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COMPLEX ANALYSIS

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CONTENTS

1. ANALYTIC FUNCTIONS

A *domain* in the complex plane $\mathbb{C}$ is an open, connected subset of $\mathbb{C}$. For example, every open disc:

$$\mathbb{D}(w,r) = \{z \in \mathbb{C} : |z-w| < r\}$$

is a domain. Throughout this course we will consider functions defined on domains.

Suppose that D is a domain and $f : D \to \mathbb{C}$ a function. This function is *complex differentiable* at a point $z \in D$ if the limit

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

exists. The value of the limit is the derivative $f'(z)$. The function $f : D \to \mathbb{C}$ is *analytic* if it is complex differentiable at each point z of the domain D. (The terms *holomorphic* and *regular* are more commonly used in place of *analytic*.)

For example, $f : z \mapsto z^n$ is analytic on all of $\mathbb{C}$ with $f'(z) = nz^{n-1}$ but $g : z \mapsto \overline{z}$ is not complex differentiable at any point and so g is not analytic.

It is important to observe that asking for a function to be complex differentiable is much stronger than asking for it to be real differentiable. To see this, first recall the definition of real differentiability. Let D be a domain in $\mathbb{R}^2$ and write the points in D as $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Let $f : D \to \mathbb{R}^2$ be a function. Then we can write

$$f(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ f_2(\boldsymbol{x}) \end{pmatrix}$$

with $f_1, f_2 : D \to \mathbb{R}$ as the two components of f. The function f is *real differentiable* at a point $\boldsymbol{a} \in D$ if there is a real linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$||f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - T(\boldsymbol{h})|| = o(||\boldsymbol{h}||) \qquad \text{as } \boldsymbol{h} \to \boldsymbol{0}\ .$$

This means that

$$\frac{||f(\boldsymbol{a}+\boldsymbol{h}) - f(\boldsymbol{a}) - T(\boldsymbol{h})||}{||\boldsymbol{h}||} \to 0 \qquad \text{as } \boldsymbol{h} \to \boldsymbol{0}\ .$$

We can write this out in terms of the components. Let T be given by the 2×2 real matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$\left|\left|\begin{pmatrix} f_1(\boldsymbol{a}+\boldsymbol{h}) \\ f_2(\boldsymbol{a}+\boldsymbol{h}) \end{pmatrix} - \begin{pmatrix} f_1(\boldsymbol{a}) \\ f_2(\boldsymbol{a}) \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right|\right| = o(||\boldsymbol{h}||) \qquad \text{as } \boldsymbol{h} \to \boldsymbol{0}\ .$$

This means that

$$\begin{aligned} &|f_1(\boldsymbol{a}+\boldsymbol{h}) - f_1(\boldsymbol{a}) - (ah_1 + bh_2)| = o(||\boldsymbol{h}||) \qquad \text{and} \\ &|f_2(\boldsymbol{a}+\boldsymbol{h}) - f_2(\boldsymbol{a}) - (ch_1 + dh_2)| = o(||\boldsymbol{h}||) \end{aligned}$$

as $\boldsymbol{h} \to \boldsymbol{0}$. By taking one of the components of $\boldsymbol{h}$ to be 0 in this formula, we see that the matrix for T must be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{a}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{a}) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{a}) \end{pmatrix}\ .$$

We can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ by letting $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ correspond to $x_1 + ix_2$. Then f gives a map $f : D \to \mathbb{C}$. This is **complex** differentiable if it is **real** differentiable and the map T is linear over the complex numbers. The complex linear maps $T : \mathbb{C} \to \mathbb{C}$ are just multiplication by a complex number $w = w_1 + iw_2$, so T must be

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w_1 & -w_2 \\ w_2 & w_1 \end{pmatrix}\ .$$

In particular, this shows that a complex differentiable function must satisfy the *Cauchy – Riemann equations*:

$$\frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) = \frac{\partial f_2}{\partial x_2}(\boldsymbol{a}) \quad \text{and} \quad \frac{\partial f_1}{\partial x_2}(\boldsymbol{a}) = -\frac{\partial f_2}{\partial x_1}(\boldsymbol{a}) \ .$$

There are also more direct ways to obtain the Cauchy – Riemann equations. For example, if $f : D \to \mathbb{C}$ is complex differentiable at a point a with derivative $f'(a)$, then we can consider the functions

$$x_1 \mapsto f(a + x_1) \quad \text{and} \quad x_2 \mapsto f(a + ix_2)$$

for real values of x_1 and x_2. These must also be differentiable and so

$$f'(a) = \frac{\partial f}{\partial x_1}(a) = \frac{\partial f_1}{\partial x_1}(a) + i\frac{\partial f_2}{\partial x_1}(a) \quad \text{and} \quad f'(a) = \frac{1}{i}\frac{\partial f}{\partial x_2}(a) = -i\frac{\partial f_1}{\partial x_2}(a) + \frac{\partial f_2}{\partial x_2}(a) \ .$$

2. POWER SERIES

A *power series* is an infinite sum of the form $\sum_{n=0}^{\infty} a_n(z-z_o)^n$. Recall that a power series converges on a disc.

Proposition 2.1 Radius of convergence
For the sequence of complex numbers (a_n) define $R = \sup\{r : a_n r^n \to 0 \text{ as } n \to \infty\}$. Then the power series $\sum a_n z^n$ converges absolutely on the open disc $\mathbb{D}(z_o, R)$ and diverges outside the corresponding closed disc $\overline{\mathbb{D}(z_o, R)}$. Indeed, the power series converges uniformly on each disc $B(z_o, r)$ with r strictly less than R.

We call R the *radius of convergence of the power series $\sum a_n(z-z_o)^n$. It can take any value from 0 to $+\infty$ including the extreme values. The series may converge or diverge on the circle $\partial\mathbb{D}(z_o, R)$.*

Proof:
It is clear that if $\sum a_n(z-z_o)^n$ converges then the terms $a_n(z-z_o)^n$ must tend to 0 as $n \to \infty$. Therefore, $a_n r^n \to 0$ as $n \to \infty$ for each $r \leqslant |z-z_o|$. Hence $R \geqslant |z-z_o|$ and we see that the power series diverges for $|z-z_o| > R$.

Suppose that $|z-z_o| < R$. Then we can find r with $|z-z_o| < r < R$ and $a_n r^n \to 0$ as $n \to \infty$. This means that there is a constant K with $|a_n| r^n \leqslant K$ for each $n \in \mathbb{N}$. Hence

$$\sum |a_n||z-z_o|^n \leqslant \sum K \left(\frac{|z-z_o|}{r}\right)^n .$$

The series on the right is a convergent geometric series, and $\sum a_n z^n$ converges, absolutely, by comparison with it. Also, this convergence is uniform on $\mathbb{D}(z_o, r)$. $\square$

We wish to prove that a power series can be differentiated term-by-term within its disc of convergence.

Proposition 2.2 Power series are differentiable.
Let R be the radius of convergence of the power series $\sum a_n(z-z_o)^n$. The sum $s(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$ is complex differentiable on the disc $\mathbb{D}(z_o, R)$ and has derivative $t(z) = \sum_{n=1}^{\infty} n a_n (z-z_o)^{n-1}$.

Proof:
We may assume that $z_o = 0$. For a fixed point w with $|w| < R$, we can choose r with $|w| < r < R$. We will consider h satisfying $|h| < r - |w|$ so that $|w+h| < r$.

Consider the function (curve):

$$\gamma : [0,1] \to \mathbb{C} \ ; \quad t \mapsto n(n-1)(w+th)^{n-2}h^2 \ .$$

Straightforward integration shows that

$$\int_0^s \gamma(t)\, dt = n(w+th)^{n-1}h \Big|_0^s = n(w+sh)^{n-1}h - nw^{n-1}h$$

and

$$\int_0^1 \int_0^s \gamma(t)\, dt\, ds = (w+sh)^n - nw^{n-1}sh \Big|_0^1 = (w+h)^n - w^n - nw^{n-1}h \ .$$

For each $t \in [0,1]$ we have $|w+th| < r$, so $|\gamma(t)| \leqslant n(n-1)r^{n-2}|h|^2$. This implies that

$$|(w+h)^n - w^n - nw^{n-1}| \leqslant \int_0^1 \int_0^s n(n-1)r^{n-2}|h|^2 \, dt \, ds = \tfrac{1}{2}n(n-1)r^{n-2}|h|^2$$

Hence,

$$\begin{aligned} |s(w+h) - s(w) - t(w)h| &= \left| \sum_{n=0}^{\infty} a_n \left((w+h)^n - w^n - nw^{n-1}h \right) \right| \\ &\leqslant \sum_{n=0}^{\infty} |a_n| |(w+h)^n - w^n - nw^{n-1}h| \\ &\leqslant \frac{1}{2} \left(\sum_{n=0}^{\infty} n(n-1)|a_n|r^{n-2} \right) |h|^2 \ . \end{aligned}$$

The series $\sum n(n-1)|a_n|r^{n-2}$ converges by comparison with $\sum |a_n|s^n$ for any s with $r < s < R$. Therefore, s is differentiable at w and $s'(w) = t(w)$. □

The derivative of the power series s is itself a power series, so s is twice differentiable. Repeating this shows that s is infinitely differentiable, that is we can differentiate it as many times as we wish.

Corollary 2.3 Power series are infinitely differentiable
Let R be the radius of convergence of the power series $\sum a_n(z-z_o)^n$. Then the sum

$$s(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$$

is infinitely differentiable on $B(z_o, R)$ with

$$s^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n(z-z_o)^{n-k} \ .$$

In particular, $s^{(k)}(z_o) = k!a_k$, so the power series is the Taylor series for s.

□

The Exponential Function

One of the most important applications of power series is to the exponential function. This is defined as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \ .$$

The ratio test shows that the series converges for all complex numbers z. Hence, it defines a function

$$\exp : \mathbb{C} \to \mathbb{C} \ .$$

We know, from Proposition 2.1 , that the exponential function is differentiable with

$$\exp'(z) = \sum_{n=0}^{\infty} n \frac{1}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \exp(z) \ .$$

This is the key property of the function and we will use it to establish the other properties.

Proposition 2.4 Products of exponentials
For any complex numbers w, z we have

$$\exp(z+w) = \exp(z)\exp(w) \ .$$

Proof:

Let a be a fixed complex number and consider the function

$$g(z) = \exp(z)\exp(a - z) \ .$$

This is differentiable and its derivative is

$$g'(z) = \exp(z)\exp(a - z) - \exp(z)\exp(z - a) = 0 \ .$$

This implies that g is constant. (For consider the function $\gamma : t \mapsto g(tz)$ defined on the unit interval $[0, 1] \subset \mathbb{R}$. This has derivative 0 and so the mean value theorem shows that it is constant. Therefore, $g(z) = g(0)$ for each z.) The value of g at 0 is $\exp(0)\exp(a) = \exp(a)$, so we see that

$$\exp(z)\exp(a - z) = \exp(a) \ .$$

$\square$

This Proposition allows us to establish many of the properties of the exponential function very easily.

Corollary 2.5 Properties of the exponential

(a) The exponential function has no zeros.

(b) For any complex number z we have $\exp \overline{z} = \overline{\exp z}$.

(c) $e : x \mapsto \exp x$ is a strictly increasing function from $\mathbb{R}$ onto $(0, \infty)$.

(d) For real numbers y, the map $f : y \mapsto \exp iy$ traces out the unit circle, at unit speed, in the positive direction.

Proof:

(a) For $\exp(z)\exp(-z) = \exp(0) = 1$.

(b) Is an immediate consequence of the power series.

(c) For $x \in \mathbb{R}$ it is clear from the power series that $\exp x$ is real. Moreover $\exp x = \left(\exp \frac{1}{2}x\right)^2 > 0$. This shows that $e'(x) = e(x) > 0$ and so e is a strictly increasing positive function. The power series also shows that

$$e(x) = \exp x > x \qquad \text{for } x > 1$$

so $e(x) \nearrow +\infty$ as $x \nearrow +\infty$. Finally,

$$e(x) = \frac{1}{e(-x)} \searrow 0 \qquad \text{as } x \searrow -\infty \ .$$

(d) Part (b) shows that $|\exp iy|^2 = \exp iy \ \exp -iy = 1$, so f maps into the unit circle. Moreover, f is differentiable with $f'(y) = i \exp iy$, so f traces out the unit circle at unit speed in the positive direction. $\square$

Any complex number w can be written as $r(\cos\theta + i\sin\theta)$ for some modulus $r \geqslant 0$ and some argument $\theta \in \mathbb{R}$. The modulus $r = |z|$ is unique but the argument is only determined up to adding an integer multiple of 2π (and is completely arbitrary when $w = 0$).

Part (a) of the Corollary shows that $\exp z$ is never 0. Suppose that $w \neq 0$. Then (c) shows that we can find a unique real number x with $\exp x = |w|$. Part (d) shows that $\exp i\theta = \cos\theta + i\sin\theta$. Hence

$$w = |w| \exp i\theta = \exp x \exp i\theta = \exp(x + i\theta) \ .$$

So there is a complex number $z_o = x + i\theta$ with $\exp z_o = w$. Furthermore, parts (c) and (d) show that the only solutions of $\exp z = w$ are $z = z_o + 2n\pi i$ for an integer $n \in \mathbb{Z}$.

Logarithms

Corollary 2.5(c) shows that the exponential function on the real line gives a strictly increasing map $e : \mathbb{R} \to (0, \infty)$ from $\mathbb{R}$ onto $(0, \infty)$. This map must then be invertible and we call its inverse the *natural logarithm* and denote it by $\ln : (0, \infty) \to \mathbb{R}$. We want to consider analogous complex logarithms that are inverse to the complex exponential function.

We know that $\exp z$ is never 0, so we can not hope to define a complex logarithm of 0. For any non-zero complex number w we have seen that there are infinitely many complex numbers z with $\exp z = w$ and any two differ by an integer multiple of $2\pi i$. Therefore, the exponential function can not be invertible.

However, if we restrict our attention to a suitable domain D in $\mathbb{C} \setminus \{0\}$, then we can try to find a continuous function $\lambda : D \to \mathbb{C}$ with $\exp \lambda(z) = z$ for each $z \in D$. Such a map is called a *branch of the logarithm* on D. If one branch λ exists, then $z \mapsto \lambda(z) + 2n\pi i$ is another branch of the logarithm.

Consider, for example, the domain

$$D = \{z = r \exp i\theta : 0 < r \text{ and } \alpha < \theta < \alpha + 2\pi\}$$

that is obtained by removing a half-line from $\mathbb{C}$. The map

$$\lambda : D \to \mathbb{C} \ ; \quad r \exp i\theta \mapsto \ln r + i\theta$$

for $r > 0$ and $\alpha < \theta < \alpha + 2\pi$ is certainly continuous and satisfies $\exp \lambda(z) = z$ for each $z \in D$. Hence it is one of the branches of the logarithm on D.

As remarked above, the point 0 is special and there is no branch of the logarithm defined at 0. We call 0 a *logarithmic singularity*. Many authors abuse the notation by writing $\log z$ for $\lambda(z)$. However, it is important to remember that there are many branches of the logarithm and that there is none defined on all of $\mathbb{C} \setminus \{0\}$.

The branches of the logarithm are important and we will use them throughout this course. Note that, for any branch λ of the logarithm, we have

$$\lambda(z) = \ln|z| + i\theta$$

where θ is an argument of z. The real part is unique and clearly continuous. However, the imaginary part is only determined up to an additive integer multiple of 2π. The choice of a branch of the logarithm on D corresponds to a continuous choice of the argument $\theta : D \to \mathbb{R}$.

Since the branch $\lambda : D \to \mathbb{C}$ is inverse to the exponential function, the inverse function theorem shows that λ is differentiable with

$$\lambda'(w) = \frac{1}{\exp' \lambda(w)} = \frac{1}{\exp \lambda(w)} = \frac{1}{w} \ .$$

To do this more carefully, let w be a point of D. Choose $k \neq 0$ so small that $w + k \in D$. Then set $z = \lambda(w)$ and $z + h = \lambda(w + k)$. Since λ is continuous, $h \to 0$ as $k \to 0$. Hence

$$\frac{\lambda(w+k) - \lambda(w)}{k} = \frac{(z+h) - z}{\exp \lambda(w+k) - \exp \lambda(w)} = \frac{h}{\exp(z+h) - \exp z}$$

tends to $\frac{1}{\exp' z}$ as $K \to 0$. This shows that λ is complex differentiable at w with

$$\lambda'(w) = \frac{1}{\exp z} = \frac{1}{w} \ .$$

Thus every branch of the logarithm is analytic.

Let Ω be the complex plane cut along the negative real axis: $\Omega = \mathbb{C} \setminus (-\infty, 0]$. Every $z \in \Omega$ can be written uniquely as

$$r \exp i\theta \qquad \text{with } r > 0 \ \text{ and } \ -\pi < \theta < \pi.$$

We call this θ the *principal branch of the argument of* z and denote it by $\mathrm{Arg}(z)$. In a similar way, the *principal branch of the logarithm* is:

$$\mathrm{Log} : \Omega \to \mathbb{C} \ ; \quad z \mapsto \ln |z| + i \, \mathrm{Arg}(z) \ .$$

Powers

We can also define branches of powers of complex numbers. Suppose that $n \in \mathbb{Z}$, z a complex number and $\lambda : D \to \mathbb{C}$ any branch of the complex logarithm defined at z. Then

$$z^n = (\exp \lambda(z))^n = \exp(n\lambda(z))$$

and the value of the right side does not depend on which branch λ we choose. When α is a complex number but not an integer, we may define a *branch* of the αth power on D by

$$p_\alpha : D \to \mathbb{C} \ ; \quad z \mapsto (\exp \alpha\lambda(z)) \ .$$

This behaves as we would expect an αth power to, for example,

$$p_\alpha(z) p_\beta(z) = p_{\alpha+\beta}(z)$$

analogously to $z^\alpha z^\beta = z^{\alpha+\beta}$ for integers α and β. Moreover, p_α is analytic on D since exp and λ are both analytic with

$$p'_\alpha(z) = \exp'(\alpha\lambda(z))\alpha\lambda'(z) = (\exp \alpha\lambda(z)) \frac{\alpha}{z} = \alpha \exp((\alpha - 1)\lambda(z)) = \alpha p_{\alpha-1}(z) \ .$$

However, there are many different branches of the αth power coming from different branches of the logarithm.

For example, on the cut plane $\Omega = \mathbb{C} \setminus (-\infty, 0]$ the principal branch of the αth power is given by

$$z \mapsto \exp(\alpha \, \mathrm{Log} \, z) = \exp(\alpha(\ln |z| + i \, \mathrm{Arg}(z))) \ .$$

When $\alpha = \frac{1}{2}$ this is

$$r \exp i\theta \mapsto r^{1/2} \exp \tfrac{1}{2} i\theta \qquad \text{for } r > 0 \text{ and } -\pi < \theta < \pi.$$

Note that none of these branches of powers is defined at 0 since no branch of the logarithm is defined there. The point 0 is called a *branch point* for the power. The only powers that can be defined to be analytic at 0 are the non-negative integer powers.

If we set $e = \exp 1 = 2.71828\ldots$, then $\exp z$ is one of the values for the zth power of e. Despite the fact that there are other values (unless $z \in \mathbb{Z}$) we often write this as e^z. In particular, it is very common to write $e^{i\theta}$ for $\exp i\theta$.

Conformal Maps

A *conformal map* is an analytic map $f : D \to \Omega$ between two domains D, Ω that has an analytic inverse $g : \Omega \to D$. This certainly implies that f is a bijection and that $f'(z)$ is never 0, since the chain rule gives $g'(f(z))f'(z) = 1$. When there is a conformal map $f : D \to \Omega$ then the complex analysis on D and Ω are the same, for we can transform any analytic map $h : D \to \mathbb{C}$ into a map $h \circ g : \Omega \to \mathbb{C}$ and *vice versa.*

You have already met Möbius transformations as examples of conformal maps. For instance, $z \mapsto \dfrac{1+z}{1-z}$ is a conformal map from the unit disc $\mathbb{D}$ onto the right half-plane $H = \{x + iy : x > 0\}$. Its inverse is $w \mapsto \dfrac{w-1}{w+1}$. Powers also give useful examples, for instance:

$$\{x + iy : x, y > 0\} \to \{u + iv : v > 0\} \ ; \quad z \mapsto z^2$$

is a conformal map. Its inverse is a branch of the square root. Similarly, the exponential map gives us examples. The map

$$\{x + iy : -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi\} \to \{u + iv : u > 0\} \ ; \quad z \mapsto \exp z$$

is conformal. Its inverse is the principal branch of the logarithm.

Conformal maps preserve the angles between curves. For consider the straight line $\beta : t \mapsto z_o + t\omega$ where $|\omega| = 1$. The analytic map f sends this to the curve

$$f \circ \beta : t \mapsto f(z_o + t\omega) \ .$$

The tangent to this curve at $t = 0$ is in the direction of

$$\lim_{t \to 0} \frac{f(z_o + t\omega) - f(z_o)}{|f(z_o + t\omega) - f(z_o)|} = \frac{f'(z_o)\omega}{|f'(z_o)\omega|} \ .$$

Provided that $f'(z_o) \neq 0$, this shows that $f \circ \beta$ is a curve through $f(z_o)$ in the direction of $f'(z_o)\omega$. Consequently, such a function f preserves the angle between two curves, in both magnitude and orientation. This shows that conformal maps preserve the angles between any two curves.

3. INTEGRATION ALONG CURVES

We have seen that it is a much stronger condition on a function to be complex differentiable than to be real differentiable. The reason for this is that we can apply the fundamental theorem of calculus when we integrate f along a curve in D that starts and ends at the same point. This will show that, for suitable curves, the integral is 0 — a result we call Cauchy's theorem. This theorem has many important consequences and is the key to the rest of the course.

We therefore wish to integrate functions along curves in D. First recall some of the properties of integrals along intervals of the real line. If $\phi : [a,b] \to \mathbb{C}$ is a continuous function, then the Riemann integral

$$I = \int_a^b \phi(t)\, dt$$

exists. For any angle θ, we have

$$Ie^{i\theta} = \Re\left(\int_a^b \phi(t)e^{i\theta}\, dt\right) = \int_a^b \Re\left(\phi(t)e^{i\theta}\right)\, dt \leqslant \int_a^b |\phi(t)|\, dt$$

so we have the inequality

$$\left|\int_a^b \phi(t)\, dt\right| \leqslant \int_a^b |\phi(t)|\, dt\ .$$

A *continuously differentiable curve* in D is a map $\gamma : [a,b] \to D$ defined on a compact interval $[a,b] \subset \mathbb{R}$ that is continuously differentiable at each point of $[a,b]$. (At the endpoints a,b we demand a one-sided derivative.) The image $\gamma([a,b])$ will be denoted by $[\gamma]$. We think of the parameter t as time and the point $z = \gamma(t)$ traces out the curve as time increases. The direction that we move along the curve is important and is often denoted by an arrow.

As the time increases by a small amount δt, so the point $z = \gamma(t)$ on the curve moves by $\delta z = \gamma(t+\delta t) - \gamma(t) \approx \gamma'(t)\, \delta t$. Hence, it is natural to define the integral of a continuous function $f : D \to \mathbb{C}$ along γ to be

$$\int_\gamma f(z)\, dz \ = \ \int_a^b f(\gamma(t))\gamma'(t)\, dt\ .$$

We can also define integrals with respect to the arc-length s along γ where $\frac{ds}{dt} = |\gamma'(t)|$. This is usually denoted by:

$$\int_\gamma f(z)\, |dz| = \int_a^b f(\gamma(t))\, |\gamma'(t)|\, dt\ .$$

In particular, the *length* of γ is:

$$L(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\, dt\ .$$

Then we have the important inequality:

Proposition 3.1
Let $\gamma : [a,b] \to D$ be a continuously differentiable curve in the domain D and let $f : D \to \mathbb{C}$ be a continuous function. Then

$$\left|\int_\gamma f(z)\, dz\right| = \left|\int_a^b f(\gamma(t))\gamma'(t)\, dt\right| \leqslant \int_a^b |f(\gamma(t))||\gamma'(t)|\, dt \leqslant L(\gamma).\sup\{|f(z)| : z \in [\gamma]\}\ .$$

$\square$

Example: The straight-line curve $[w_0, w_1]$ between two points of $\mathbb{C}$ is given by

$$[0,1] \to \mathbb{C} \; ; \quad t \mapsto (1-t)w_o + tw_1 \; .$$

This has length $|w_1 - w_0|$. The unit circle c is given by

$$c : [0, 2\pi] \to \mathbb{C} \; ; \quad t \mapsto z_o + r \exp it$$

and has length 2π. For any integer n we have

$$\int_c z^n \, dz = \int_0^{2\pi} \exp int \; i \exp it \; dt = \begin{cases} 0 & \text{if } n \neq -1; \\ 2\pi i & \text{if } n = -1. \end{cases}$$

It is possible to re-parametrise a curve $\gamma : [a, b] \to D$. Suppose that $h : [c, d] \to [a, b]$ is a continuously differentiable, strictly increasing function with a continuously differentiable inverse $h^{-1} : [a, b] \to [c, d]$. Then $\gamma \circ h : [c, d] \to D$ is a curve and the substitution rule for integrals shows that

$$\int_{\gamma \circ h} f(z) \, dz = \int_c^d f(\gamma(h(s))\gamma'(h(s))h'(s) \, ds = \int_a^b f(\gamma(t))\gamma'(t) \, dt = \int_\gamma f(z) \, dz$$

and similarly that $L(\gamma \circ h) = L(\gamma)$. Sometimes it is useful to reverse the orientation of the curve. For any curve $\gamma : [a, b] \to D$, the *reversed curve* $-\gamma$ is given by

$$-\gamma : [-b, -a] \to D \; ; \quad t \mapsto \gamma(-t) \; .$$

This traces out the same image as γ but in the reverse direction.

It is useful to generalise the definition of a curve slightly. A *piecewise continuously differentiable curve* is a map $\gamma : [a, b] \to D$ for which there is a subdivision

$$a = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = b$$

with each of the restrictions $\gamma| : [t_n, t_{n+1}] \to D$ $(n = 0, 1, \ldots, N)$ being a continuously differentiable curve. The integral along γ is then

$$\int_\gamma f(z) \, dz \;=\; \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f(\gamma(t))\gamma'(t) \, dt$$

and

$$\int_\gamma f(z) \, |dz| = \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} f(\gamma(t))|\gamma'(t)| \, dt \; .$$

We clearly have

$$\left| \int_\gamma f(z) \, dz \right| \leqslant \int_\gamma |f(z)| \, |dz| \leqslant L(\gamma). \sup\{|f(z)| : z \in [\gamma]\} \; .$$

From now on, we will suppose, tacitly, that all the curves we consider are piecewise continuously differentiable.

Proposition 3.2 Fundamental Theorem of Calculus
Let $f : D \to \mathbb{C}$ be an analytic function. If f is the derivative of another analytic function $F : D \to \mathbb{C}$, then

$$\int_\gamma f(z) \, dz = F(\gamma(b)) - F(\gamma(a))$$

for any piecewise continuously differentiable curve $\gamma : [a, b] \to D$.

We call $F : D \to \mathbb{C}$ an *antiderivative* of f if $F'(z) = f(z)$ for all $z \in D$.

Proof:

The fundamental theorem of calculus show that

$$\int_\gamma f(z)\ dz = \int_a^b f(\gamma(t))\gamma'(t)\ dt = \int_a^b F'(\gamma(t))\gamma'(t)\ dt = \int_a^b (F \circ \gamma)'(t)\ dt = F(\gamma(b)) - F(\gamma(a))$$

for any continuously differentiable curve γ. The result follows for piecewise continuously differentiable curves by adding the results for each continuously differentiable section. $\square$

A curve $\gamma : [a, b] \to D$ is *closed* if $\gamma(b) = \gamma(a)$. In this case, the Proposition shows that

$$\int_\gamma f(z)\ dz = 0$$

provided that f is the derivative of a function $F : D \to \mathbb{C}$. This is our first form of Cauchy's theorem.

For the sake of variety, we use many different names for curves, such as paths or routes. Closed curves are sometimes called contours.

Example: Let A be the domain $\mathbb{C} \setminus \{0\}$ and γ the closed curve

$$\gamma : [0, 2\pi] \to A\ ;\quad t \mapsto \exp it$$

that traces out the unit circle in a positive direction. Let $f(z) = z^n$ for $n \in \mathbb{Z}$. Then

$$\int_\gamma z^n\ dz = \int_0^{2\pi} \exp int\ (2\pi i \exp it)\ dt = \begin{cases} 2\pi i & \text{when } n = -1; \\ 0 & \text{otherwise.} \end{cases}$$

For each function $f(z) = z^n$ with $n \neq -1$ there is a function $F(z) = z^{n+1}/(n + 1)$ with $F'(z) = f(z)$ on A, so the integral around γ should be 0. However, for $n = -1$ the Proposition shows that there can be no such function $F : A \to \mathbb{C}$ with $F'(z) = \frac{1}{z}$. This means that there is no branch of the logarithm f defined on all of A.

Winding Numbers

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve that does not pass through 0. A *continuous choice of the argument on* γ is a continuous map $\theta : [a, b] \to \mathbb{R}$ with $\gamma(t) = |\gamma(t)| \exp i\theta(t)$ for each $t \in [a, b]$. The change $\theta(b) - \theta(a)$ measures the angle about 0 turned through by γ. We call $(\theta(b) - \theta(a))/2\pi$ the *winding number* $n(\gamma, 0)$ *of* γ *about* 0. Suppose that ϕ is another continuous choice of the argument on γ. Then $\theta(t) - \phi(t)$ must be an integer multiple of 2π. Since $\theta - \phi$ is continuous on the connected interval $[a, b]$, we see that there is an integer k with $\phi(t) - \theta(t) = 2k\pi$ for all $t \in [a, b]$. Hence $\theta(b) - \theta(a) = \phi(b) - \phi(a)$ and the winding number is well defined.

When γ is a piecewise continuously differentiable curve, we can give a continuous choice of $\theta(t)$ explicitly and hence find an expression for the winding number. Let

$$h(t) = \int_{\gamma|[a,t]} \frac{1}{z}\ dz = \int_a^t \frac{\gamma'(t)}{\gamma(t)}\ dt$$

for $t \in [a, b]$. The chain rule shows that

$$\frac{d}{dt}\left(\gamma(t) \exp -h(t)\right) = \gamma'(t)(\exp -h(t)) - \gamma(t)h'(t)(\exp -h(t)) = \left(\gamma'(t) - \gamma(t)\frac{\gamma'(t)}{\gamma(t)}\right) \exp -h(t) = 0\ .$$

Hence $\gamma(t)\exp -h(t)$ is constant. Therefore,

$$\gamma(t) = \gamma(a)\exp h(t) = \gamma(a)\exp \Re h(t) \exp i\Im h(t) \ .$$

This means that $\theta(t) = \arg\gamma(a) + \Im h(t)$ gives a continuous choice of the argument of $\gamma(t)$. Consequently, the total angle turned through by γ is

$$\Im\left(\int_\gamma \frac{1}{z}\,dz\right) \ .$$

If γ is piecewise continuously differentiable, we can apply this argument to each section of γ and so find that the final formula still holds.

The formula is particularly important when γ is a closed curve. Then $\gamma(b) = \gamma(a)$, so $\exp h(b) = 1$ and we must have $h(b) = 2N\pi i$ for some integer N. The number N counts the number of times γ winds positively around 0. We have the formula:

$$N = \frac{h(b)}{2\pi i} = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz \ .$$

We can also consider how many times a closed curve γ winds around any point w_o that does not lie on γ. By translating w_o to 0 we see that this is

$$n(\gamma; w_o) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z - w_o}\,dz \ ,$$

which is called the *winding number of γ about w_o*.

Example: The curve $\gamma : [0,1] \to \mathbb{C};\ t \mapsto z_o + re^{2\pi i t}$ has winding number

$$n(\gamma; w_o) = \begin{cases} 1 & \text{when } |w_o - z_o| < r; \\ 0 & \text{when } |w_o - z_o| > r. \end{cases}$$

It is not defined when $|w_o - z_o| = r$.

Lemma 3.3
Let γ be a piecewise continuously differentiable closed curve taking values in the disc $B(z_o, R)$. Then $n(\gamma; w_o) = 0$ for all points $w_o \notin B(z_o, R)$.

Proof:
By translating and rotating the curve, we may assume that $w_o = 0$ and z_o is a positive real number no smaller than R. For z in the disc $B(z_o, R)$, we can find an unique real number $\phi(z) \in (-\pi, \pi)$ with $z = |z|e^{i\phi(z)}$. (This is the principal branch of the argument of z.) The map $\phi : B(1,1) \to \mathbb{R}$ is then continuous. Hence, $t \mapsto \phi(\gamma(t))$ is a continuous choice of the argument on γ. So

$$n(\gamma; 0) = \frac{\phi(\gamma(b)) - \phi(\gamma(a))}{2\pi} \ .$$

Since $\gamma(b) = \gamma(a)$, this winding number must be 0. $\square$

The winding number $n(\gamma; w)$ is unchanged if we perturb γ by a sufficiently small amount.

Proposition 3.4 Winding numbers under perturbation
Let $\alpha, \beta : [a,b] \to \mathbb{C}$ be two closed curve and w a point not on $[\alpha]$. If

$$|\beta(t) - \alpha(t)| < |\alpha(t) - w| \qquad \textit{for each } t \in [a,b]$$

then $n(\beta; w) = n(\alpha; w)$.

Proof:

By translating the curves, we may assume that $w = 0$. Then $|\beta(t) - \alpha(t)| < |\alpha(t)|$ for $t \in [a, b]$. This certainly implies that $\beta(t) \neq 0$, so the winding number $n(\beta; 0)$ exists. Write

$$\beta(t) = \alpha(t)\left(1 + \frac{\beta(t) - \alpha(t)}{\alpha(t)}\right) = \alpha(t)\gamma(t) \ .$$

Since the argument of a product is the sum of the arguments, this implies that

$$n(\beta; 0) = n(\alpha; 0) + n(\gamma; 0) \ .$$

However the inequality in the proposition shows that γ takes values in the disc $B(1, 1)$ so the lemma proves that $n(\gamma; 0) = 0$. □

Proposition 3.5 Winding number constant on each component
Let γ be a piecewise continuously differentiable closed curve in $\mathbb{C}$. The winding number $n(\gamma; w)$ is constant for w in each component of $\mathbb{C} \setminus [\gamma]$ and is 0 on the unbounded component.

Proof:

The image $[\gamma]$ is a compact subset of $\mathbb{C}$, so it is bounded, say $[\gamma] \subset B(0, R)$. The complement $U = \mathbb{C} \setminus [\gamma]$ is open, so each component of the complement is also open. One component contains $\mathbb{C} \setminus B(0, R)$, so it is the unique unbounded component that contains all points of sufficiently large modulus.

Let $w_o \in U = \mathbb{C} \setminus [\gamma]$. Then there is a disc $B(w_o, r) \subset U$. For w with $|w - w_o| < r$ we have

$$|(\gamma(t) - w) - (\gamma(t) - w_o)| = |w - w_o| < r \leqslant |\gamma(t) - w_o| \ .$$

Proposition 3.4 then shows that $n(\gamma; w) = n(\gamma; w_o)$. So the function $w \mapsto n(\gamma; w)$ is continuous (indeed constant) at w_o. It follows that $w \mapsto n(\gamma; w)$ is a continuous integer-valued function on U. It must therefore be constant on each component of U.

Lemma 3.3 shows that $n(\gamma; w) = 0$ for w outside the disc $B(0, R)$. So the winding number must be 0 on the unbounded component of U. □

Homotopy

Let $\gamma_0, \gamma_1 : [a, b] \to D$ be two piecewise continuously differentiable closed curves in the domain D. A homotopy from γ_0 to γ_1 is a family of piecewise continuously differentiable closed curves γ_s for $s \in [0, 1]$ that vary continuously from γ_0 to γ_1. This means that the map

$$h : [0, 1] \times [a, b] \to D \ ; \quad (s, t) \mapsto \gamma_s(t)$$

is continuous. More formally, we define a *homotopy* to be a continuous map $h : [0, 1] \times [a, b] \to D$ with

$$h_s : [a, b] \to D \ ; \quad t \mapsto h(s, t)$$

being a piecewise continuously differentiable closed curve in D for each $s \in [0, 1]$. We then say that the curves h_0 and h_1 are homotopic and write $h_0 \simeq h_1$. This gives an equivalence relation between closed curves in D.

Example: Suppose that $\gamma_0, \gamma_1 : [0, 1] \to D$ are closed paths in the domain D and that, for each $t \in [0, 1]$, the line segment $[\gamma_0(t), \gamma_1(t)]$ lies within D. Then the map

$$h : [0, 1] \times [0, 1] \to D \ ; \quad (s, t) \mapsto (1 - s)\gamma_0(t) + s\gamma_1(t)$$

is continuous and defines a homotopy from γ_0 to γ_1. We sometimes call such a homotopy a *linear* homotopy.

A closed curve γ in D is *null-homotopic* if it is homotopic in D to a constant curve. The domain D is *simply-connected* if every closed curve in D is null-homotopic. For example, a disc $B(z_o, r)$ is simply-connected since there is a linear homotopy from any curve γ in the disc to z_o.

A domain $D \subset \mathbb{C}$ is called a *star with centre* z_o if, for each point $w \in D$ the entire line segment $[z_o, w]$ lies within D. A domain D is a *star domain* if it is a star with some centre z_o. Clearly every disc is a star domain but such domains as $\mathbb{C} \setminus \{0\}$ are not. Every star domain is simply-connected because a curve is linearly homotopic to the constant curve at the centre.

Proposition 3.6 Winding number and homotopy
If two closed curves γ_0 and γ_1 are homotopic in a domain D and $w \in \mathbb{C} \setminus D$, then $n(\gamma_0; w) = n(\gamma_1; w)$.

Proof:
By translating the curves and the domain, we may assume that $w = 0$.

Let $h : [0, 1] \times [a, b] \to D$ be the homotopy with $\gamma_0 = h_0$ and $\gamma_1 = h_1$. Since $[0, 1] \times [a, b]$ is a compact subset of D, there is an $\varepsilon > 0$ with $|h_s(t)| > \varepsilon$ for each $(s, t) \in [0, 1] \times [a, b]$. The homotopy h is uniformly continuous. Hence there is a $\delta > 0$ with

$$|h_s(t) - h_u(t)| < \varepsilon \qquad \text{whenever} \qquad |s - u| < \delta \ .$$

This means that

$$|h_s(t) - h_u(t)| < |h_u(t)| \qquad \text{whenever} \qquad |s - u| < \delta \ .$$

Hence Proposition 3.4 shows that

$$n(h_s; 0) = n(h_u; 0) \qquad \text{whenever} \qquad |s - u| < \delta \ .$$

This clearly establishes the result. □

Chains and Cycles

Let D be a domain in $\mathbb{C}$. A *chain* in D is a finite collection $\gamma_n : [a_n, b_n] \to D$ (for $n = 1, 2, 3, \ldots, N$) of piecewise continuously differentiable curves in D. We will write $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ for this collection. The empty chain will be written as 0. We can add two chains and obtain another chain. The integral of a continuous function $f : D \to \mathbb{C}$ around Γ is then defined to be the sum

$$\int_\Gamma f(z)\, dz = \sum_{n=1}^{N} \int_{\gamma_n} f(z)\, dz \ .$$

In particular, the *winding number* $n(\Gamma; w)$ of a chain Γ about any point $w \notin [\Gamma]$ is

$$n(\Gamma; w) = \sum_{n=1}^{N} n(\gamma_n; w) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - w}\, dz \ .$$

A *cycle* in D is a chain $\Gamma = \gamma_1 + \gamma_2 + \ldots + \gamma_N$ where each point $w \in \mathbb{C}$ occurs the same number of times as an initial point $\gamma_n(a_n)$ as it does as an final point $\gamma_n(b_n)$. This means that a cycle consists of a finite number of closed curves, each of which may be made up from a number of the curves γ_n. The winding number $n(\Gamma; w)$ of a cycle Γ is therefore an integer.

Proposition 3.2 shows that any analytic function $f : D \to \mathbb{C}$ that has an antiderivative on D must satisfy

$$\int_\Gamma f(z)\, dz = 0$$

for every cycle Γ in the domain D.

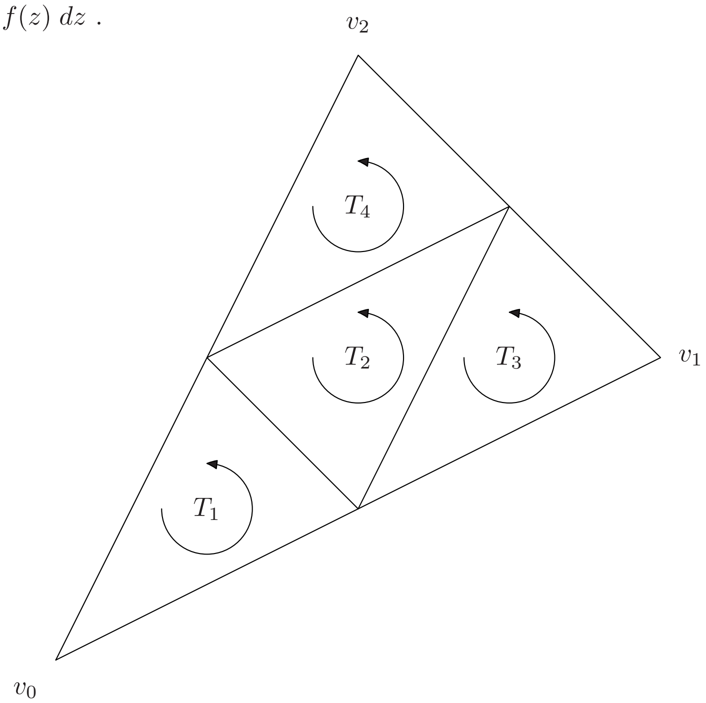
Let T be a closed triangle that lies inside the domain D. Let v_0, v_1, v_2 be the vertices labelled in anti-clockwise order around T. Then the edges $[v_0, v_1]$, $[v_1, v_2]$, $[v_2, v_0]$ are straight-line paths in D. The three sides taken in order give a closed curve $[v_0, v_1] + [v_1, v_2] + [v_2, v_0]$in D that we denote by ∂T.

Proposition 4.1 Cauchy's theorem for triangles
Let $f : D \to \mathbb{C}$ be an analytic function and T a closed triangle that lies within D. Then

$$\int_{\partial T} f(z)\, dz = 0 \ .$$

This proof is due to Goursat and relies on repeated bisection. It underlies all the stronger versions of Cauchy's theorem that we will prove later.

Proof:
Set $I = \displaystyle\int_{\partial T} f(z)\, dz$.



Subdivide T into four similar triangles T_1, T_2, T_3, T_4 as shown. Then we have

$$\sum_{k=1}^{4} \int_{\partial T_k} f(z)\, dz = \int_{\partial T} f(z)\, dz$$

because the integrals along the sides of T_k in the interior of T cancel. At least one the integrals

$$\int_{\partial T_k} f(z)\, dz$$

must have modulus at least $\frac{1}{4}|I|$. Choose one of the triangles with this property and call it T'. Repeating this procedure we obtain sequence of triangles $(T^{(n)})$ nested inside one another with

$$\left| \int_{\partial T^{(n)}} f(z)\, dz \right| \geqslant \frac{|I|}{4^n} \ .$$

Let $L(\gamma)$ denote the length of a path γ and set $L = L(\partial T)$. Then each T_k has $L(\partial T_k) = \frac{1}{2}L$. Therefore, $L(\partial T^{(n)}) = L/2^n$.

The triangle T is a compact subset of $\mathbb{C}$ with $T^{(n)}$ closed subsets. If the intersection $\bigcap_{n\in\mathbb{N}} T^{(n)}$ of these sets were empty, then the complements $T \setminus T^{(n)}$ would form an open cover of T with no finite subcover. Therefore, we must have $\bigcap_{n\in\mathbb{N}} T^{(n)}$ non-empty. Choose a point $z_o \in \bigcap_{n\in\mathbb{N}} T^{(n)}$.

The function f is differentiable at z_o. So, for each $\varepsilon > 0$, there is a $\delta > 0$ with

$$\left| \frac{f(z) - f(z_o)}{z - z_o} - f'(z_o) \right| < \varepsilon$$

whenever $z \in B(z_o, \delta)$. This means that

$$f(z) = f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)$$

with $|\eta(z)| < \varepsilon$ for $z \in B(z_o, \delta)$. For n sufficiently large, we have $T^{(n)} \subset B(z_o, \delta)$, so

$$\left| \int_{\partial T^{(n)}} f(z)\, dz \right| = \left| \int_{\partial T^{(n)}} f(z_o) + f'(z_o)(z - z_o) + \eta(z)(z - z_o)\, dz \right| .$$

The integrals

$$\int_{\partial T^{(n)}} f(z_o)\, dz \quad \text{and} \quad \int_{\partial T^{(n)}} f'(z_o)(z - z_o)\, dz$$

can be evaluated explicitly and are both zero, so

$$\left| \int_{\partial T^{(n)}} f(z)\, dz \right| \leqslant \int_{\partial T^{(n)}} \varepsilon |z - z_o|\, dz \leqslant \varepsilon L(\partial T^{(n)}) \sup\{|z - z_o| : z \in \partial T^{(n)}\} \leqslant \varepsilon L(\partial T^{(n)})^2 = \varepsilon \frac{L^2}{4^n} .$$

This gives

$$|I| = \left| \int_{\partial T} f(z)\, dz \right| \leqslant 4^n \left| \int_{\partial T^{(n)}} f(z)\, dz \right| \leqslant \varepsilon L^2 .$$

This is true for all $\varepsilon > 0$, so we must have $I = 0$. □

We can use this proposition to prove Cauchy's theorem for discs. The proof actually works for any star domain.

Theorem 4.2 Cauchy's theorem for a star domain
Let $f : D \to \mathbb{C}$ be an analytic function on a star domain $D \subset \mathbb{C}$ and let γ be a piecewise continuously differentiable closed curve in D. Then

$$\int_\gamma f(z)\, dz = 0 .$$

Proof:
Let D be the star domain with centre z_o then each line segment $[z_o, z]$ to a point $z \in D$ lies within D. By Proposition 3.1 we need only show that there is an antiderivative F of f, that is a function with $F'(z) = f(z)$ for $z \in D$. Define $F : D \to \mathbb{C}$ by

$$F(w) = \int_{[z_o, w]} f(z)\, dz .$$

Since D is open, each $w \in D$ is contained in a disc $\mathbb{D}(w, r)$ that lies within D. This implies that the triangle with vertices $z_o, w, w + h$ lies within the star domain D provided that $|h| < r$. Then Cauchy's theorem for this triangle gives

$$F(w + h) - F(w) = \int_{[w, w+h]} f(z)\, dz .$$

Consequently,

$$|F(w + h) - F(w) - f(w)h| = \left| \int_{[w, w+h]} f(z) - f(w)\, dz \right| \leqslant |h|. \sup\{|f(z) - f(w)| : z \in [w, w + h]\} .$$

The continuity of f at w shows that $\sup\{|f(z) - f(w)| : z \in [w, w + h]\}$ tends to 0 as h tends to 0. Hence F is differentiable at w and $F'(w) = f(w)$. □

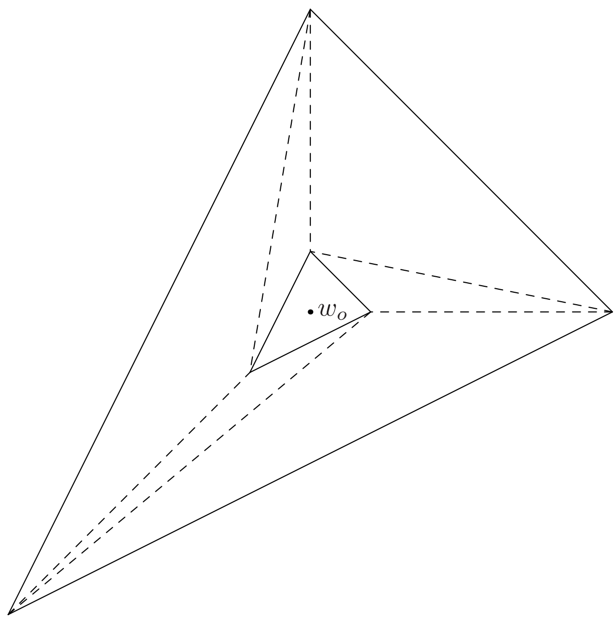
We wish to apply Theorem 4.2 under slightly weaker conditions on f. We want to allow there to be a finite number of exceptional points in D where f is not necessarily differentiable but is continuous. Later we will see that such a function must, in fact, be differentiable at each exceptional point.

Proposition 4.1′ Cauchy's theorem for triangles
Let $f : D \to \mathbb{C}$ be a continuous function that is complex differentiable at every point except $w_o \in D$. Let T be a closed triangle that lies within D. Then

$$\int_{\partial T} f(z)\, dz = 0 \ .$$

Proof:
If $w_o \notin T$, then this result is simply Proposition 4.1 . Hence, we may assume that $w_o \in T$. Let T^ε be the triangle obtained by enlarging T with centre w_o by a factor $\varepsilon < 1$. Then we can divide $T \setminus T^\varepsilon$ into triangles that lie entirely within $T \setminus \{w_o\}$. The integral around each of these triangles is 0 by Proposition 4.1. Adding these results we see that

$$\int_{\partial T} f(z)\, dz = \int_{\partial T^\varepsilon} f(z)\, dz \ .$$



Since f is continuous on D, there is a constant K with $|f(z)| \leqslant K$ for every $z \in T$. Therefore,

$$\left| \int_{\partial T} f(z)\, dz \right| = \left| \int_{\partial T^\varepsilon} f(z)\, dz \right| \leqslant L(\partial T^\varepsilon)K = \varepsilon L(\partial T)K \ .$$

This is true for every $\varepsilon > 0$, so we must have $\int_{\partial T} f(z)\, dz = 0$ as required. □

This proposition allows us to extend Cauchy's Theorem 4.2 to functions that fail to be differentiable at one point (or, indeed, at a finite number of points).

Theorem 4.2′ Cauchy's theorem for a star domain
Let $f : D \to \mathbb{C}$ be a continuous function on a star domain $D \subset \mathbb{C}$ that is complex differentiable at every point except $w_o \in D$. Let γ be a piecewise continuously differentiable closed curve in D. Then

$$\int_{\gamma} f(z)\, dz = 0 \ .$$

Proof:

We argue exactly as in the proof of Theorem 4.2 . Let z_o be a centre for the star domain D and define $F(z)$ to be the integral of f along the straight line path $[z_o, z]$ from z_o to z. The previous proposition shows that

$$F(z+h) - F(z) = \int_{[z,z+h]} f(z)\, dz \ .$$

So F is differentiable with $F'(z) = f(z)$ for each $z \in D$. Now Proposition 3.1 gives the result. □

The crucial application of this corollary is the following. Suppose that $f : D \to \mathbb{C}$ is an analytic function on a disc $D = B(z_o, R) \subset \mathbb{C}$ and $w_o \in D$. Then we can define a new function $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} \dfrac{f(z) - f(w_o)}{z - w_o} & \text{for } z \neq w_o; \\ \\ f'(w_o) & \text{for } z = w_o. \end{cases}$$

This is certainly complex differentiable at each point of D except w_o. At w_o we know that f is differentiable, so g is continuous. We can now apply Theorem 4.2′ to g and obtain

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz$$

for any closed curve γ in D that does not pass through w_o. Now

$$0 = \int_\gamma g(z)\, dz = \int_\gamma \frac{f(z) - f(w_o)}{z - w_o}\, dz = \int_\gamma \frac{f(z)}{z - w_o}\, dz - f(w_o) \int_\gamma \frac{1}{z - w_o}\, dz \ .$$

So we obtain

$$f(w_o) n(\gamma; w_o) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w_o}\, dz \ . \qquad (*)$$

This applies, in particular, when γ is the boundary of a circle contained in D.

Theorem 4.3 Cauchy's Representation Formula
Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$ and let $\overline{B(z_o, R)}$ be a closed disc in D. Then

$$f(w) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - w}\, dz \qquad \text{for } w \in D(z_o, R)$$

when $C(z_0, R)$ is the circular path $C(z_0, R) : [0, 2\pi] \to \mathbb{C}$; $t \mapsto z_o + Re^{it}$.

Proof:

This follows immediately from formula $(*)$ above since the winding number of $C(z_o, R)$ about any $w \in B(z_o, R)$ is 1. □

Cauchy's representation formula is immensely useful for proving the local properties of analytic functions. These are the properties that hold on small discs rather then the global properties that require we study a function on its entire domain. The next chapter will use the representation formula frequently but, as a first example:

Example: Let $f : D \to \mathbb{C}$ be an analytic function on a domain D. For $z_o \in D$ there is a closed disc $\overline{B(z_o, R)}$ within D and Cauchy's representation formula gives

$$f(z_o) = \frac{1}{2\pi i} \int_{C(z_o,R)} \frac{f(z)}{z - z_o}\, dz = \int_0^{2\pi} f(z_o + Re^{i\theta})\, \frac{d\theta}{2\pi} \ .$$

So the value of f at the centre of the circle is the average of the values on the circle C.

Theorem 4.4 Liouville's theorem
Any bounded analytic function $f : \mathbb{C} \to \mathbb{C}$ defined on the entire complex plane is constant.

Proof:

Let w, w' be any two points of $\mathbb{C}$ and let M be an upper bound for $|f(z)|$ for $z \in \mathbb{C}$. Then Cauchy's representation formula gives

$$f(w) = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w}\, dz \qquad \text{for each } r > |w| \ .$$

Hence,

$$f(w) - f(w') = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)}{z-w} - \frac{f(z)}{z-w'}\, dz = \frac{1}{2\pi i} \int_{C(0,r)} \frac{f(z)(w-w')}{(z-w)(z-w')}$$

for $r > \max\{|w|, |w'|\}$. Consequently,

$$|f(w) - f(w')| \leqslant \frac{L(C(0,r))}{2\pi} \sup \left\{ \frac{|f(z)||w-w'|}{|z-w||z-w'|} : |z| = r \right\} \leqslant r \left(\frac{M|w-w'|}{(r-|w|)(r-|w'|)} \right) \ .$$

The right side tends to 0 as $r \nearrow +\infty$, so the left side must be 0. Thus $f(w) = f(w')$. $\square$

Exercise: Show that an analytic function $f : \mathbb{C} \to \mathbb{C}$ that never takes values in the disc $D(w_o, R)$ is constant.

For the function

$$g : \mathbb{C} \to \mathbb{C} \ ; \quad z \mapsto \frac{1}{f(z) - w_o}$$

is bounded by $1/R$ and so is constant by Liouville's theorem.

Corollary 4.5 The Fundamental Theorem of Algebra
Every non-constant polynomial has a zero in $\mathbb{C}$.

Proof:

Suppose that $p(z) = z^N + a_{N-1}z^{N-1} + \dots a_1 z + a_0$ is a polynomial that has no zero in $\mathbb{C}$. Then $f(z) = 1/p(z)$ is an analytic function. As $z \to \infty$ so $f(z) \to 0$. Hence f is bounded. By Liouville's theorem, p must be constant. $\square$

By dividing a polynomial by $z - z_o$ for each zero z_o we see that the total number of zeros of p, counting multiplicity, is equal to the degree of p.

Homotopy form of Cauchy's Theorem.

Let $f : D \to \mathbb{C}$ be an analytic function on a domain D. We wish to study how the integral

$$\int_\gamma f(z)\, dz$$

varies as we vary the closed curve γ in D. Recall that two closed curves $\beta, \gamma : [a, b] \to D$ are *linearly homotopic in D* if, for each $t \in [a, b]$ the line segment $[\beta(t), \gamma(t)]$ is a subset of D.

Theorem 4.6 Homotopy form of Cauchy's Theorem.
Let $f : D \to \mathbb{C}$ be an analytic map on a domain $D \subset \mathbb{C}$. If the two piecewise continuously differentiable closed curves α, β are homotopic in D, then

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz \ .$$

Proof:

Let $h : [0,1] \times [a,b] \to D$ be the homotopy. So each map $h_s : [a,b] \to D\ ;\quad t \mapsto h(s,t)$ is a piecewise continuously differentiable closed curve in D, $h_0 = \alpha$ and $h_1 = \beta$. This means that h is piecewise continuously differentiable on each "vertical" line $\{s\} \times [a,b]$. Initially we will assume that h is also continuously differentiable on each "horizontal" line $[0,1] \times \{t\}$. For any rectangle

$$Q = \{(s,t) \in [0,1] \times [a,b] : s_1 \leqslant s \leqslant s_2 \text{ and } t_1 \leqslant t \leqslant t_2\}$$

let ∂Q denote the boundary of Q positively oriented. Then h is piecewise continuously differentiable on each segment of the boundary, so $h(\partial Q)$ is a piecewise continuously differentiable closed curve in D. If we divide Q into two smaller rectangles Q_1, Q_2 by drawing a horizontal or vertical line ℓ then the segments of the integrals $\int_{h(\partial Q_1)} f(z)\, dz$ and $\int_{h(\partial Q_2)} f(z)\, dz$ along ℓ cancel, so

$$\int_{h(\partial Q)} f(z)\, dz = \int_{h(\partial Q_1)} f(z)\, dz + \int_{h(\partial Q_2)} f(z)\, dz\ .$$

For the original rectangle $R = [0,1] \times [a,b]$ the image of the horizontal sides $[0,1] \times \{a\}$ and $[0,1] \times \{b\}$ are the same since each h_s is closed. Hence

$$\int_{h(\partial R)} f(z)\, dz = \int_\beta f(z)\, dz - \int_\alpha f(z)\, dz\ .$$

We need to show that this is 0.

Define $\rho(z) = \inf\{|z-w| : w \in \mathbb{C} \setminus D\}$ to be the distance from $z \in D$ to the complement of D. Since D is open, $\rho(z) > 0$ for each $z \in D$. Moreover, ρ is continuous since $|\rho(z) - \rho(z')| \leqslant |z - z'|$. Hence, ρ attains a minimum value on the compact set $h(R)$, say

$$\rho(h(s,t)) \geqslant r > 0 \qquad \text{for every } s \in [0,1],\ t \in [a,b]\ .$$

This means that each disc $B(h(s,t), r)$ is contained in D.

Furthermore, we know that h is uniformly continuous on the compact set $[0,1] \times [a,b]$. So there is a $\delta > 0$ with

$$|h(u,v) - h(s,t)| \leqslant r \qquad \text{whenever} \quad ||(u,v) - (s,t)|| < \delta\ . \tag{$*$}$$

Suppose that Q is a rectangle in R with diameter less than δ and P_o a point in Q. Then $h(Q) \subset B(h(P_o), r)$ and the disc $B(h(P_o), r)$ is a subset of D. Cauchy's theorem for star domains (4.2) can now be applied to this disc to see that

$$\int_{h(\partial Q)} f(z)\, dz = 0\ .$$

We can divide R into rectangles $(Q_n)_{n=1}^N$ each with diameter less than δ. So

$$\int_{h(\partial R)} f(z)\, dz = \sum_{n=1}^N \int_{h(\partial Q_n)} f(z)\, dz = 0$$

as required.

It remains to deal with the case where the homotopy h is not continuously differentiable on each horizontal line. Choose a subdivision

$$0 = s(0) < s(1) < \ldots < s(N-1) < s(N) = 1$$

of $[0,1]$ with $|s(k+1) - s(k)| < \delta$ for $k = 0, 1, \ldots, N-1$. Then equation $(*)$ above shows that $|h(s(k),t) - h(s(k+1),t)| < r$ for each $t \in [a,b]$. Hence the entire line segment $[h(s(k),t), h(s(k+1),t)]$ lies in the disc $B(h(s(k),t), r)$ and hence in D. So $h_{s(k)}$ and $h_{s(k+1)}$ are **LINEARLY** homotopic in D. We can certainly apply the above argument to linear homotopies, so we see that

$$\int_{h_{s(k)}} f(z)\, dz = \int_{h_{s(k+1)}} f(z)\, dz\ .$$

Adding these results gives

$$\int_\alpha f(z)\, dz = \int_\beta f(z)\, dz\ .$$

$\square$

Corollary 4.7 Cauchy's Theorem for null-homotopic curves
Let $f : D \to \mathbb{C}$ be an analytic map on a domain D and γ a piecewise continuously differentiable closed curve in D that is null-homotopic in D. Then

$$\int_\gamma f(z)\, dz = 0 \ .$$

$\square$

If the domain D is simply connected, then any closed curve in D is null-homotopic, so Cauchy's theorem will apply.

5. CONSEQUENCES OF CAUCHY'S THEOREM

Cauchy Transforms

Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable path in $\mathbb{C}$ and $\phi : [\gamma] \to \mathbb{C}$ a continuous function on $[\gamma]$. Then the integral
$$\Phi(w) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{z-w}\, dz$$
exists for each $w \in \mathbb{C} \setminus [\gamma]$. This is the *Cauchy transform of* ϕ. We will show that it defines a function analytic everywhere except on $[\gamma]$.

Proposition 5.1 Cauchy transforms have power series
Let Φ be the Cauchy transform of a continuous function $\phi : [\gamma] \to \mathbb{C}$. For $z_o \in \mathbb{C} \setminus [\gamma]$ let R be the radius of the largest disc $B(z_o, R)$ that lies within $\mathbb{C} \setminus [\gamma]$. Then
$$\Phi(w) = \sum_{n=0}^{\infty} a_n (w - z_o)^n \qquad \textit{for} \quad |w - z_o| < R$$
where the coefficients a_n are given by
$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{(z - z_o)^{n+1}}\, dz\ .$$

Proof:
We may assume, by translating γ, that $z_o = 0$. The formula for the sum of a geometric series shows that
$$\frac{1}{z-w} = \frac{1}{z} + \frac{w}{z^2} + \ldots + \frac{w^{N-1}}{z^N} + \frac{w^N}{z^N(z-w)}\ .$$
Integrating this gives
$$\Phi(w) = a_0 + a_1 w + \ldots + a_{N-1} w^{N-1} + E_N(w)$$
where
$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z)}{z^{k+1}}\, dz \qquad \text{and} \qquad E_N(w) = \frac{1}{2\pi i} \int_\gamma \frac{\phi(z) w^N}{z^N(z-w)}\, dz\ .$$
Let $||\phi||_\infty = \sup\{|\phi(z)| : z \in [\gamma]\}$. For $z \in [\gamma]$ we have $|z| \geqslant R$ and $|z - w| \geqslant R - |w|$, so
$$|E_N(w)| \leqslant \frac{L(\gamma)}{2\pi} \frac{||\phi||_\infty}{(R - |w|)} \left(\frac{|w|}{R}\right)^N\ .$$
This shows that, for $|w| < R$,
$$\left| \Phi(w) - \sum_{n=0}^{N-1} a_n w^n \right| = |E_N(w)| \to 0 \qquad \text{as } N \to \infty\ .$$
Therefore the power series $\sum a_n w^n$ converges on $B(0, R)$ to Φ. □

Corollary 5.2 Cauchy transforms are infinitely differentiable
The Cauchy transform Φ of a continuous function $\phi : [\gamma] \to \mathbb{C}$ is infinitely differentiable on $\mathbb{C} \setminus [\gamma]$ with
$$\Phi^{(n)}(z_o) = \frac{n!}{2\pi i} \int_\gamma \frac{\phi(z)}{(z - z_o)^{n+1}}\, dz\ .$$

Proof:

We know that Φ is given by a power series $\Phi(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$ on the disc $B(z_o, R)$. By Corollary 2.3 this power series is infinitely differentiable. Moreover,

$$\Phi^{(n)}(z_o) = n! a_n = \frac{n!}{2\pi i} \int_\gamma \frac{\phi(z)}{(z - z_o)^{n+1}} \, dz$$

as required. □

If we apply these results to the Cauchy representation formula we obtain the following theorem.

Theorem 5.3 Analytic functions have power series
Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. For each point $z_o \in D$, let R be the radius of the largest disc $B(z_o, R)$ that lies within D. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n \qquad \text{for} \quad |z - z_o| < R$$

where the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_o)^{n+1}} \, dz$$

for C_r the circle of radius r $(0 < r < R)$ about z_o. Therefore, f is infinitely differentiable on D and we have representation formulae

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^{n+1}} \, dz$$

for w with $|w - z_o| < r$.

Proof:

For $0 < r < R$, let C_r be the circle of radius r with centre z_o. The Cauchy representation formula (Theorem 4.3) shows that f is the Cauchy transform

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w} \, dz$$

for $w \in B(z_o, r)$. Hence, f must be given by a power series $\sum_{n=0}^{\infty} a_n (w - z_o)^n$ on this disc $B(z_o, r)$. The coefficients a_n must be

$$a_n = \frac{f^{(n)}(z_o)}{n!} \ ,$$

which is independent of r. This holds for all $r < R$, so the series $\sum_{n=0}^{\infty} a_n (w - z_o)^n$ must converge on all of $B(z_o, R)$.

Also Corollary 5.2 shows that the Cauchy transform satisfies

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - w)^{n+1}} \, dz \ .$$

□

Note that the expression for the nth derivative clearly implies that

$$|f^{(n)}(w)| \leqslant \frac{n! r}{(r - |w|)^{n+1}} \sup\{|f(z)| : |z| = r\}$$

for each $w \in \mathbb{D}(0, r)$. These are *Cauchy's inequalities.*

This theorem has many useful consequences. Our first will be a partial converse of Cauchy's theorem.

Proposition 5.4 Morera's theorem
Let $f : D \to \mathbb{C}$ be a continuous function on a domain $D \subset \mathbb{C}$. If, for every closed triangle $T \subset D$, the integral $\displaystyle\int_{\partial T} f(z) \, dz$ is 0, then f is analytic.

Proof:

Let $z_o \in D$ and choose $R > 0$ so that $B(z_o, R) \subset D$. Then we can define a function $F : B(z_o, R) \to \mathbb{C}$ by

$$F(z) = \int_{[z_o,z]} f(z)\, dz\ .$$

Since f is continuous, the fundamental theorem of calculus shows that F is complex differentiable at each point of $B(z_o, R)$ with $F'(z) = f(z)$ (compare Theorem 4.2). Now F is analytic on the disc $B(z_o, R)$ and so the previous theorem shows that it is twice continuously differentiable. Thus $f'(z) = F''(z)$ exists. □

Note that the result fails if we do not insist that f is continuous. For example the function $f : \mathbb{C} \to \mathbb{C}$ that is 0 except at at a single point is not analytic.

The Local Behaviour of Analytic Functions

The power series expansion for an analytic function is very useful for describing the local behaviour of analytic functions. A key result is that the zeros of an non-constant analytic function are isolated. This means that if $f : D \to \mathbb{C}$ is a non-constant analytic function and $f(z_o) = 0$, then there is a neighbourhood V of z_o on which f has no other zeros.

Theorem 5.5 Isolated Zeros
The zeros of a non-constant analytic function are isolated.

Proof:

Let $f : D \to \mathbb{C}$ be an analytic function. For each $z \in D$ we know that there is a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n$$

that converges to $f(w)$ on some disc $B(z, R)$. The coefficients a_n are given by $f^{(n)}(z)/n!$. If all the coefficients a_n are 0, then f is zero on the entire disc $B(z, R)$. Conversely, if f is zero on some neighbourhood V of z, then each derivative $f^{(n)}(z)$ is 0 and so each coefficient a_n is 0.

Let A be the set: $\{z \in D :$ there is a neighbourhood V of z with $f(w) = 0$ for all $w \in V\ \}$. This is clearly open. However, we have shown that $A = \{z \in D : f^{(n)}(z) = 0 \text{ for all } n = 0, 1, 2, \dots\ \}$. If $z \in B = D \setminus A$, then there is a natural number n with $f^{(n)}(z) \neq 0$. Since $f^{(n)}$ is continuous, $f^{(n)}(w) \neq 0$ on some neighbourhood of z. Therefore, B is also open. Since D is connected, one of the two sets A, B must be empty. If B is empty, then f is constantly 0 on D. If A is empty, we will show that the zeros of f are isolated.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function with $f(z) = 0$ for some $z \in D$. Since f is not constant, the set B can not be all of D and must therefore be empty. This means that at least one of the coefficients of the power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z)^n \qquad \text{for} \quad w \in B(z, r)$$

is non-zero. Let a_N be the first such coefficient. Then

$$f(w) = (w - z)^N \left(\sum_{n=N}^{\infty} a_n (w - z)^{n-N} \right)\ .$$

Since the power series $\sum a_n (w - z)^n$ converges on $B(z, r)$, so does $\sum a_n (w - z)^{n-N}$ and it gives an analytic function $F : B(z, r) \to \mathbb{C}$. Note that $F(z) = a_N \neq 0$. Since F is continuous, there is an r with $0 < r < R$ and $F(w) \neq 0$ for $w \in B(z_o, r)$. This means that $f(w) = (w - z_o)^N F(w)$ is not 0 on $B(z_o, r)$ except at z_o. Thus z_o is an isolated zero. □

Corollary 5.6 Identity Theorem
Let $f, g : D \to \mathbb{C}$ be two analytic functions on a domain D. If the set $E = \{z \in D : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ everywhere on D.

Proof:

E is the set of zeros of the analytic function $f - g$. $\square$

This corollary gives us the *principle of analytic continuation*: If $f : D \to \mathbb{C}$ is an analytic function on a (non-empty) domain D and f extends to an analytic function $F : \Omega \to \mathbb{C}$ on some larger domain Ω, then F is unique. For, if $\widetilde{F} : \Omega \to \mathbb{C}$ were another extension of f, then F and $\widetilde{F}$ would agree on D and hence on all of Ω. However, there may not be any extension of f to a larger domain.

Let $f : D \to \mathbb{C}$ be a non-constant analytic function on a domain $D \subset \mathbb{C}$. For any point $z_o \in D$, we know that $f(z)$ is represented by a power series

$$f(w) = \sum_{n=0}^{\infty} a_n (w - z_o)^n$$

on some disc $B(z_o, R)$. Clearly $a_0 = f(z_o)$. Since the zeros of $f - f(z_o)$ are isolated, there must be a first coefficient (after a_0) that is non-zero, say a_N. We call N the *degree of f at z_o* and write it $\deg(f; z_o)$. We can write f as

$$f(w) = f(z_o) + (w - z_o)^N g(w)$$

for $w \in B(z_o, R)$ and some analytic function $g : B(z_o, R) \to \mathbb{C}$ with $g(z_o) \neq 0$. Indeed, we can define a function F on all of D by

$$F(w) = \begin{cases} \frac{f(w) - f(z_o)}{(w - z_o)^N} & \text{when } w \in D \setminus \{z_o\}; \\ g(w) & \text{when } w \in B(z_o, R). \end{cases}$$

These definitions agree on $B(z_o, R) \setminus \{z_o\}$ and so do define an analytic function $F : D \to \mathbb{C}$ with $f(w) = f(z_o) + (w - z_o)^N F(w)$ on all of D.

Locally Uniform Convergence

Let f_n and f be functions from a domain D into $\mathbb{C}$. We say that $f_n \to f$ *locally uniformly* on D if, for each $z_o \in D$, there is a neighbourhood V of z_o in D with $f_n(z) \to f(z)$ uniformly for $z \in V$.

Example: Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

converge locally uniformly on $B(0, R)$ to $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This was proven in Proposition 2.1 .

Suppose that $f_n \to f$ on the domain D. Then, for each $z_o \in D$, there is an open disc $\Delta(z_o)$ in D, centred on z_o, with $f_n(z) \to f(z)$ uniformly on $\Delta(z_o)$. If K is any compact subset of D, then K is covered by these sets $\Delta(z_o)$ for $z_o \in K$. Hence, there is a finite subcover. This shows that $f_n \to f$ uniformly on the compact set K. We will use this particularly when K is the image $[\gamma]$ of a curve γ.

Suppose that each of the functions f_n is continuous on D. The uniform limit of continuous functions is continuous, so f is continuous on each $\Delta(z_o)$ and hence on all of D. We will now prove the the locally uniform limit of analytic functions is analytic.

Proposition 5.7 Locally uniform convergence of analytic functions
Let $f_n : D \to \mathbb{C}$ be a sequence of analytic functions on a domain D that converges locally uniformly to a function f. Then f is analytic on D. Moreover, the derivatives $f_n^{(k)}$ converge locally uniformly on D to $f^{(k)}$.

Proof:

Let $z_o \in D$. Then there is a disc $\Delta = B(z_o, r)$ on which f_n converge uniformly to f. The functions f_n are continuous so the uniform limit f is also continuous on Δ. Also, the uniform convergence implies that

$$\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$$

for any closed curve γ in Δ. Since f_n is analytic, Cauchy's theorem for the disc Δ implies that $\int_\gamma f_n(z)\,dz = 0$. Therefore, $\int_\gamma f(z)\,dz = 0$. Morera's theorem now shows that f is analytic on Δ. Since z_o is arbitrary, f is analytic on all of D.

Now let $C(z_o, s)$ be the circle of radius $s < r$ about z_o. For $|w| < s$ Cauchy's representation formula (4.3) gives

$$f_n^{(k)}(w) = \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z)}{(z-w)^{k+1}}\,dz$$

and a similar formula for f, which we now know is analytic. Therefore,

$$\begin{aligned}
|f_n^{(k)}(w) - f^{(k)}(w)| &= \left| \frac{k!}{2\pi i} \int_{C(z_o,s)} \frac{f_n(z)-f(z)}{(z-w)^{k+1}}\,dz \right| \\
&\leqslant \frac{k!}{2\pi} L(C(z_o,s)) \sup\left\{ \left| \frac{f_n(z)-f(z)}{(z-w)^{k+1}} \right| : |z-z_o| = s \right\} \\
&\leqslant \frac{k!s}{(s-|w-z_o|)^k} \sup\{|f_n(z)-f(z)| : |z-z_o| = s\}
\end{aligned}$$

and we see that $f_n^{(k)}(w) \to f^{(k)}(w)$ uniformly on any disc $D(z_o, t)$ with $t < s$. □

This theorem gives us an alternative proof of Proposition 2.2 , which showed that a power series could be differentiated term by term inside its radius of convergence. For suppose that $s(z) = \sum a_n(z-z_o)^n$ is a power series with radius of convergence $R > 0$. Then the partial sums

$$S_N(z) = \sum_{n=0}^{N} a_n(z-z_o)^n$$

converge locally uniformly to s on $B(z_o, R)$. Each S_N is a polynomial and so is certainly analytic. Therefore s is analytic on $B(z_o, R)$. Moreover,

$$s'(z) = \lim_{N\to\infty} S_N'(z) = \lim_{N\to\infty} \sum_{n=0}^{N} na_n(z-z_o)^{n-1} = \sum_{n=0}^{\infty} na_n(z-z_o)^{n-1} .$$

Isolated Singularities

Let D be a domain and z_o a point of D. We are concerned about an analytic function $f : D\setminus\{z_o\} \to \mathbb{C}$ that is not defined at the point z_o. We call z_o an *isolated singularity of* f. It is defined and analytic at every point of some disc $B(z_o, R)$ except the centre z_o. We will study the behaviour of f as we approach the singular point.

The simplest possibility for f is that we can extend it to a function analytic on all of D, even at the point z_o. If this is the case, we say that f has a *removable singularity at* z_o. Usually we replace f by the analytic extension:

$$F(z) = \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

Since F is to be continuous, the value w_o it takes at z_o must be $\lim\limits_{z\to z_o} f(z)$ and F is unique. We will now show that f has a removable singularity at z_o if and only if the limit $\lim\limits_{z\to z_o} f(z)$ exists.

Example: The function

$$s : \mathbb{C} \setminus \{0\} \to \mathbb{C} \; ; \quad z \mapsto \frac{\sin z}{z}$$

has a removable singularity at 0. For the power series for the sine function shows that

$$s(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+1)!} \; .$$

So we can extend s to 0 by sending 0 to 1. This extension is given by a power series and so is analytic on all of $\mathbb{C}$.

Proposition 5.8 Removable singularities
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a removable singularity at $z_o \in D$ if and only if there is a finite limit $w_o \in \mathbb{C}$ with $f(z) \to w_o$ as $z \to z_o$.

Proof:
If f has a removable singularity at z_o, then there is an analytic extension $F : D \to \mathbb{C}$. This extension is continuous, so $f(z) = F(z) \to F(z_o)$ as $z \to z_o$.

For the converse, suppose that $f(z) \to w_o$ as $z \to z_o$. Then we can define

$$F : D \to \mathbb{C} \; ; \quad z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ w_o & \text{when } z = z_o. \end{cases}$$

This is certainly continuous at z_o and analytic elsewhere on D. Therefore, we can apply Cauchy's theorem to any triangle T within D using Proposition 4.1$'$ and obtain $\displaystyle\int_{\partial T} f(z)\, dz = 0$. Morera's theorem now shows that F is analytic on all of D. $\square$

When we proved Cauchy's theorem we considered a function $f : D \to \mathbb{C}$ that was analytic except at one point z_o where it was continuous. The last proposition shows that such a function is actually analytic even at z_o. So the exceptional point is no different from any other.

It is useful to strengthen the last proposition a little.

Corollary 5.9 Riemann's Removable Singularity Criterion
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a removable singularity at $z_o \in D$ if and only if $\lim\limits_{z\to z_o} (z - z_o) f(z) = 0$.

Note that when f is bounded in a neighbourhood of z_o, then the limit $\lim (z - z_o) f(z)$ certainly exists and is 0 and so there must be a removable singularity at z_o.

Proof:
The function $g(z) = (z - z_o) f(z)$ is analytic on $D \setminus \{z_o\}$ and tends to 0 as $z \to z_o$. Hence the previous proposition tells us that g has a removable singularity at z_o. Let $G : D \to \mathbb{C}$ be the analytic extension of g. We certainly have $G(z_o) = \lim_{z\to z_o} g(z) = 0$. Hence

$$f(z) = \frac{G(z) - G(z_o)}{z - z_o} \to G'(z_o) \qquad \text{as} \qquad z \to z_o \; .$$

Therefore, the previous proposition shows that f has a removable singularity at z_o. $\square$

So far we have only considered functions $f : D \to \mathbb{C}$ taking values in the finite complex plane $\mathbb{C}$. However, in the Algebra and Geometry course you considered functions taking values in the Riemann sphere (or extended complex plane) $\mathbb{C}_\infty$. The Riemann sphere consists of the complex plane $\mathbb{C}$ and one extra point ∞. You saw that the extra point ∞ behaved in the same way as the finite points in $\mathbb{C}$ and that the Möbius transformations $z \mapsto (az+b)/(cz+d)$ permuted the points of $\mathbb{C}_\infty$. We now wish to explain what it means for a function $f : D \to \mathbb{C}_\infty$ that takes values in the Riemann sphere to be analytic.

Let $f : D \to \mathbb{C}_\infty$ be a function defined on a domain $D \subset \mathbb{C}$ and $z_o \in D$. If $f(z_o) \in \mathbb{C}$, then f is complex differentiable at z_o if the limit $\lim_{z \to z_o} \dfrac{f(z) - f(z_o)}{z - z_o}$ exists and is a point of $\mathbb{C}$. If $f(z_o) = \infty$, we use the Möbius transformation $J : w \mapsto 1/w$ to send ∞ to a finite point and then ask if $J \circ f$ is complex differentiable at z_o. Thus we say that f *is complex differentiable at the point z_o with $f(z_o) = \infty$* if $z \mapsto 1/f(z)$ is complex differentiable at z_o. (It is not useful to define a value for $f'(z_o)$ at points where $f(z_o) = \infty$.) We call a point z_o where $f(z_o) = \infty$ and f is complex differentiable a *pole* of f. A function $f : D \to \mathbb{C}_\infty$ that is is not identically ∞ but is complex differentiable at each point of D is *meromorphic on D*. Since the zeros of a non-constant analytic function are isolated, the poles of a meromorphic function are also isolated. Thus a meromorphic function is analytic on its domain except for a set of poles each of which is isolated. For example, if $f : D \to \mathbb{C}$ is an analytic function and is not identically 0, then $z \mapsto 1/f(z)$ is meromorphic. This implies that each rational function is meromorphic on $\mathbb{C}$.

Suppose that $f : D \to \mathbb{C}$ is a meromorphic function and has a pole at z_o. The function f is certainly continuous at z_o so there is a neighbourhood V of z_o with $|f(z)| > 1$ for $z \in V$. Now the function $g : z \mapsto 1/f(z)$ is complex differentiable and finite at each point of V and it has a zero at z_o. Since f is not identically ∞, g can not be identically 0. Therefore, the zero at z_o is isolated. This means that we can write $g(z) = (z - z_o)^N G(z)$ for some natural number $N \geqslant 1$ and some function G that is analytic near z_o and has $G(z_o) \neq 0$. Therefore $f(z) = (z - z_o)^{-N} F(z)$ where $F(z) = 1/G(z)$ is analytic near z_o and has $F(z_o) \neq 0, \infty$. This show how the meromorphic function f behaves near a pole. We write $N = \deg(f; z_o)$ and call z_o a *pole of order N for f*.

We will say that an analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a *pole* at $z_o \in D$ if there is a meromorphic function $F : D \to \mathbb{C}_\infty$ that extends f and F has a pole at z_o. This is similar to f having a removable singularity at z_o except that the correct value to put for $f(z_o)$ is ∞.

Proposition 5.10 Poles as isolated singularities
The analytic function $f : D \setminus \{z_o\} \to \mathbb{C}$ has a pole at z_o if and only if $f(z) \to \infty$ as $z \to z_o$.

Proof:
If f has an extension F with a pole at z_o, then $f(z) = F(z) \to F(z_o) = \infty$ as $z \to z_o$.

For the converse, suppose that $f(z) \to \infty$ as $z \to z_o$. There is a neighbourhood V of z_o with $|f(z)| > 1$ for $z \in V \setminus \{z_o\}$. Hence, $g : z \mapsto 1/f(z)$ is bounded, analytic on $V \setminus \{z_o\}$ and has $g(z) \to 0$ as $z \to z_o$. Corollary 5.9 shows that g has a removable singularity at z_o so there is a function $G : V \to \mathbb{C}$ extending g. Now the function

$$F : z \mapsto \begin{cases} f(z) & \text{when } z \in D \setminus \{z_o\}; \\ 1/G(z) & \text{when } z \in V. \end{cases}$$

is well-defined and gives a meromorphic extension of f. □

There remain some isolated singularities that are neither removable singularities nor poles. We call these *essential singularities*. Functions behave very dramatically near an essential singularity.

Example: The function $f : z \mapsto \exp(1/z)$ has an essential singularity at 0. For real values of t we have

$$\exp(1/t) \to \infty \quad \text{as} \quad t \searrow 0+ \qquad \text{while} \qquad \exp(1/t) \to 0 \quad \text{as} \quad t \nearrow 0-$$

so the limit $\lim_{z\to 0} f(z)$ can not exist either as a finite complex number or as ∞. Therefore, f can not have either a removable singularity or a pole at 0.

Exercise: The function $g : z \mapsto (\cos z)\exp(1/z)$ has an essential singularity at 0.

> *The function $\cos z$ is analytic and non-zero near 0. If g had a removable singularity or a pole at 0, then $\exp(1/z) = g(z)/\cos z$ would also have a removable singularity or a pole at 0. We know that this is not true.*

Proposition 5.11 Weierstrass - Casorati Theorem
An analytic function takes values arbitrarily close to any complex number on any neighbourhood of an essential singularity.

Proof:
Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at z_o. Suppose that there is some neighbourhood of z_o on which f does not take values arbitrarily close to $w_o \in \mathbb{C}$. Say

$$|f(z) - w_o| > \varepsilon \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then the function $g : z \mapsto 1/(f(z) - w_o)$ is bounded by $1/\varepsilon$ for $0 < |z - z_o| < R$. Therefore, g has a removable singularity at z_o by Corollary 5.9 . Consequently, $f(z) = w_o + 1/g(z)$ will have a removable singularity or a pole at z_o.

A similar argument applies for $w_o = \infty$. Suppose that

$$|f(z)| > K \qquad \text{for} \quad 0 < |z - z_o| < R\ .$$

Then $g : z \mapsto 1/f(z)$ is bounded by $1/K$ for $0 < |z - z_o| < R$. Therefore, g has a removable singularity at z_o and f will have a removable singularity or a pole. □

(In fact much more is true. Picard showed that in every neighbourhood of an essential singularity the function takes each value $w \in \mathbb{C}_\infty$ with at most two exceptions. The example $z \mapsto \exp(1/z)$ takes every value except 0 and ∞.)

Analytic Functions on an Annulus

Let $A = \{z \in \mathbb{C} : R_1 < |z| < R_2\}$ be an annulus or ring-shaped domain and let $f : A \to \mathbb{C}$ be an analytic function. We have seen that $\int_\gamma f(z)\,dz$ can be non-zero, for example when $f(z) = 1/z$. In this section we want to study what values the integral can take.

Proposition 5.12 Cauchy's theorem on an annulus
For each analytic function $f : A \to \mathbb{C}$ there is a constant K_f with

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = n(\gamma; 0)K_f$$

for every closed, piecewise continuously differentiable path γ in A.

Note that this is certainly true when f is analytic on the entire disc $\{z : |z| < R_2\}$ because of Cauchy's theorem. In this case $K_f = 0$. Also, it is true for $f(z) = 1/z$ because of the definition of the winding number $n(\gamma; 0)$. In this case, $K_f = 1$.

Proof:

Let S be the strip $\{w = u + iv \in \mathbb{C} : \log R_1 < u < \log R_2\}$, which is a star with any point as a centre. The exponential mapping $\exp : S \to A;\ w \mapsto e^w$ maps S onto A. Cauchy's theorem for star domains (4.2) shows that the analytic function $\phi : S \to \mathbb{C}$; $\phi(w) = f(e^w)e^w$ has an antiderivative Φ. Now $e^{w+2\pi i} = e^w$ so $\phi(w + 2\pi i) = \phi(w)$ and hence $\Phi'(w + 2\pi i) = \Phi'(w)$. Hence, there is a constant K_f with

$$\Phi(w + 2\pi i) = \Phi(w) + 2\pi i K_f \ .$$

Let C_r be the circle $C_r : [0, 2\pi] \to A$, $t \mapsto re^{it}$ for $R_1 < r < R_2$. Then

$$\int_{C_r} f(z)\ dz = \int_0^{2\pi} f(re^{it}) ire^{it}\ dt = i \int_0^{2\pi} \phi(\log r + it)\ dt = \Phi(\log r + 2\pi i) - \Phi(\log r) = 2\pi i K_f$$

so we can determine K_f from this integral.

Consider first the case where $K_f = 0$. Then we have $\Phi(w + 2n\pi i) = \Phi(w)$ for each $n \in \mathbb{Z}$. So we can define a function $F : A \to \mathbb{C}$ unambiguously by $F(z) = \Phi(w)$ for any w with $z = e^w$. The derivative of this satisfies $F'(e^w)e^w = \Phi'(w) = \phi(w)w = f(e^w)e^w$. Hence, $F'(z) = f(z)$ and f has an antiderivative on A. Consequently,

$$\int_\gamma f(z)\ dz = 0$$

for any closed curve γ in A by Proposition 3.2 .

Now suppose that $K_f \neq 0$. Then we can replace f by the function

$$g(z) = f(z) - \frac{K_f}{z} \ .$$

This has

$$K_g = \frac{1}{2\pi i} \int_{C_r} g(z)\ dz = \frac{1}{2\pi i} \int_{C_r} g(z)\ dz - \frac{K_f}{2\pi i} \int_{C_r} \frac{1}{z}\ dz = K_f - n(C_r; 0)K_f = 0 \ .$$

Therefore, we can apply the previous argument to g and obtain

$$\frac{1}{2\pi i} \int_\gamma f(z)\ dz = \frac{1}{2\pi i} \int_\gamma g(z)\ dz + \frac{K_f}{2\pi i} \int_\gamma \frac{1}{z}\ dz = 0 + n(\gamma; 0)K_f$$

as required. □

We can also apply this result to an annulus $A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\}$ centred at some other point z_o. Then we have

$$\frac{1}{2\pi i} \int_\gamma f(z)\ dz = n(\gamma; z_o)K_f$$

for any closed curve γ in A. This result is particularly useful when $R_1 = 0$. Then we call the constant K_f the *residue of f at z_o* and denote it by $\mathrm{Res}(f; z_o)$.

Proposition 5.13 Analytic functions on an annulus
For each analytic function $f : A \to \mathbb{C}$ there are analytic functions

$$F_1 : \{z : |z| > R_1\} \to \mathbb{C} \qquad \text{and} \qquad F_2 : \{z : |z| < R_2\} \to \mathbb{C}$$

with $f(z) = F_2(z) - F_1(z)$ for each $z \in A$.

Proof:

We proceed as in the proof of the Cauchy Representation Theorem (4.3). Let w be a fixed point in A and set

$$g(z) = \frac{f(z) - f(w)}{z - w} \qquad \text{for } z \in A \setminus \{w\} .$$

Then $g(z) \to f'(w)$ as $z \to w$, so g has a removable singularity at w (Proposition 5.8). If we set $g(w) = f'(w)$ then we obtain a function g analytic on all of the annulus A. For any closed curve γ in $A \setminus \{w\}$ we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz - \frac{f(w)}{2\pi i}\int_\gamma \frac{1}{z-w}\,dz = \frac{1}{2\pi i}\int_\gamma g(z)\,dz ,$$

which gives

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,dz = n(\gamma;0)f(w) + \frac{1}{2\pi i}\int_\gamma g(z)\,dz .$$

We can apply this when γ is the circle C_r for $r \neq |w|$. For this the previous proposition shows that

$$\frac{1}{2\pi i}\int_\gamma g(z)\,dz = K_g$$

is independent of r. Hence

$$\begin{aligned} \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz &= K_g \qquad &&\text{when } R_1 < r < |w| \\ \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz &= f(w) + K_g \qquad &&\text{when } |w| < r < R_2 . \end{aligned} \tag{$*$}$$

Let

$$F_1(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz \qquad \text{for } R_1 < r < |w| .$$

Corollary 5.2 shows that F_1 is an analytic function of w on $\{w : r < |w|\}$. Since $f(z)/(z-w)$ is analytic on the annulus $\{z : R_1 < |z| < |w|\}$ the value of $F_1(w)$ is independent of $r \in (R_1, |w|)$. This means that F_1 is an analytic function on $\{w : R_1 < |w|\}$. Similarly,

$$F_2(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\,dz \qquad \text{for } |w| < r < R_2$$

gives an analytic function on $\{w : |w| < R_2\}$.

Finally, equations $(*)$ shows that

$$f(w) = F_2(w) - F_1(w) .$$

□

We already know that analytic functions on discs have power series expansions. The last proposition gives similar expansions for analytic functions on an annulus.

Corollary 5.14 Laurent expansions
For each analytic function $f : A = \{z \in \mathbb{C} : R_1 < |z - z_o| < R_2\} \to \mathbb{C}$ there are coefficients a_n for $n \in \mathbb{Z}$ with

$$f(w) = \sum_{n=-\infty}^{\infty} a_n (w - z_o)^n \qquad \text{for } w \in A .$$

This series converges locally uniformly on the annulus A. Moreover,

$$n(\gamma;0)a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_o)^{n+1}}\,dz$$

for every $n \in \mathbb{Z}$ and any piecewise continuously differentiable closed curve γ in A.

Proof:

By translating A we may ensure that $z_o = 0$. Then we know that $f(w) = F_2(w) - F_1(w)$ for analytic functions $F_1 : \{w : R_1 < |w|\} \to \mathbb{C}$ and $F_2 : \{w : |w| < R_2\} \to \mathbb{C}$. The function F_2 is analytic on a disc, so it has a power series expansion $F_2(w) = \sum_{n=0}^{\infty} b_n w^n$ that converges locally uniformly on $\{w : |w| < R_2\}$.

The argument for F_1 is similar but the disc is centred on ∞ in $\mathbb{C}_\infty$ rather than on 0. Hence we must begin by using a Möbius transformation to move ∞ to 0. First note that

$$F_1(w) = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{z-w}\, dz \qquad \text{has} \qquad |F_1(w)| \leqslant \frac{r \sup\{|f(z)| : |z| = r\}}{|w| - r}$$

so $F_1(w) \to 0$ as $w \to \infty$. Let $G(z) = F_1(1/z)$ then $G(z) \to 0$ as $z \to 0$. Therefore G has a removable singularity at 0 and so gives us an analytic function $G : \{z : |z| < 1/R_1\} \to \mathbb{C}$. This has a power series expansion $G(z) = \sum_{n=1}^{\infty} c_n z^n$ that converges locally uniformly on $\{z : |z| < 1/R_1\}$. (The constant term is 0 since $G(0) = 0$.) Thus $F_1(w) = \sum_{n=1}^{\infty} c_n w^{-n}$ and the series converges locally uniformly on $\{w : R_1 < |w|\}$.

Putting these power series together we obtain

$$f(w) = \sum_{n=0}^{\infty} b_n w^n \; - \; \sum_{n=1}^{\infty} c_n w^{-n} \; .$$

Both parts of this sum converge locally uniformly on the annulus A. This gives the Laurent series we wanted.

Since the Laurent series for f converges uniformly on the compact set $[\gamma]$, we see that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_o)^{n+1}}\, dz = \sum_{k=-\infty}^{\infty} a_k \frac{1}{2\pi i}\int_\gamma (z-z_o)^{k-n-1}\, dz \; .$$

We can easily evaluate the integrals $\int_\gamma (z-z_o)^m\, dz$ and see that they are 0 except when $m = -1$. Hence,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_o)^{n+1}}\, dz = a_n n(\gamma; z_o) \; .$$

□

Laurent Series about isolated singularities

Let z_o be a point in the domain D and let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function. So f has an isolated singularity at z_o. There will be a disc $B(z_o, R)$ that lies within D. So f is analytic on the annulus $A = \{z : 0 < |z - z_o| < R\}$ and has a Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

on this annulus. Corollary 5.14 shows that the residue of f at z_o is $\mathrm{Res}(f; z_o) = a_{-1}$.

Proposition 5.15 Laurent series for isolated singularities
Let $f : D \setminus \{z_o\} \to \mathbb{C}$ be an analytic function with an isolated singularity at z_o and let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$$

be its Laurent expansion that converges for $0 < |z - z_o| < R$. Then

(a) f has a removable singularity at z_o if and only if $a_n = 0$ for $n < 0$.

(b) f has a pole at z_o of order N if and only if $a_n = 0$ for $n < -N$ and $a_{-N} \neq 0$.

(c) f has an essential singularity at z_o if and only if $a_n \neq 0$ for infinitely many negative values of n.

Proof:

(a) Suppose that f has a removable singularity at z_o. then there is an analytic function $F : D \to \mathbb{C}$ extending f. For γ a closed curve in the annulus A we have

$$a_n = \frac{1}{2\pi i} \int_C \frac{F(z)}{(z - z_o)^{n+1}}\, dz$$

and Cauchy's theorem shows that this is 0 for $n < 0$. Conversely, if $a_n = 0$ for $n < 0$, then the Laurent series reduces to a power series and defines an analytic extension of f.

(b) Suppose that f has a pole of order N at z_o. Then $f(z) = (z - z_o)^{-N} G(z)$ for some function G analytic near z_o and with $G(z_o) \neq 0$. The Laurent series for G is

$$G(z) = \sum_{n=-\infty}^{\infty} a_{n-N} (z - z_o)^n \ .$$

This has a removable singularity at z_o, so part (a) implies that $a_n = 0$ for $n < -N$. We also have $a_{-N} = G(z_o) \neq 0$. Conversely, if $a_n = 0$ for $n < -N$ and $a_{-N} = 0$, then

$$f(z) = (z - z_o)^{-N} \sum_{n=0}^{\infty} a_{n-N} (z - z_o)^n$$

so f has a pole of order N at z_o.

(c) The singularity is essential if and only if it is neither removable nor a pole. Similarly, the Laurent series has $a_n \neq 0$ for infinitely many negative n if and only if there is no integer N with $a_n = 0$ for $n < -N$. Thus (a) and (b) imply (c). □

Laurent series give us a quick proof of the Residue theorem at least for simply connected domains. Suppose that f has an isolated singularity at z_o and has Laurent series $\sum_{n=-\infty}^{\infty} a_n (z - z_o)^n$. The part

$$P(z) = \sum_{n=-\infty}^{-1} a_n (z - z_o)^n$$

of this series is called the *principal part of f at z_o*. The principal part is a power series in $1/(z - z_o)$ and converges for z sufficiently close to z_o. Therefore, it must converge for all $z \in \mathbb{C} \setminus \{z_o\}$. The difference $f - P$ is analytic at z_o.

The Maximum Modulus Principle

Theorem 5.16 Maximum Modulus Principle
Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. If the modulus $|f|$ has a local maximum at a point $z_o \in D$, then f is constant on D.

Proof:

The hypothesis means that there is a disc $\mathbb{D}(z_o, R) \subset D$ with

$$|f(z)| \leqslant |f(z_o)| \qquad \text{for all } z \in \mathbb{D}(z_o, R) \ .$$

For any r with $0 < r < R$ let $C(r)$ be the circle of radius r centred on z_o. Cauchy's representation theorem (4.3) shows that

$$f(z_o) = \int_{C(r)} f(z)\, dz = \int_0^{2\pi} f(z_o + re^{i\theta})\, \frac{d\theta}{2\pi} \ .$$

Set $M = |f(z_o)|$ and write $f(z_o) = Me^{i\alpha}$. Then we see that

$$M = \Re\left(f(z_o)e^{-i\alpha}\right) \leqslant \int_0^{2\pi} \Re\left(f(z_o + re^{i\theta})e^{-i\alpha}\right)\, \frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} |f(z_o + re^{i\theta})|\, \frac{d\theta}{2\pi} \leqslant \int_0^{2\pi} M\, \frac{d\theta}{2\pi} = M \ .$$

There must be equality throughout this and, since the integrand is continuous, this means that

$$\Re\left(f(z_o + re^{i\theta})e^{-i\alpha}\right) = |f(z_o + re^{i\theta})| = M$$

for all θ. Therefore, $f(z_o + re^{i\theta}) = M$.

The function $z \mapsto f(z) - f(z_o)$ is analytic and has zeros at every point of $\mathbb{D}(z_o, R)$. So Theorem 5.5 shows that it is constant. $\square$

The maximum modulus principle is very useful indeed. The commonest way to apply it is as follows. Suppose that K is a compact set and f is analytic on some domain D containing K. Then $|f|$ is continuous on K and so is bounded above and attains its supremum at some point z_o. If z_o is an interior point of K, then the maximum modulus principle tells us that f is constant. In this case, f achieves its maximum modulus at each point of K. Thus we see that in every case the maximum modulus of f is achieved on the boundary of K.

6. THE HOMOLOGY FORM OF CAUCHY'S THEOREM

*The proof of the Homology Form of Cauchy's Theorem is **not** examinable.*

The aim of this chapter is to prove the following theorem.

Theorem 6.1 Homology form of Cauchy's Theorem
Let Γ be a cycle in the domain $D \subset \mathbb{C}$ and suppose that $n(\Gamma; w) = 0$ for each $w \in \mathbb{C} \setminus D$. Then, for any analytic function $f : D \to \mathbb{C}$ we have

$$\int_\Gamma f(z)\, dz = 0\ .$$

Note that, for each point $w \in \mathbb{C} \setminus D$, the map $p_w : z \mapsto 1/(z - w)$ is analytic on D and

$$n(\Gamma; w) = \frac{1}{2\pi i} \int_\Gamma p_w(z)\, dz\ .$$

So we must certainly have $n(\Gamma; w) = 0$ if the integral of any analytic function around Γ is to vanish.

We will prove the theorem is two stages. First we will prove it for "grid cycles" that are made up of straight line segments parallel to the axes. Then we will approximate any cycle by such a "grid cycle".

Fix a number $\delta > 0$ and consider the grid of $\delta \times \delta$-squares

$$Q = \{x + iy : m\delta \leqslant x \leqslant (m+1)\delta,\ n\delta \leqslant y \leqslant (n+1)\delta\}$$

for $m, n \in \mathbb{Z}$. The centre of this square will be denoted by $c(Q)$ and the boundary by ∂Q. The boundary is made up of the four edges of Q followed anti-clockwise. For simplicity we will call a cycle made up from the edges of these squares a *grid cycle*.

Lemma 6.2
Any grid cycle B satisfies $B = \sum n(B; c(Q))\partial Q$ where the sum is over all $\delta \times \delta$ squares Q in the grid.

Proof:
The set $[B]$ of points on the cycle B is compact and so contained in some disc. Lemma 3.3 shows that the winding number $n(B; c(Q))$ is zero for Q outside this disc. Hence the sum $A = \sum n(B; c(Q))\partial Q$ in the lemma has only finitely many non-zero terms and does give a grid cycle.

We can write the cycle B as a sum $\sum k(\eta)\eta$ over all of the edges η of the squares, with the coefficients $k(\eta)$ being (positive or negative) integers. A particular edge η separates two squares, say Q^+ and Q^- with η having coefficient $+1$ in ∂Q^+ and -1 in ∂Q^-.

Suppose that the coefficient $k(\eta)$ is 0. Then the two points $c(Q^+)$ and $c(Q^-)$ both lie in the same component of $\mathbb{C} \setminus [B]$. So Proposition 3.5 shows that $n(B; c(Q^+)) = n(B; c(Q^-))$.

More generally, if $k(\eta) = k$, then the coefficient of η in $B - k\partial Q^+$ is 0. So

$$n(B - k\partial Q^+; c(Q^+)) = n(B - k\partial Q^+; c(Q^-))\ .$$

Hence, we have $n(B; c(Q^+)) - k = n(B; c(Q^+)) - kn(\partial Q^+; c(Q^+)) = n(B; c(Q^-))$. Therefore,

$$k(\eta) = n(B; c(Q^+)) - n(B; c(Q^-))\ . \qquad (*)$$

Exactly the same argument applies to the cycle A.

Consider one of the squares P. For this we have

$$n(A; c(P)) = \sum n(B; c(Q))n(\partial Q; c(P))\ .$$

Clearly the winding number $n(\partial Q; c(P))$ is zero except when $P = Q$, when it is 1. Therefore $n(A; c(P)) = n(B; c(P))$. Now the identity $(*)$ shows that the coefficient of each edge η is the same in A as in B. □

Lemma 6.3 Cauchy's theorem for grids
Let B be a grid cycle and let $f : D \to \mathbb{C}$ be an analytic function. If, for each closed square Q that is not a subset of D, the winding number $n(B; c(Q))$ is zero, then $\int_B f(z)\,dz = 0$.

Proof:
The previous lemma shows that $B = \sum n(B; c(Q))\partial Q$. If the winding number $n(B; c(Q)) \neq 0$, then the closed square Q must be a subset of D. Hence, Cauchy's Theorem for star domains (4.2), shows that
$$\int_{\partial Q} f(z)\,dz = 0 \ .$$
Summing over all such squares gives the result. □

Let Γ be any cycle. We will show that we can approximate Γ by a grid cycle B. To do this, it is sufficient to consider each of the component closed curves in Γ. So, let γ be a piecewise continuously differentiable closed curve.

First, it is useful to introduce some notation. If α is a curve that ends at w and β a curve that begins at w then $\beta \cdot \alpha$ will denote the curve α followed by β. Also, α^{-1} will denote the curve α reversed. For each point $z = x + iy \in \mathbb{C}$, set
$$\widehat{z} = \left\lfloor \frac{x}{\delta} \right\rfloor \delta + i \left\lfloor \frac{y}{\delta} \right\rfloor \delta \ ,$$
which is the lower left corner of the square containing z.

Subdivide the closed curve γ into curves γ_j for $j = 1, 2, 3, \ldots, J$ each of length at most δ. Let γ_j begin at z_j and ends at z_{j+1}. Since γ is a closed curve, we have $z_{J+1} = z_1$. For each j, let α_j be the straight line path from z_j to $\widehat{z}_j$. Then α_j has length at most $\sqrt{2}\delta$.

Since $|z_j - z_{j+1}| \leqslant \delta$, the points $\widehat{z}_j$ and $\widehat{z}_{j+1}$ must both be corners of one of the squares Q. Hence there is a path β_j from $\widehat{z}_j$ to $\widehat{z}_{j+1}$ along the sides of Q with length at most 2δ. Now the path $\alpha_{j+1}^{-1} \cdot \beta_j \cdot \alpha_j$ goes from z_j to z_{j+1} and lies inside the disc $\mathbb{D}(z_j, 4\delta)$.

Set $\beta = \beta_J \cdot \beta_{J-1} \cdot \ldots \cdot \beta_2 \cdot \beta_1$. This is a closed curve along the edges of the grid, so it is a grid cycle. The following result shows that the integral of an analytic function around γ is the same as the integral around β.

Lemma 6.4
Let γ be a piecewise continuously differentiable closed curve in the domain $D \subset \mathbb{C}$. For δ sufficiently small, there is a grid cycle β in D with
$$\int_\gamma f(z)\,dz = \int_\beta f(z)\,dz$$
for every analytic function $f : D \to \mathbb{C}$.

Proof:
The set $[\gamma]$ is compact and so
$$d([\gamma], \mathbb{C} \setminus D) = \inf\{|z - w| : z \in [\gamma],\ w \in \mathbb{C} \setminus D\} > 0 \ .$$
Choose δ with $10\delta < d([\Gamma], \mathbb{C} \setminus D)$. The grid cycle β we constructed above lies within a distance 4δ of Γ so it will certainly lie within D.

For each point z_j, both of the curves γ_j and $\alpha_{j+1}^{-1} \cdot \beta_j \cdot \alpha_j$ go from z_j to z_{j+1} and lie inside the disc $\mathbb{D}(z_j, 4\delta)$. Furthermore, we have chosen δ so that this disc lies within D. Hence, Cauchy's Theorem for star domains (4.2) shows that
$$\int_{\gamma_j} f(z)\,dz = \int_{\alpha_j} f(z)\,dz + \int_{\beta_j} f(z)\,dz - \int_{\alpha_{j+1}} f(z)\,dz \ .$$
Adding these over all j gives
$$\int_\gamma f(z)\,dz = \int_\beta f(z)\,dz$$
since the integrals along the paths α_j occur in opposing directions and so cancel out. This establishes the result. □

These results enable us to prove Theorem 6.1.

Proof of Theorem 6.1

The cycle Γ consists of finitely many closed curves γ. For each of these we can construct a grid cycle β for which Lemma 6.4 holds. Let B be the grid cycle obtained by adding together these β.

Suppose that Q is one of the closed squares in the $\delta \times \delta$ grid and that Q is not a subset of D, say $w_o \in Q \setminus D$. Recall that $10\delta < d([\Gamma], \mathbb{C} \setminus D)$, so the entire square Q is disjoint from Γ. Furthermore, B lies within a distance 4δ of Γ so the entire square Q is disjoint from B. This means that Q lies in one component of $\mathbb{C} \setminus [B]$ and so, by Proposition 3.5 , the winding number $n(B; w)$ is constant on Q. The hypothesis of Theorem 6.1 shows that $n(\Gamma; w_o) = 0$, and Lemma 6.4 shows that

$$n(B; w_o) = \frac{1}{2\pi i} \int_B \frac{1}{z - w_o} \, dz = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z - w_o} \, dz = n(\Gamma; w_o) = 0 \ .$$

Hence, $n(B; c(Q)) = 0$.

Therefore, the conditions of Lemma 6.2 are satisfied and

$$\int_B f(z) \, dz = 0 \ .$$

Finally, Lemma 6.3 shows that $\int_\Gamma f(z) \, dz = 0$ as required. □

We say that a cycle Γ in the domain D is *homologous to* 0 *in* D when the winding number $n(\Gamma; w)$ is zero for each point $w \in \mathbb{C} \setminus D$. Then Theorem 6.1 shows that

$$\int_\Gamma f(z) \, dz = 0$$

for any analytic function $f : D \to \mathbb{C}$ and any cycle Γ that is homologous to 0 in D.

The simplest case, and the one we use most often, is when the cycle Γ is the boundary of a subset of D. Suppose that U is a subset of D bounded by finitely many disjoint, piecewise continuously differentiable closed curves in D. We orient these boundary curves so that the set U lies to the left as we go along the curve. Then ∂U is a cycle in D. The Jordan curve theorem shows that the winding number $n(\partial U; w)$ is 0 for $w \notin U$ and 1 for $w \in U$.

A closed curve $\gamma : [a, b] \to \mathbb{D}$ is *simple* if it does not cross itself, so $\gamma(s) = \gamma(t)$ for two distinct points s, t only when s and t are the endpoints a and b. The Jordan curve theorem shows that such a curve divides the plane into two connected components: the inside and the outside of γ. The winding number $n(\gamma; w)$ is 1 for w inside γ and 0 for w outside γ. (We will not prove this.)

It is usual to apply Cauchy's theorem when the cycle Γ is a simple closed curve bounding a region in D. However, we will make a slightly more general definition: A cycle Γ *bounds a domain* Ω if the winding number $n(\Gamma; w)$ is 1 for all points $w \in \Omega$ and either 0 or undefined for all points not in Ω. It is clear that a cycle Γ in D that bounds a domain $\Omega \subset D$ is homologous to 0 in D. Hence Theorem 6.1 implies that:

Theorem 6.1' Cauchy's Theorem
If $f : D \to \mathbb{C}$ is an analytic function on a domain $D \subset \mathbb{C}$ and the cycle Γ bounds a domain $\Omega \subset D$, then

$$\int_\Gamma f(z) \, dz = 0 \ .$$

□

Note that, in most of the cases where this is applied, we can divide the region D into pieces each of which is a star domain and then deduce the result from Cauchy's Theorem for star-domains (4.2).

The Residue Theorem

Let D be a domain in $\mathbb{C}$ and f a function that is analytic on D except for isolated singularities at the points $z_1, z_2, \dots, z_K$. This means that, for each $k = 1, 2, 3, \dots, K$, there is a closed disc $\overline{B(z_k, R_k)}$ that lies within D and contains only the singularity at z_k. Then f is analytic on $B(z_k, R_k) \setminus \{z_k\}$ and has a residue $\mathrm{Res}(f; z_k)$ at z_k.

Theorem 6.5 Residue theorem
Let D be a domain in $\mathbb{C}$ and f a function that is analytic on D except for isolated singularities at the points $z_1, z_2, \dots, z_K$. For any cycle Γ in $D \setminus \{z_1, z_2, \dots, z_K\}$ that is homologous to 0 in D we have

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{k=1}^{K} n(\Gamma; z_k) \mathrm{Res}(f; z_k) \ .$$

Proof:
Let C_k denote the positively oriented circle bounding the disc $B(z_k, r_k)$. Then $n(C_k; w) = 1$ if $w \in B(z_k, r_k)$ and $n(C_k; w) = 0$ for any $w \notin \overline{B(z_k, r_k)}$. In particular, $n(C_k; z_k) = 1$ but $n(C_k; z_j) = 0$ for any $j \neq k$. Hence the cycle

$$\Delta = \Gamma - \sum_{k=1}^{K} n(\Gamma; z_k) C_k$$

is homologous to 0 in $D \setminus \{z_1, z_2, \dots, z_K\}$. Now the homology form of Cauchy's theorem (Theorem 6.1) shows that

$$0 = \int_\Delta f(z)\, dz = \int_\Gamma f(z)\, dz - \sum_{k=1}^{K} n(\Gamma; z_k) \int_{C_k} f(z)\, dz \ .$$

Finally, the definition of the residue shows that $\displaystyle\int_{C_k} f(z)\, dz = 2\pi i \mathrm{Res}(f; z_k)$. □

We can restate the residue theorem for cycles that bound subdomains:

Theorem 6.5' Residue theorem
Let D be a domain and f a function that is analytic on D except for isolated singularities. Let Γ be a cycle that bounds a subdomain Ω of D and does not pass through any singularity. Then

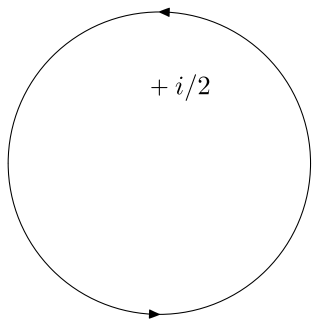
$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum \mathrm{Res}(f; w)$$

where the sum is over all the singularities in Ω.

Proof:
The set $\overline{\Omega}$ is compact since it is bounded by $[\Gamma]$. For each $w \in \overline{\Omega}$ there is an open neighbourhood that contains at most one singularity of f, because the singularities are isolated. These open neighbourhoods form an open cover for $\overline{\Omega}$ so there is a finite subcover. Hence there can be only a finite number of singularities within Ω. Now we can apply Theorem 6.5 . □

Examples

Example 1 $\displaystyle\int_0^{2\pi} \frac{1}{5 - 4\sin\theta}\, d\theta$



Let Γ be the circular curve: $\Gamma : [0, 2\pi] \to \mathbb{C}$; $\theta \mapsto e^{i\theta}$. Then we have

$$\int_\Gamma f(z)\, dz = \int_0^{2\pi} f(e^{i\theta}) i e^{i\theta}\, d\theta$$

so we should choose f with

$$ie^{i\theta} f(e^{i\theta}) = \frac{1}{5 - 4\sin\theta} = \frac{i}{5i - 2e^{2i\theta} + 2e^{-i\theta}} \ .$$

Therefore set

$$f(z) = \frac{1}{-2z^2 + 5iz + 2} = \frac{-1}{(2z - i)(z - 2i)} \ .$$

This is meromorphic on all of $\mathbb{C}$ with simple poles at $\frac{1}{2}i$ and $2i$. Only the pole at $\frac{1}{2}i$ lies within Γ, and there the residue is

$$\mathrm{Res} f(\tfrac{1}{2}i) = \lim_{z \to \frac{1}{2}i} (z - \tfrac{1}{2}i) f(z) = \lim_{z \to \frac{1}{2}i} \frac{-(z - \frac{1}{2}i)}{(2z - i)(z - 2i)} = \frac{-1}{2(\frac{1}{2}i - 2i)} = -\tfrac{1}{3}i \ .$$

Hence the residue theorem gives

$$\int_0^{2\pi} \frac{1}{5 - 4\sin\theta}\, d\theta = 2\pi i \mathrm{Res} f(\tfrac{1}{2}i) = \tfrac{2}{3}\pi \ .$$

Example 2 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx$

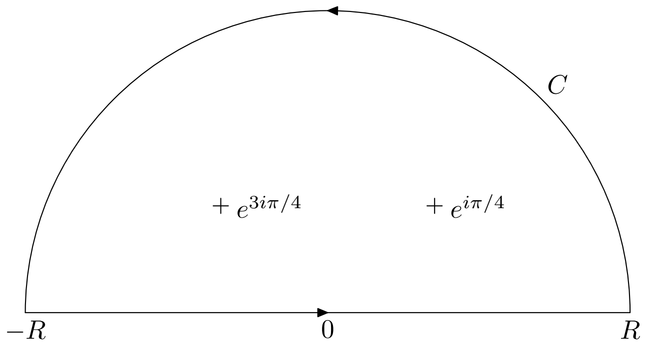
Note first that this improper integral certainly converges. Hence we know that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx\ .$$

(The following argument only shows that the symmetric integrals $\displaystyle\int_{-R}^{R} \frac{1}{1+x^4}\,dx$ converge to a limit as $R \nearrow \infty$. We would need a slightly different argument to show that

$$\lim_{R,S\to\infty} \int_{S}^{R} \frac{1}{1+x^4}\,dx$$

exists and hence that the improper integral we are concerned with converges. However, since we already know that the improper integral converges, the argument certainly enables us to evaluate it.)



Let Γ be the semi-circular contour made up of straight line $[-R,+R]$ and the semi-circle $C : [0,\pi] \to \mathbb{C};\ \theta \mapsto Re^{i\theta}$. Let f be the meromorphic function

$$f : z \mapsto \frac{1}{1+z^4}\ .$$

This has simple poles at the four 4th roots of -1. The residue at one of these, say ω, is

$$\mathrm{Res} f(\omega) = \lim_{z\to\omega} (z-\omega) f(z) = \lim_{z\to\omega} \frac{z-\omega}{z^4+1} = \lim_{z\to\omega} \frac{1}{4z^3} = -\frac{\omega}{4}\ .$$

Hence the residue theorem gives

$$\int_{\Gamma} \frac{1}{1+z^4}\,dz = 2\pi i \left(\mathrm{Res} f(e^{i\pi/4}) + \mathrm{Res} f(e^{3i\pi/4})\right) = \tfrac{1}{2}\sqrt{2}\pi$$

provided that $R > 1$.

For $z = Re^{i\theta}$ on the semi-circle C we have

$$|f(z)| \leqslant \frac{1}{R^4-1}$$

so

$$\left|\int_C f(z)\,dz\right| \leqslant \int_0^{\pi} |f(Re^{i\theta})| R\,d\theta \leqslant \frac{\pi R}{R^4-1} \to 0 \qquad \text{as } R \nearrow \infty\ .$$

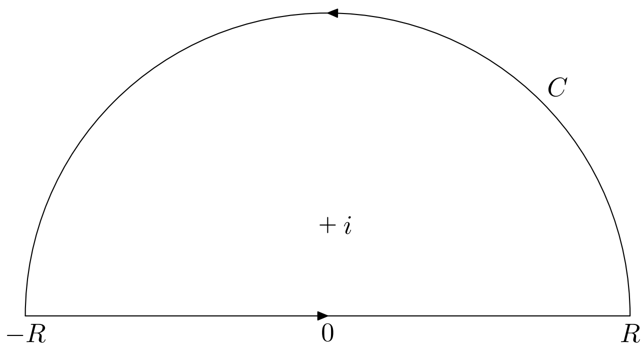
Now

$$\int_{\Gamma} \frac{1}{1+z^4}\,dz = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_C f(z)\,dz\ .$$

Hence we see that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^4}\,dx = 2\sqrt{2}\pi\ .$$

Example 3 $\displaystyle\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx$ for $t \in \mathbb{R}$.



It is simple to see that this improper integral converges. For $t \geqslant 0$ take the semi-circular contour used in Example 2 and set

$$f(z) = \frac{e^{itz}}{1+z^2}\ .$$

This has simple poles at $\pm i$ with residues $\mathrm{Res} f(\pm i) = e^{\mp t}/2i$. For $z = Re^{i\theta} = R\cos\theta + iR\sin\theta$ on the semi-circle C we have

$$|f(z)| = \frac{e^{-tR\sin\theta}}{R^2-1}$$

so,

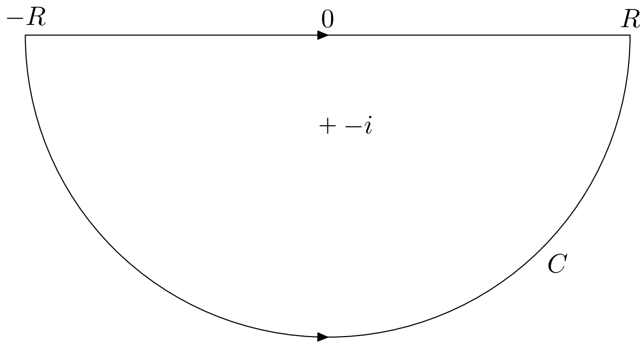
$$\left|\int_C f(z)\,dz\right| = \left|\int_0^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,d\theta\right| \leqslant \int_0^{\pi} \frac{e^{-tR\sin\theta}}{R^2-1}\,R\,d\theta \to 0 \quad \text{as } R \nearrow \infty\ .$$

Hence we get

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{itx}}{1+x^2}\,dx = 2\pi i \mathrm{Res} f(i) = \pi e^{-t}\ .$$

For $t \leqslant 0$ we instead take the semicircle in the lower half-plane so that the integrand is small on that semi-circle. This gives

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{1+x^2}\,dx = \pi e^{t} \qquad \text{for } t \leqslant 0\ .$$

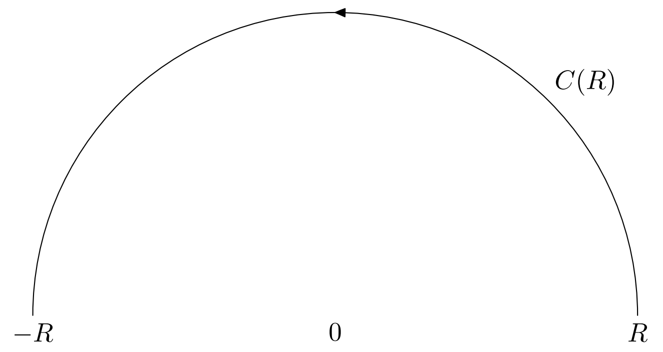


The argument in this example is very useful, especially when evaluating Fourier transforms. We can improve it to obtain:

Lemma 6.6 Jordan's Lemma
Let $f : H^+ = \{x + iy : y > 0\} \to \mathbb{C}$ be a meromorphic function with $f(z) \to 0$ as $z \to \infty$ in the half=plane H^+. Let $C(R)$ be the semi-circular path $C(R) : (0, \pi) \to H^+$; $\theta \mapsto Re^{i\theta}$. Then

$$\int_{C(R)} f(z)e^{itz}\, dz \to 0 \qquad \text{as } R \nearrow \infty$$

provided that $t > 0$.



Proof:

For any $\varepsilon > 0$, we can find R_o so that $|f(z)| < \varepsilon$ whenever $|z| > R_o$. For $z = Re^{i\theta}$ on $C(R)$, with $R > R_o$ we then have

$$|f(z)e^{itz}| = |f(z)|e^{-tR\sin\theta} \leqslant \varepsilon e^{-tR\sin\theta}\ .$$

Now

$$\sin\theta \geqslant \frac{2}{\pi}\theta \qquad \text{for} \qquad 0 < \theta < \tfrac{1}{2}\pi$$

so

$$|f(z)e^{itz}| \leqslant \begin{cases} \varepsilon e^{-2tR\theta/\pi} & \text{for } 0 < \theta \leqslant \frac{1}{2}\pi; \\ \varepsilon e^{-2tR(\pi-\theta)/\pi} & \text{for } \frac{1}{2}\pi \leqslant \theta < \pi. \end{cases}$$

Therefore,

$$\left|\int_{C(R)} f(z)e^{itz}\, dz\right| = \left|\int_0^\pi f(Re^{i\theta})e^{itRe^{i\theta}}\ iRe^{i\theta}\, d\theta\right|$$
$$\leqslant 2\int_0^{\pi/2} \varepsilon e^{-2tR\theta/\pi}\ R\, d\theta = \pi\varepsilon\frac{1-e^{-tR}}{t} \leqslant \frac{\pi\varepsilon}{t}$$

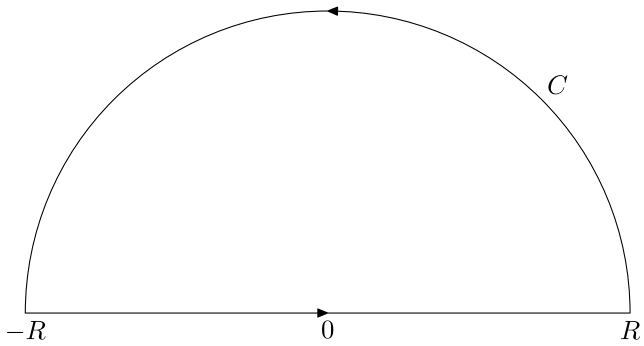
This certainly shows that $\int_{C(R)} f(z)e^{itz}\, dz \to 0$ as $R \nearrow \infty$. $\square$

Example 4 $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

The integrand is an even function of x, so the integral will converge if and only if the limit

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin x}{x}\,dx$$

exists. We will prove that it does using the residue theorem. (Note, we could also prove convergence using the alternating series test but this would not give the value of the integral.)



Let Γ be the semi-circular contour consisting of the straight line $[-R, R]$ and the semi-circle C as shown. Let f be the map

$$f : z \mapsto \frac{e^{iz} - 1}{z} .$$

This has a removable singularity at 0 and so is analytic on all of $\mathbb{C}$. Consequently

$$0 = \int_\Gamma f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_C \frac{e^{iz}}{z}\,dz - \int_C \frac{1}{z}\,dz .$$

Jordan's lemma shows that $\int_C \frac{e^{iz}}{z}\,dz \to 0$ as $R \to \infty$. The final integral is:

$$\int_C \frac{1}{z}\,dz = \int_0^\pi \frac{1}{Re^{i\theta}}\,iRe^{i\theta}\,d\theta = \pi i .$$

Thus we see that

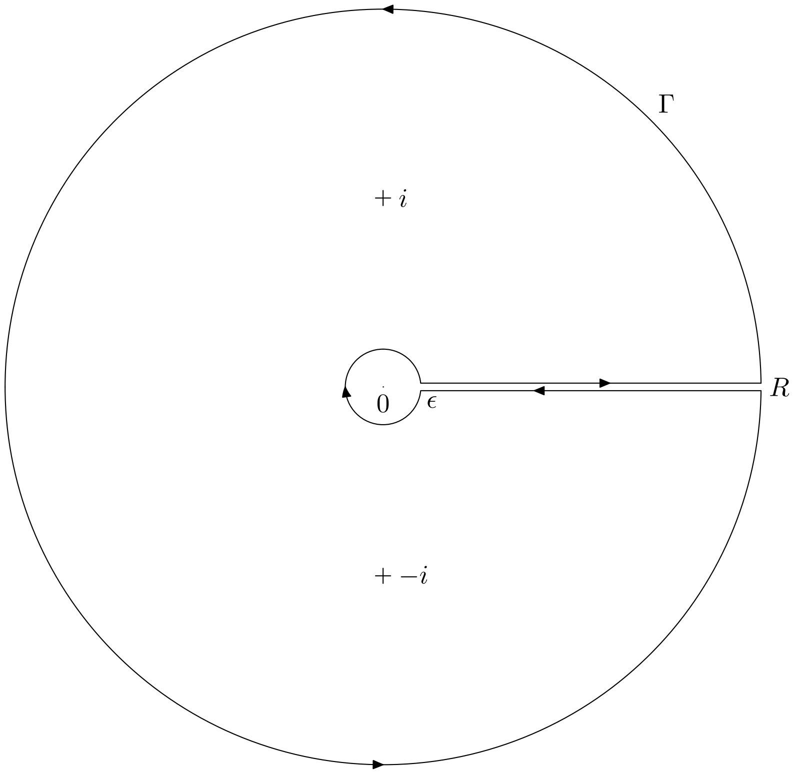
$$\int_{-R}^{R} f(x)\,dx \to \pi i \qquad \text{as } R \to \infty .$$

Taking the imaginary part of this gives

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi .$$

Example 5 $\quad I = \displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$

Let $f(z) = \dfrac{z^{1/2}}{1+z^2}$ with $z^{1/2}$ denoting the principal branch of the square root. So $(re^{i\theta})^{1/2} = r^{1/2}e^{i\theta/2}$ for $0 < \theta < 2\pi$. Let Γ be the "keyhole" contour shown.



This contour consists of an anticlockwise circle $C(R)$ of radius R, a clockwise circle $C(\varepsilon)$ of radius ε, a straight line slightly above the real axis, say $[\varepsilon + i\delta, R + i\delta]$ and a similar line just below the real axis $[R - i\delta, \varepsilon - i\delta]$. (The function f is not defined on the positive real axis itself.)

On $C(R)$ we have $|f(z)| \leqslant \dfrac{R^{1/2}}{R^2-1}$, so

$$\int_{C(R)} f(z)\,dz \to 0 \qquad \text{as } R \to \infty\ .$$

On $C(\varepsilon)$ we have $|f(z)| \leqslant \dfrac{\varepsilon^{1/2}}{1-\varepsilon^2}$, so

$$\int_{C(\varepsilon)} f(z)\,dz \to 0 \qquad \text{as } \varepsilon \to 0\ .$$

On $[\varepsilon + i\delta, R + i\delta]$ we have $f(x+i\delta) \approx \dfrac{x^{1/2}}{1+x^2}$, so

$$\int_{\varepsilon+i\delta}^{R+i\delta} f(z)\,dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$$

as $R \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$. Similarly, on $[\varepsilon - i\delta, R - i\delta]$ we have $f(x-i\delta) \approx \dfrac{-x^{1/2}}{1+x^2}$, so

$$\int_{\varepsilon-i\delta}^{R-i\delta} f(z)\,dz \to \int_0^\infty \frac{-x^{1/2}}{1+x^2}\,dx$$

as $R \to \infty$, $\varepsilon \to 0$ and $\delta \to 0$.

The function f has residues at the two points i and $-i$ with residues $\frac{1}{2}\exp -i\pi/4$ and $\frac{1}{2}\exp 5i\pi/4$ respectively. Therefore, the residue theorem gives

$$\int_\Gamma f(z)\, dz = 2\pi i(\tfrac{1}{2}\exp -i\pi/4 + \tfrac{1}{2}\exp 5i\pi/4) = \frac{\pi}{\sqrt{2}}\ .$$

Consequently,

$$2\int_0^\infty \frac{x^{1/2}}{1+x^2}\, dx = \frac{\pi}{\sqrt{2}}\ .$$

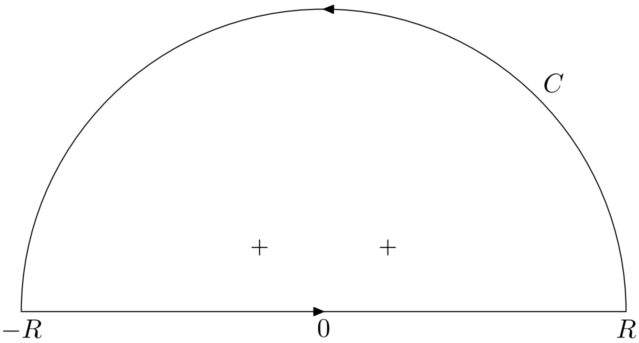
So the integral I converges to $\pi/2\sqrt{2}$.

There are various alternative ways to compute this integral by making a change of variables. These allow us to avoid the complicated keyhole contour.

For example, the change of variables $x = u^2$ changes the integral I to

$$I = \int_0^\infty \frac{2u^2}{1+u^4}\, du = \tfrac{1}{2}\int_{-\infty}^\infty \frac{2u^2}{1+u^4}\, du\ .$$

To evaluate this we take the semi-circular contour:



and the function $g(z) = \dfrac{2z^2}{1+z^4}$. This has simple poles at the fourth roots of -1 with the residue at ω being

$$\mathrm{Res}g(\omega) = \lim_{z\to\omega} (z-\omega)\frac{2z^2}{1+z^4} = \frac{1}{2\omega}\ .$$

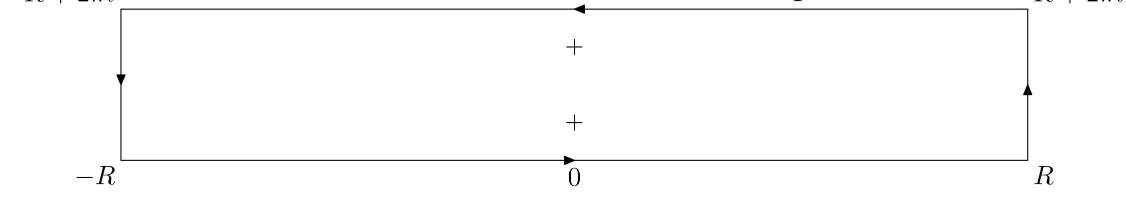
Hence,

$$I = \tfrac{1}{2}2\pi i\left(\frac{1}{2e^{i\pi/4}} + \frac{1}{2e^{i3\pi/4}}\right) = \frac{\pi}{2\sqrt{2}}\ .$$

An alternative is to make the change of variables $x = e^s$, which changes the integral I to

$$I = \int_{-\infty}^\infty \frac{e^{3s/2}}{1+e^{2s}}\, ds\ .$$

For this we use the rectangular contour



and the function $h(z) = \dfrac{e^{3z/2}}{1+e^{2z}}$. The contour winds around the two poles at $\frac{1}{2}\pi i$ and $\frac{3}{2}\pi i$.

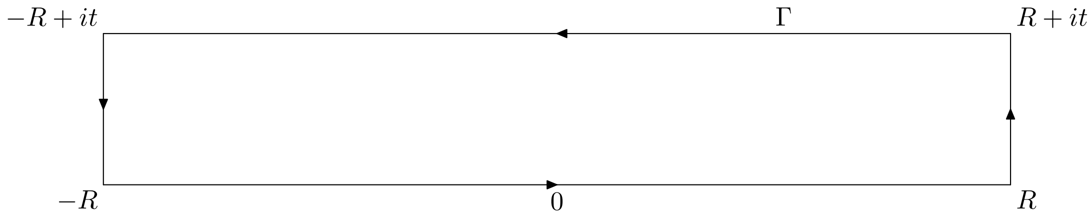
Example 6 $\displaystyle\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-itx}\, dx$

Note that we already know the normal function integral:

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}\ ,$$

which gives the value when $t = 0$.

Take the rectangular contour Γ shown below and the analytic function $f(z) = \exp -\frac{1}{2}z^2$.



There are no singularities for this function so Cauchy's theorem shows that $\int_\Gamma f(z)\, dz = 0$. If $z = \pm R + iy$ with $0 \leqslant y \leqslant |t|$, then

$$|f(z)| = \exp -\tfrac{1}{2}(R^2 - y^2) \leqslant (\exp -\tfrac{1}{2}R^2)(\exp \tfrac{1}{2}t^2)\ .$$

Hence the integral

$$\int_{[R,R+it]} f(z)\, dz = \int_R^{R+it} f(R+iy)\, dy \to 0 \quad \text{as} \quad R \to \infty$$

and

$$\int_{[-R,-R+it]} f(z)\, dz \to 0 \qquad \text{as} \quad R \to \infty$$

similarly. This implies that

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} f(x+it)\, dx\ .$$

Now

$$\exp -\tfrac{1}{2}(x+it)^2 = \exp -\tfrac{1}{2}(x^2 + 2itx - t^2) = e^{-x^2/2} e^{itx} e^{t^2/2}\ .$$

So we see that

$$\int_{-\infty}^{\infty} e^{-x^2/2}\, e^{-itx}\, dx = \sqrt{2\pi}\, e^{-t^2/2}\ .$$

7. THE ARGUMENT PRINCIPLE

Let $f : D \to \mathbb{C}$ be an analytic map and Γ a cycle in D. Then $f \circ \Gamma$ is also a cycle. If $w \in \mathbb{C} \setminus [f \circ \Gamma]$ then the winding number $n(f \circ \Gamma; w)$ is given by

$$n(f \circ \Gamma; w) = \frac{1}{2\pi i} \int_{f \circ \Gamma} \frac{1}{z - w} \, dz = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z) - w} \, dz \; .$$

The integrand $f'(z)/(f(z) - w)$ is meromorphic with poles at the points z_k where $f(z_k) = w$. Near such a point we have

$$f(z) = w + (z - z_k)^N F(z)$$

where $N = \deg(f; z_k)$ and F is analytic on a neighbourhood of z_k with $F(z_k) \neq 0$. Hence,

$$\frac{f'(z)}{f(z) - w} = \frac{N}{z - z_k} + \frac{F'(z)}{F(z)}$$

and hence there is a simple pole at z_k with residue N. Thus the residue theorem (6.5) gives

Theorem 7.1 Argument Principle
Let $f : D \to \mathbb{C}$ be a non-constant analytic function and Γ a cycle in D that is homologous to 0 in D. Suppose that f does not take the value w on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z : f(z) = w} \deg(f; z) n(\Gamma; z)$$

where the sum is taken over all points $z \in D$ with $f(z) = w$.

Proof:
The points where $f(z) = w$ are isolated in D and the set $[\Gamma] \cup \{z \in D : n(\Gamma; z) \neq 0\}$ is compact, so there are only a finite number of non-zero terms in the sum.

The residue theorem shows that

$$n(f \circ \Gamma; w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z) - w} \, dz = \sum_{z : f(z) = w} \mathrm{Res}(f'/(f - w); z) n(\Gamma; z) = \sum_{z : f(z) = w} \deg(f; z) n(\Gamma; z) \; .$$

□

It is usual to apply the argument principle to a cycle Γ that bounds a subdomain of D. Then the winding numbers are all 0 or 1 and we obtain:

Theorem 7.1' Argument Principle
Let $f : D \to \mathbb{C}$ be a non-constant analytic function and Γ a cycle in D that bounds a subdomain Ω of D. Suppose that f does not take the value w on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z \in \Omega : f(z) = w} \deg(f; z) \; .$$

The sum on the right side is the number of solutions of $f(z) = w$ in Ω, counting multiplicity. □

We can also apply this argument when f is a meromorphic function, that is a function that is analytic except for isolated poles. If f has a pole of order N at z_o then

$$f(z) = w + (z - z_o)^{-N} F(z)$$

on a neighbourhood of z_o with F analytic and $F(z_o) \neq 0$. Hence

$$\frac{f'(z)}{f(z) - w} = \frac{-N}{z - z_o} + \frac{F'(z)}{F(z)}$$

and we see that $f'(z)/(f(z) - w)$ has a simple pole at z_o with residue $-N$. This proves:

Theorem 7.2 Argument Principle for meromorphic functions
Let $f : D \to \mathbb{C}$ be a non-constant meromorphic function and Γ a cycle in D that bounds a subdomain Ω of D. Suppose that f takes neither the value w nor ∞ on $[\Gamma]$. Then

$$n(f \circ \Gamma; w) = \sum_{z \in \Omega : f(z) = w} \deg(f; z) \; - \sum_{z \in \Omega : f(z) = \infty} \deg(f; z) \; .$$

□

Exercise: Show that the polynomial $p(z) = z^4 + 2z^2 - 2z + 2$ has exactly 2 zeros with positive real part.

Consider the closed curve Γ obtained by following the imaginary axis from iR to $-iR$ and then the semi-circle $\sigma : [0, 1] \to \mathbb{C}$; $t \mapsto -iRe^{\pi it}$. For $z = iy$ we have $p(iy) = (y^4 - 2y + 2) - 2iy$. This always has strictly positive real part. Moreover, for R large, $p(\pm iR)$ has small argument. Hence, the winding number of the curve $p([-iR, iR])$ about 0 tends to 0 as $R \nearrow +\infty$. Also, the term z^4 dominates in $p(z)$ on the semi-circle σ. Indeed

$$p(\sigma(t)) = R^4 e^{4\pi it} \left(1 + \frac{-2R^2 e^{2\pi it} + 2iRe^{\pi it} + 2}{R^4 e^{4\pi it}} \right) = R^4 e^{4\pi it} \phi(t)$$

and $\phi(t) \in B(1, 1)$ for R large enough. Hence $p \circ \sigma$ winds approximately the same number of times about 0 as does $t \mapsto R^4 e^{4\pi it}$, that is 2 times. Putting these together gives $n(p \circ \Gamma; 0) = 2$ when R is sufficiently large. So the argument principle shows that there are 2 zeros of p within the half-disc bounded by Γ. Thus there are 2 zeros with positive real part.

However, we are counting these zeros with multiplicity, so we need to see that there are no multiple zeros. Suppose that z_o were a double zero of p. Then $p(z_o) = 0$ and $p'(z_o) = 0$. This implies that $4p(z_o) - z_o p'(z_o) = 4z_o^2 - 6z_o + 2 = 0$. We can solve this quadratic to find that $z_o = \frac{1}{2}$ or 1. But neither $\frac{1}{2}$ nor 1 is a zero of p, so all the zeros of p are simple. (Alternatively, observe that the zeros must be in conjugate pairs and that no zero is real.)

Rouché's theorem formalises this type of argument.

Proposition 7.3 Rouché's Theorem
Let Γ be a cycle in a domain D that bounds a subdomain Ω. If $f, g : D \to \mathbb{C}$ are analytic functions with

$$|f(z) - g(z)| < |g(z)| \qquad \text{for all} \quad z \in [\Gamma]$$

then f and g have the same number of zeros within Ω, counting multiplicity.

Proof:

The inequality shows that neither f nor g has a zero on $[\Gamma]$. We may therefore apply Proposition 3.4 to the component curves of $f \circ \Gamma$ and $g \circ \Gamma$ to obtain $n(f \circ \Gamma; 0) = n(g \circ \Gamma; 0)$. Now the argument principle (7.1') completes the proof. □

Exercise: Show that all 4 zeros of $p(z) = z^4 + 2z^2 - 2z + 2$ have modulus between 2 and $\frac{1}{2}$.

On the circle $|z| = 2$ we expect the leading term of p to dominate, so take $f(z) = p(z)$ and $g(z) = z^4$. Then

$$|f(z) - g(z)| = |2z^2 - 2z + 2| \leqslant 2|z|^2 + 2|z| + 2 < 2^4 = |z|^4 \qquad \textit{for} \quad |z| = 2$$

so Rouché's theorem shows that p and z^4 have the same number of zeros within $|z| = 2$. This is 4 zeros. Similarly, on $|z| = \frac{1}{2}$ the constant term dominates so

$$|z^4 + 2z^2 - 2z| < |2| \qquad \textit{for} \quad |z| = \tfrac{1}{2}$$

implies that p and 2 have the same number of zeros within $|z| = \frac{1}{2}$, that is 0.

Local Mapping Theorem

We can now complete our study of the local behaviour of analytic functions.

Theorem 7.4 Local Mapping Theorem
Let $f : D \to \mathbb{C}$ be a non-constant analytic function, $z_o \in D$, $w_o = f(z_o)$ and $K = \deg(f; z_o)$. Then there are $r, s > 0$ such that, for each $w \in B(w_o, s) \setminus \{w_o\}$ there are exactly K points $z \in B(z_o, r)$ with $f(z) = w$.

Proof:

We know that there is an analytic function $F : D \to \mathbb{C}$ with $f(z) = w_o + (z - z_o)^K F(z)$ and $F(z_o) \neq 0$. Hence, we can choose $r > 0$ so that the closed disc $\overline{B(z_o, r)}$ lies within D and $F(z) \neq 0$ on $\overline{B(z_o, r)}$. Let C be the circle $\partial B(z_o, r)$. Then $[f \circ C]$ is a compact subset of $\mathbb{C}$ that does not contain w_o. Choose $s > 0$ so that $B(w_o, s)$ does not meet $[f \circ C]$.

The winding number $n(f \circ C; w)$ is constant on each component of $\mathbb{C} \setminus [f \circ C]$ and hence it is constant on $B(w_o, s)$. The argument principle shows that $n(f \circ C; w)$ is the number of solutions of $f(z) = w$ in $B(z_o, r)$, counting multiplicity. For $w = w_o$, this number is K. Therefore, there are K solutions of $f(z) = w$ in $B(z_o, r)$ for each $w \in B(w_o, s)$.

The derivative of f is $f'(z) = (z - z_o)^{K-1} \left(KF(z) + (z - z_o)F'(z) \right)$, so we can choose r sufficiently small that $f'(z) \neq 0$ on $B(z_o, r) \setminus \{z_o\}$. Then $f - w$ can not have any multiple zeros in $B(z_o, r) \setminus \{z_o\}$. Hence, there are exactly K distinct solutions of $f(z) = w$ in $B(z_o, r)$ for each $w \in B(w_o, s)$ except w_o. For $w = w_o$, the only solution of $f(z) = w$ in $B(z_o, r)$ is at z_o where it has multiplicity K. $\square$

Corollary 7.5 Open Mapping Theorem
A non-constant analytic function $f : D \to \mathbb{C}$ maps open sets in D to open sets in $\mathbb{C}$.

Proof:

If U is an open subset of D and $z_o \in U$, then we wish to prove that there is a disc about $f(z_o)$ that lies within $f(U)$. The local mapping theorem shows that we can choose $r, s > 0$ so that $B(z_o, r) \subset U$ and $B(f(z_o), s) \subset f(U)$. $\square$

Note that this certainly implies the maximum modulus principle. For suppose that $f : D \to \mathbb{C}$ is a non-constant analytic function with a local maximum for $|f(z)|$ at the point z_o. Then there is a disc $\mathbb{D}(z_o, r) \subset D$ with $|f(z)| \leqslant |f(z_o)|$ for $z \in \mathbb{D}(z_o, r)$. However, the open mapping theorem shows that $f(\mathbb{D}(z_o, r))$ is open, so it contain a disc $\mathbb{D}(f(z_o), s)$. This disc can not lie inside the set $\{w : |w| \leqslant |f(z_o)|\}$.

8. FOURIER TRANSFORMS

The results in this section are not proved properly. The comments given are meant to be suggestive rather than accurate. In particular, the major results rely on reversing the order of integration (Fubini's Theorem) which is not proved at all.

We will wish to consider functions $f : \mathbb{R} \to \mathbb{C}$ with the following properties:

(a) f is continuous with $f(x) \to 0$ as $x \to \pm\infty$;

(b) the improper Riemann integral $\displaystyle\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

When f satisfies these conditions we will say that $f \in C_o \cap L_1$.

The *Fourier transform* $\widehat{f}$ of a function $f : \mathbb{R} \to \mathbb{C}$ is defined by:

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx$$

for each real number t. For this to make sense, the product $f(x)e^{-itx}$ must be integrable over the entire real line. This is true, for example, if $f \in C_o \cap L_1$.

The value of the Fourier transform at t measures the part of f that has period $2\pi/t$ — the same as e^{itx}. Hence, we might hope that the complete Fourier transform determines f. This is true and the content of the Inversion Theorem. In this section we will not provide full proofs of the results. This is partly because the proofs are quite delicate and partly because it is more natural to write the proofs in terms of the Lebesgue integral rather than the improper Riemann integral.

We begin with a few trivial properties of the Fourier transform.

Proposition 8.1
Let $f, f_1, f_2 \in C_o \cap L_1$. Then

(i) If $g = \lambda_1 f_1 + \lambda_2 f_2$, then $\widehat{g}(t) = \lambda_1 \widehat{f_1}(t) + \lambda_2 \widehat{f_2}(t)$ for each $t \in \mathbb{R}$.

(ii) If $g(x) = f(-x)$, then $\widehat{g}(t) = \widehat{f}(-t)$.

(iii) If $g(x) = f(\lambda x)$ with $\lambda > 0$, then $\widehat{g}(t) = \lambda^{-1}\widehat{f}(\lambda^{-1}t)$.

(iv) If $g(x) = \overline{f(x)}$, then $\widehat{g}(t) = \overline{\widehat{f}(-t)}$.

(v) If $g(x) = f(x+a)$ for some $a \in \mathbb{R}$, then $\widehat{g}(t) = \widehat{f}(t)e^{ita}$.

(vi) If $g(x) = f(x)e^{iax}$ for some $a \in \mathbb{R}$, then $\widehat{g}(t) = \widehat{f}(t-a)$.

(vii) Suppose that f is differentiable with $f' \in C_o \cap L_1$. Then $\widehat{f'}(t) = it\widehat{f}(t)$.

Proof:

Parts (i) to (vi) are all simple changes of variables in the integrals. For example, in (v) we have

$$\widehat{g}(t) = \int_{-\infty}^{\infty} f(x+a)e^{-itx}\, dx = \int_{-\infty}^{\infty} f(x')e^{-it(x'-a)}\, dx' = \widehat{f}(t)e^{ita}$$

by making the change of variables $x' = x + a$.

Part (vii) follows from integration by parts:

$$\widehat{f'}(t) = \int_{-\infty}^{\infty} f'(x)e^{-itx}\, dx = \lim_{R\to\infty}\left(f(x)e^{-itx}\Big|_{-R}^{R} + \int_{-R}^{R} f(x)ite^{-itx}\, dx\right) = \int_{-\infty}^{\infty} itf(x)e^{-itx}\, dx$$

□

Suppose that X is a real-valued, random variable with density $f : \mathbb{R} \to \mathbb{R}$. Then the *characteristic function of* X is $\mathbb{E}(e^{itX})$. This is

$$\mathbb{E}(e^{itX}) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \widehat{f}(-t)\ .$$

So the Fourier transform and the characteristic function are very closely related. As an example, consider the standard normal distribution $N(0,1)$. This has distribution

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}x^2\right)\ .$$

So example 6 in Chapter 6 shows that it's Fourier transform is

$$\widehat{f}(t) = \exp\left(-\tfrac{1}{2}t^2\right)\ .$$

More generally the $N(\mu, \sigma^2)$ normal distribution has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\tfrac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

and this has Fourier transform

$$\widehat{f}(t) = \exp\left(-\tfrac{1}{2}\sigma^2 t^2 - i\mu t\right)\ .$$

Suppose that X and Y are two independent random variables with densities f and g respectively. The sum $X+Y$ has density

$$h(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\, du\ .$$

The independence of X and Y ensures that

$$\mathbb{E}(e^{it(X+Y)}) = \mathbb{E}(e^{itX}e^{itY}) = \mathbb{E}(e^{itX})\mathbb{E}(e^{itY})\ .$$

So

$$\widehat{h}(t) = \widehat{f}(t)\widehat{g}(t)\ .$$

This motivates the following definition and Proposition.

The *convolution* $f * g$ of two functions $f, g : \mathbb{R} \to \mathbb{C}$ is given by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-u)g(u)\, du$$

whenever this integral converges.

Proposition 8.2 Fourier transform of convolutions
Let $f * g$ be the convolution of two functions $f, g \in C_o \cap L_1$. Then

$$\widehat{f * g}(t) = \widehat{f}(t)\widehat{g}(t)\ .$$

Proof:

Observe that

$$\widehat{f * g}(t) = \int_{-\infty}^{\infty} (f*g)(x) e^{-itx}\, dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u)g(u)\, du\, e^{-itx}\, dx\ .$$

We can switch the order of integration without changing the value of this integral. This is called *Fubini's Theorem* and requires proof, which we will not give here. Hence

$$\begin{aligned}
\widehat{f * g}(t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u)g(u)e^{-itx}\, dx\, du \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-u)e^{-it(x-u)}\, dx\, g(u)e^{-itu}\, du \\
&= \int_{-\infty}^{\infty} \widehat{f}(t)\, g(u)e^{-itu}\, du = \widehat{f}(t)\widehat{g}(t)\ .
\end{aligned}$$

$\square$

Theorem 8.3 Parseval's Theorem
For $f, g \in C_o \cap L_1$ we have

$$\int_{-\infty}^{\infty} \widehat{f}(t)g(t)\ dt = \int_{-\infty}^{\infty} f(x)\widehat{g}(x)\ dx\ .$$

Proof:

First note that

$$\int_{-\infty}^{\infty} \widehat{f}(t)g(t)\ dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-itx}g(t)\ dx\ dt$$

Fubini's theorem allows us to reverse the order of the double integral to obtain:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)e^{-itx}g(t)\ dt\ dx = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} e^{-itx}g(t)\ dt\ dx = \int_{-\infty}^{\infty} f(x)\widehat{g}(x)\ dx$$

as required. □

We are now in a position to prove the inversion theorem.

Theorem 8.4 Fourier Inversion Theorem
Let $f \in C_o \cap L_1$. Then

$$f(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(t)e^{ita}\ dt\ .$$

for each $a \in \mathbb{R}$.

Proof:

Note first that the function $g(t) = \dfrac{1}{2\pi}\exp -\frac{1}{2}\sigma^2 t^2$ has Fourier transform

$$\widehat{g}(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\tfrac{1}{2}\frac{x^2}{\sigma^2}\right)\ .$$

Therefore $\widehat{g}$ is the density for an $N(0,\sigma^2)$ random variable.

Let N be an $N(0,\sigma^2)$ random variable and consider the average value of f at $a+N$, that is

$$f_\sigma(a) = \frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} f(a+u)\exp\left(-\tfrac{1}{2}\frac{u^2}{\sigma^2}\right)\ du\ .$$

Since $f \in C_o \cap L_1$, we can show that

$$f_\sigma(a) \to f(a) \qquad \text{as } \sigma \searrow 0\ .$$

Parseval's Theorem (Theorem 8.3) shows that

$$f_\sigma(a) = \int_{-\infty}^{\infty} f(a+u)\widehat{g}(u)\ du = \int_{-\infty}^{\infty} \widehat{f}(t)e^{ita}g(t)\ dt\ .$$

Here we have used Proposition 8.1(v) to see that the Fourier transform of $h(u) = f(a+u)$ is $\widehat{h}(t) = \widehat{f}(t)e^{ita}$.

Therefore

$$f_\sigma(a) = \int_{-\infty}^{\infty} \widehat{f}(t)e^{ita}g(-t)\ dt = \int_{-\infty}^{\infty} \widehat{f}(t)e^{ita}\frac{1}{2\pi}\exp -\tfrac{1}{2}\sigma^2 t^2\ dt\ .$$

Now the function $\widehat{f}(t)e^{ita}$ is integrable on $\mathbb{R}$ and the functions $\exp -\frac{1}{2}\sigma^2 t^2$ increase locally uniformly to 1 as $\sigma \searrow 0$, so we see that

$$f_\sigma(a) \to \int_{-\infty}^{\infty} \widehat{f}(t)e^{ita}\frac{1}{2\pi}\ dt$$

as $\sigma \searrow 0$. □

The function

$$\check{h}(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(t)e^{ita} \, dt$$

is called the *inverse Fourier transform of* h. It is very similar to the Fourier transform, with

$$\check{h}(x) = \frac{1}{2\pi} \widehat{h}(-x) \; .$$

The Inversion Theorem shows that the inverse Fourier transform of a Fourier transform is the original function.

Corollary 8.5 Plancherel's Formula
For $f, \hat{f} \in C_o \cap L_1$ *we have*

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(t)|^2 \, dt \; .$$

Proof:

The Fourier inversion theorem shows that

$$\widehat{\widehat{f}}(x) = 2\pi f(-x) \; .$$

Hence, Plancherel's theorem gives

$$\int_{-\infty}^{\infty} \overline{f(x)} f(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{f(x)} \widehat{\widehat{f}}(x) \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{\widehat{f}(-t)} \widehat{f}(-t) \, dt$$

which proves the desired result. $\square$