COMPLEX ANALYSIS — Example Sheet 3

TKC Lent 2006

-1. Show that, for two analytic functions f, g we have $\deg(g \circ f; z_o) = \deg(f; z_o).\deg(g; f(z_o))$.

2. Let $p(z) = z^5 + z$. Find all z such that $|z| = 1$ and $\operatorname{Im} p(z) = 0$. Calculate $\operatorname{Re} p(z)$ for such z. Hence sketch the contour $p \circ \gamma$, where $\gamma(t) = e^{2\pi it}$ and use your sketch to determine the number of z (counted with multiplicity) such that $|z| < 1$ and $p(z) = x$ for each real number x.

3. Calculate $\int_C \sqrt{z}\, dz$ where $\sqrt{z}$ is the principal branch of the square root and C is the unit circle. Do this first by choosing an explicit parametrization of the unit circle and then by using a keyhole contour to relate the integral to an integral along the unit interval.

4. (i) Deduce the fundamental theorem of algebra from Rouché's theorem, including a statement about the number of roots.

   (ii) Find the number of roots of $p(z) = z^7 - 2z^6 + 6z^3 - z + 1$ in the unit disc.

   (iii) Find the number of roots of $q(z) = z^4 - 6z + 3$ in $\{z : 1 < |z| < 2\}$.

5. Let $f : D \to \mathbb{C}$ be a non-constant analytic function on a domain D with $f(0) = 0$.

   (a) Show that we can find $r, s > 0$ with $|f(z)| \geqslant s$ for each z with $|z| = r$.

   (b) Suppose that $f(z) \neq w$ for any $z \in B(0, r)$. Use the maximum modulus principle to show that $|w| > \frac{1}{2}s$.

   (c) Deduce the *open mapping theorem*: a non-constant analytic function maps one domain onto another.

6.

(i) Let $p(z) = a_k z^k + a_{k-1} z^{k-1} + \ldots + a_1 z + a_0$ be a polynomial in z of degree $k > 0$. Show that there are positive real numbers M, N and R with $N|z|^k \leqslant |p(z)| \leqslant M|z|^k$ whenever $|z| \geqslant R$.

(ii) Let $g(z) = p(z)/q(z)$ be a rational function, where p and q are polynomials satisfying $\deg q \geqslant 2 + \deg p$. Show that the sum of the residues of g at its singularities is 0.

(iii) Hence or otherwise evaluate

$$\int_C \frac{(z-1)^2(z+2i)^2}{z^4(z-2)(z+2)}\, dz$$

for the unit circle contour C.

7. Let $\phi : \mathbb{R} \to \mathbb{C}$ be a smooth function with $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$ and $\int_{-\infty}^{\infty} \phi(x)\, dx = 1$. Define $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon$ for each $\varepsilon > 0$. Show that

$$\phi_\varepsilon * f(x) - f(x) = \int_{-\infty}^{\infty} \left(f(x - \varepsilon w) - f(x)\right) \phi(w)\, dw\ .$$

Now deduce that $\phi_\varepsilon * f(x) \to f(x)$ uniformly as $\varepsilon \to 0$ for each bounded, uniformly continuous function $f : \mathbb{R} \to \mathbb{C}$. [*Hint: Split the w integral into an integral over the ball $B(0, R)$ and its complement for a suitable large R.*]

8. Evaluate the following integrals.

(a) $$\int_0^{\infty} \sin x^2\, dx\ .$$

(b) $$\int_0^{\infty} \frac{x^{a-1}}{1+x}\, dx \qquad \text{for } 0 < a < 1\ .$$

(c) $$\int_{-\infty}^{\infty} e^{-ax^2} e^{-itx}\, dx \qquad \text{for } a > 0 \text{ and } t \in \mathbb{R}\ .$$

(d) $$\int_0^{\infty} \frac{\ln(x^2+1)}{x^2+1}\, dx\ .$$

(e) $$\int_0^{\infty} \frac{\cosh ax}{\cosh x}\, dx \qquad \text{for } -1 < a < 1\ .$$ [*Rectangular contour*]

(f) $$\int_{-\infty}^{\infty} \frac{\sin x}{x} e^{-itx}\, dx\ .$$

9. Show that the set of zeros of a non-constant analytic function consists of isolated points. Consider a meromorphic function $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ which satisfies $|f(z)| \to \infty$ as $z \to \infty$. Show that f can not have poles at each of the integer points.

10. Evaluate
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x \sin \pi x}{x^2 + 2x + 5}\, dx\ .$$

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11. Let
$$f_n(z) = \sum_{m=-n}^{n} \frac{1}{(z-m)^2}\ .$$
Show that the sequence $(f_n(z))$ converges to a limit $f(z)$ for each $z \in \mathbb{C} \setminus \mathbb{Z}$. Show also that $f(z+1) = f(z)$.

Prove that the convergence is uniform on any compact set that does not meet $\mathbb{Z}$.

Let $g(z) = \pi^2 \mathrm{cosec}^2 \pi z$. Show that $f(z) - g(z)$ has removable singularities at each integer and that both $f(x+iy)$ and $g(x+iy)$ tend to 0 as $y \to \pm\infty$. Deduce that f and g are identical.

By considering the constant term of the Laurent expansions for f and g about 0, deduce that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}\ .$$
[*Hint:* $|\sin(x+iy)|^2 = \sin^2 x + \sinh^2 y$.]

12. A function $u : D \to \mathbb{C}$ is *harmonic* if it is twice continuously differentiable on a domain D and
$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\ .$$
Show that the real part of any analytic function is harmonic. Conversely, suppose that u is harmonic. Show that $\phi = \dfrac{\partial u}{\partial x} - i\dfrac{\partial u}{\partial y}$ is analytic. By considering the antiderivative of ϕ, or otherwise, show that u must be the real part of an analytic function when D is simply connected but may not be otherwise.

13. Let $f : D \to \mathbb{C}$ be an analytic function on a domain D that contains the closed unit disc $\overline{\mathbb{D}}$. Let C be the unit circle. For $w \in \mathbb{D}$ with inverse point $w^* = 1/\overline{w}$ explain why
$$f(w) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w}\, dz = \int_0^{2\pi} f(e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta} - w}\, \frac{d\theta}{2\pi} \qquad \text{and}$$
$$0 = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-w^*}\, dz = \int_0^{2\pi} f(e^{i\theta}) \frac{-\overline{w}}{e^{-i\theta} - \overline{w}}\, \frac{d\theta}{2\pi}\ .$$
Deduce *Poisson's formula*: $f(w) = \displaystyle\int_0^{2\pi} f(e^{i\theta}) \frac{1-|w|^2}{|e^{i\theta} - w|^2}\, \frac{d\theta}{2\pi}$.

Show that the Poisson kernel $P_w(e^{i\theta}) = \dfrac{1-|w|^2}{|e^{i\theta}-w|^2}$ is a probability distribution on C for each w in the unit disc. (That is, $P_w(e^{i\theta}) \geqslant 0$ and $\displaystyle\int_0^{2\pi} P_w(e^{i\theta})\, \frac{d\theta}{2\pi} = 1$.)

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