o)$ at points where $f(z_o) = \infty$.) We call a point $z_o$ where $f(z_o) = \infty$ and $f$ is complex differentiable a *pole* of $f$. A function $f : D \to \mathbb{C}_\infty$ that is is not identically $\infty$ but is complex differentiable at each point of $D$ is *meromorphic on* $D$. Since the zeros of a non-constant analytic function are isolated, the poles of a meromorphic function are also isolated. Thus a meromorphic function is analytic on its domain except for a set of poles each of which is isolated. For example, if $f : D \to \mathbb{C}$ is an analytic function and is not identically 0, then $z \mapsto 1/f(z)$ is meromorphic. This implies that each rational function is meromorphic on $\mathbb{C}$.

Suppose that $f : D \to \mathbb{C}$ is a meromorphic function and has a pole at $z_o$. The function $f$ is certainly continuous at $z_o$ so there is a neighbourhood $V$ of $z_o$ with $|f(z)| > 1$ for $z \in V$. Now the function $g : z \mapsto 1/f(z)$ is complex differentiable and finite at each point of $V$ and it has a zero at $z_o$. Since $f$ is not identically $\infty$, $g$ can not be identically 0. Therefore, the zero at $z_o$ is isolated. This means that we can write $g(z) = (z - z_o)^N G(z)$ for some natural number $N \geqslant 1$ and some function $G$ that is analytic near $z_o$ and has $G(z_o) \neq 0$. Therefore $f(z) = (z - z_o)^{-N} F(z)$ where $F(z) = 1/G(z)$ is analytic near $z_o$ and has $F(z_o) \neq 0, \infty$. This show how the meromorphic function $f$ behaves near a pole. We write $N = \deg(f; z_o)$ and call $z_o$ a *pole of order* $N$ *for* $f$.

We will say that an analytic function $f : D\backslash\{z_o\} \to \mathbb{C}$ has a *pole* at $z_o \in D$ if there is a meromorphic function $F : D \to \mathbb{C}_\infty$ that extends $f$ and $F$ has a pole at $z_o$. This is similar to $f$ having a removable singularity at $z_o$ except that the correct value to put for $f(z_o)$ is $\infty$.

**Proposition 5.10** Poles as isolated singularities
*The analytic function* $f : D \setminus \{z_o\} \to \mathbb{C}$ *has a pole at* $z_o$ *if and only if* $f(z) \to \infty$ *as* $z \to z_o$.

*Proof:*
If $f$ has an extension $F$ with a pole at $z_o$, then $f(z) = F(z) \to F(z_o) = \infty$ as $z \to z_o$.

For the converse, suppose that $f(z) \to \infty$ as $z \to z_o$. There is a neighbourhood $V$ of $z_o$ with $|f(z)| > 1$ for $z \in V \setminus \{z_o\}$. Hence, $g : z \mapsto 1/f(z)$ is bounded, analytic on $V \setminus \{z_o\}$ and has $g(z) \to 0$ as $z \to z_o$. Corollary 5.9 shows that $g$ has a removable singularity at $z_o$ so there is a function $G : V \to \mathbb{C}$ extending $g$. Now the function

$$F : z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ 1/G(z) & \text{when } z \in V. \end{cases}$$

is well-defined and gives a meromorphic extension of $f$. □

There remain some isolated singularities that are neither removable singularities nor poles. We call these *essential singularities*. Functions behave very dramatically near an essential singularity.

**Example:** The function $f : z \mapsto \exp(1/z)$ has an essential singularity at 0. For real values of $t$ we have

$$\exp(1/t) \to \infty \quad \text{as} \quad t \searrow 0+ \qquad \text{while} \qquad \exp(1/t) \to 0 \quad \text{as} \quad t \nearrow 0-$$

so the limit $\lim_{z\to 0} f(z)$ can not exist either as a finite complex number or as $\infty$. Therefore, $f$ can not have either a removable singularity or a pole at 0.

**Exercise:** The function $g : z \mapsto (\cos z)\exp(1/z)$ has an essential singularity at 0.

> *The function* $\cos z$ *is analytic and non-zero near* 0. *If* $g$ *had a removable singularity or a pole at* 0, *then* $\exp(1/z) = g(z)/\cos z$ *would also have a removable singularity or a pole at* 0. *We know that this is not true.*

**Proposition 5.11** Weierstrass - Casorati Theorem
*An analytic function takes values arbitrarily close to any complex number on any neighbourhood of an essential singularity.*

*Proof:*
Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at $z_o$. Suppose that there is some neighbourhood of $z_o$ on which $f$ does not take values arbitrarily close to $w_o \in \mathbb{C}$. Say

$$|f(z) - w_o| > \varepsilon \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then the function $g : z \mapsto 1/(f(z) - w_o)$ is bounded by $1/\varepsilon$ for $0 < |z - z_o| < R$. Therefore, $g$ has a removable singularity at $z_o$ by Corollary 5.9 . Consequently, $f(z) = w_o + 1/g(z)$ will have a removable singularity or a pole at $z_o$.

A similar argument applies for $w_o = \infty$. Suppose that

$$|f(z)| > K \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then $g : z \mapsto 1/f(z)$ is bounded by $1/K$ for $0 < |z - z_o| < R$. Therefore, $g$ has a removable singularity at $z_o$ and $f$ will have a removable singularity or a pole. □

(In fact much more is true. Picard showed that in every neighbourhood of an essential singularity the function takes each value $w \in \mathbb{C}_\infty$ with at most two exceptions. The example $z \mapsto \exp(1/z)$ takes every value except 0 and $\infty$.)

## Analytic Functions on an Annulus

Let $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ be an annulus or ring-shaped domain and let $f : A \to \mathbb{C}$ be an analytic function. We have seen that $\int_\gamma f(z)\, dz$ can be non-zero, for example when $f(z) = 1/z$. In this section we want to study what values the integral can take.

**Proposition 5.12** Cauchy's theorem on an annulus
*For each analytic function* $f : A \to \mathbb{C}$ *there is a constant* $K_f$ *with*

$$\frac{1}{2\pi i}\int_\gamma f(z)\, dz = n(\gamma; 0)K_f$$

*for every closed, piecewise continuously differentiable path* $\gamma$ *in* $A$.

Note that this is certainly true when $f$ is analytic on the entire disc $\{z : |z| < R_2\}$ because of Cauchy's theorem. In this case $K_f = 0$. Also, it is true for $f(z) = 1/z$ because of the definition of the winding number $n(\gamma; 0)$. In this case, $K_f = 1$.

*Proof:*

Let $S$ be the strip $\{w = u + iv \in \mathbb{C} : \log R_1 < u < \log R_2\}$, which is a star with any point as a centre. The exponential mapping $\exp : S \to A;\ w \mapsto e^w$ maps $S$ onto $A$. Cauchy's theorem for star domains ( 4.2 ) shows that the analytic function $\phi : S \to \mathbb{C}\ ;\ \ \phi(w) = f(e^w)e^w$ has an antiderivative $\Phi$. Now $e^{w+2\pi i} = e^w$ so $\phi(w + 2\pi i) = \phi(w)$ and hence $\Phi'(w + 2\pi i) = \Phi'(w)$. Hence, there is a constant $K_f$ with

$$\Phi(w + 2\pi i) = \Phi(w) + 2\pi i K_f\ .$$

Let $C_r$ be the circle $C_r : [0, 2\pi] \to A$, $t \mapsto re^{it}$ for $R_1 < r < R_2$. Then

$$\int_{C_r} f(z)\ dz = \int_0^{2\pi} f(re^{it}) ire^{it}\ dt = i\int_0^{2\pi} \phi(\log r + it)\ dt = \Phi(\log r + 2\pi i) - \Phi(\log r) = 2\pi i K_f$$

so we can determine $K_f$ from this integral.

Consider first the case where $K_f = 0$. Then we have $\Phi(w + 2n\pi i) = \Phi(w)$ for each $n \in \mathbb{Z}$. So we can define a function $F : A \to \mathbb{C}$ unambiguously by $F(z) = \Phi(w)$ for any $w$ with $z = e^w$. The derivative of this satisfies $F'(e^w)e^w = \Phi'(w) = \phi(w)w = f(e^w)e^w$. Hence, $F'(z) = f(z)$ and $f$ has an antiderivative on $A$. Consequently,

$$\int_\gamma f(z)\ dz = 0$$

for any closed curve $\gamma$ in $A$ by Proposition 3.2 .

Now suppose that $K_f \neq 0$. Then we can replace $f$ by the function

$$g(z) = f(z) - \frac{K_f}{z}\ .$$

This has

$$K_g = \frac{1}{2\pi i}\int_{C_r} g(z)\ dz = \frac{1}{2\pi i}\int_{C_r} g(z)\ dz - \frac{K_f}{2\pi i}\int_{C_r} \frac{1}{z}\ dz = K_f - n(C_r; 0)K_f = 0\ .$$

Therefore, we can apply the previous argument to $g$ and obtain

$$\frac{1}{2\pi i}\int_\gamma f(z)\ dz = \frac{1}{2\pi i}\int_\gamma g(z)\ dz + \frac{K_f}{2\pi i}\int_\gamma \frac{1}{z}\ dz = 0 + n(\gamma; 0)K_f$$

as required. □

We can also apply this result to an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\}$ centred at some other point $z_o$. Then we have

$$\frac{1}{2\pi i}\int_\gamma f(z)\ dz = n(\gamma; z_o)K_f$$

for any closed curve $\gamma$ in $A$. This result is particularly useful when $R_1 = 0$. Then we call the constant $K_f$ the *residue of $f$ at $z_o$* and denote it by $\mathrm{Res}(f; z_o)$.

**Proposition 5.13** Analytic functions on an annulus
*For each analytic function $f : A \to \mathbb{C}$ there are analytic functions*

$$F_1 : \{z : |z| > R_1\} \to \mathbb{C} \qquad \textit{and} \qquad F_2 : \{z : |z| < R_2\} \to \mathbb{C}$$

*with $f(z) = F_2(z) - F_1(z)$ for each $z \in A$.*

*Proof:*

We proceed as in the proof of the Cauchy Representation Theorem ( 4.3 ). Let $w$ be a fixed point in $A$ and set

$$g(z) = \frac{f(z) - f(w)}{z - w} \qquad \text{for } z \in A \setminus \{w\} \ .$$

Then $g(z) \to f'(w)$ as $z \to w$, so $g$ has a removable singularity at $w$ ( Proposition 5.8 ). If we set $g(w) = f'(w)$ then we obtain a function $g$ analytic on all of the annulus $A$. For any closed curve $\gamma$ in $A \setminus \{w\}$ we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, dz - \frac{f(w)}{2\pi i} \int_\gamma \frac{1}{z - w} \, dz = \frac{1}{2\pi i} \int_\gamma g(z) \, dz \ ,$$

which gives

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w} \, dz = n(\gamma; 0) f(w) + \frac{1}{2\pi i} \int_\gamma g(z) \, dz \ .$$

We can apply this when $\gamma$ is the circle $C_r$ for $r \neq |w|$. For this the previous proposition shows that

$$\frac{1}{2\pi i} \int_\gamma g(z) \, dz = K_g$$

is independent of $r$. Hence

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz &= \qquad K_g \qquad \text{when } R_1 < r < |w| \\ \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz &= f(w) + K_g \qquad \text{when } |w| < r < R_2 \ . \end{aligned} \qquad (*)$$

Let

$$F_1(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz \qquad \text{for } R_1 < r < |w| \ .$$

Corollary 5.2 shows that $F_1$ is an analytic function of $w$ on $\{w : r < |w|\}$. Since $f(z)/(z-w)$ is analytic on the annulus $\{z : R_1 < |z| < |w|\}$ the value of $F_1(w)$ is independent of $r \in (R_1, |w|)$. This means that $F_1$ is an analytic function on $\{w : R_1 < |w|\}$. Similarly,

$$F_2(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz \qquad \text{for } |w| < r < R_2$$

gives an analytic function on $\{w : |w| < R_2\}$.

Finally, equations $(*)$ shows that

$$f(w) = F_2(w) - F_1(w) \ .$$

□

We already know that analytic functions on discs have power series expansions. The last proposition gives similar expansions for analytic functions on an annulus.

**Corollary 5.14** Laurent expansions
*For each analytic function $f : A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\} \to \mathbb{C}$ there are coefficients $a_n$ for $n \in \mathbb{Z}$ with*

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z_o)^n \qquad \text{for } w \in A \ .$$

*This series converges locally uniformly on the annulus $A$. Moreover,*

$$n(\gamma; 0) a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_o)^{n+1}} \, dz$$

*for every $n \in \mathbb{Z}$ and any piecewise continuously differentiable closed curve $\gamma$ in $A$.*

*Proof:*

By translating $A$ we may ensure that $z_o = 0$. Then we know that $f(w) = F_2(w) - F_1(w)$ for analytic functions $F_1 : \{w : R_1 < |w|\} \to \mathbb{C}$ and $F_2 : \{w : |w| < R_2\} \to \mathbb{C}$. The function $F_2$ is analytic on a disc, so it has a power series expansion $F_2(w) = \sum_{n=0}^{\infty} b_n w^n$ that converges locally uniformly on $\{w : |w| < R_2\}$.

The argument for $F_1$ is similar but the disc is centred on $\infty$ in $\mathbb{C}_\infty$ rather than on 0. Hence we must begin by using a Möbius transformation to move $\infty$ to 0. First note that

$$F_1(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\, dz \qquad \text{has} \qquad |F_1(w)| \leqslant \frac{r \sup\{|f(z)| : |z| = r\}}{|w| - r}$$

so $F_1(w) \to 0$ as $w \to \infty$. Let $G(z) = F_1(1/z)$ then $G(z) \to 0$ as $z \to 0$. Therefore $G$ has a removable singularity at 0 and so gives us an analytic function $G : \{z : |z| < 1/R_1\} \to \mathbb{C}$. This has a power series expansion $G(z) = \sum_{n=1}^{\infty} c_n z^n$ that converges locally uniformly on $\{z : |z| < 1/R_1\}$. (The constant term is 0 since $G(0) = 0$.) Thus $F_1(w) = \sum_{n=1}^{\infty} c_n w^{-n}$ and the series converges locally uniformly on $\{w : R_1 < |w|\}$.

Putting these power series together we obtain

$$f(w) = \sum_{n=0}^{\infty} b_n w^n \; - \; \sum_{n=1}^{\infty} c_n w^{-n} \; .$$

Both parts of this sum converge locally uniformly on the annulus $A$. This gives the Laurent series we wanted.

Since the Laurent series for $f$ converges uniformly on the compact set $[\gamma]$, we see that

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_o)^{n+1}}\, dz = \sum_{k=-\infty}^{\infty} a_k \frac{1}{2\pi i}\int_{\gamma} (z-z_o)^{k-n-1}\, dz \; .$$

We can easily evaluate the integrals $\int_\gamma (z-z_o)^m\, dz$ and see that they are 0 except when $m = -1$. Hence,

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{(z-z_o)^{n+1}}\, dz = a_n n(\gamma; z_o) \; .$$

□

**Laurent Series about isolated singularities**

Let $z_o$ be a point in the domain $D$ and let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function. So $f$ has an isolated singularity at $z_o$. There will be a disc $B(z_o, R)$ that lies within $D$. So $f$ is analytic on the annulus $A = \{z : 0 < |z - z_o| < R\}$ and has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_o)^n$$

on this annulus. Corollary 5.14 shows that the residue of $f$ at $z_o$ is $\mathrm{Res}(f; z_o) = a_{-1}$.

**Proposition 5.15** Laurent series for isolated singularities
*Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at $z_o$ and let*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

*be its Laurent expansion that converges for $0 < |z - z_o| < R$. Then*

*(a) $f$ has a removable singularity at $z_o$ if and only if $a_n = 0$ for $n < 0$.*

*(b) $f$ has a pole at $z_o$ of order $N$ if and only if $a_n = 0$ for $n < -N$ and $a_{-N} \neq 0$.*

*(c) $f$ has an essential singularity at $z_o$ if and only if $a_n \neq 0$ for infinitely many negative values of $n$.*

*Proof:*
(a) Suppose that $f$ has a removable singularity at $z_o$. then there is an analytic function $F : D \to \mathbb{C}$ extending $f$. For $\gamma$ a closed curve in the annulus $A$ we have

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{(z - z_o)^{n+1}} \, dz$$

and Cauchy's theorem shows that this is 0 for $n < 0$. Conversely, if $a_n = 0$ for $n < 0$, then the Laurent series reduces to a power series and defines an analytic extension of $f$.

(b) Suppose that $f$ has a pole of order $N$ at $z_o$. Then $f(z) = (z - z_o)^{-N} G(z)$ for some function $G$ analytic near $z_o$ and with $G(z_o) \neq 0$. The Laurent series for $G$ is

$$G(z) = \sum_{n=-\infty}^{\infty} a_{n-N} (z - z_o)^n \ .$$

This has a removable singularity at $z_o$, so part (a) implies that $a_n = 0$ for $n < -N$. We also have $a_{-N} = G(z_o) \neq 0$. Conversely, if $a_n = 0$ for $n < -N$ and $a_{-N} = 0$, then

$$f(z) = (z - z_o)^{-N} \sum_{n=0}^{\infty} a_{n-N} (z - z_o)^n$$

so $f$ has a pole of order $N$ at $z_o$.

(c) The singularity is essential if and only if it is neither removable nor a pole. Similarly, the Laurent series has $a_n \neq 0$ for infinitely many negative $n$ if and only if there is no integer $N$ with $a_n = 0$ for $n < -N$. Thus (a) and (b) imply (c). □

Laurent series give us a quick proof of the Residue theorem at least for simply connected domains. Suppose that $f$ has an isolated singularity at $z_o$ and has Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$. The part

$$P(z) = \sum_{n=-\infty}^{-1} a_n (z - z_o)^n$$

of this series is called the *principal part of $f$ at $z_o$*. The principal part is a power series in $1/(z - z_o)$ and converges for $z$ sufficiently close to $z_o$. Therefore, it must converge for all $z \in \mathbb{C} \setminus \{z_o\}$. The difference $f - P$ is analytic at $z_o$.