COMPLEX ANALYSIS — Example Sheet 1

TKC Lent 2006

The first set of questions are intended to be short and straightforward, the second set are longer, and the final set are additional exercises for those who have completed the earlier ones.

-1. Suppose that Ω is a domain in $\mathbb{C}$ and $\lambda : \Omega \to \mathbb{C}$ is a branch of the logarithm on Ω. Write down all of the other branches of the logarithm on Ω and show that there are no others.

2. Show that complex integration along a piecewise continuously differentiable curve γ is a linear operation over $\mathbb{C}$ on the space of all continuous function f on $\mathbb{C}$. Show that:
   (a) $\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$.
   (b) $\int_{-\gamma} f(z)\,|dz| = \int_\gamma f(z)\,|dz|$.
   (c) $\left|\int_\gamma f(z)\,dz\right| \leqslant \int_\gamma |f(z)|\,|dz|$.

3. Let $f : D \to \mathbb{C}$ be an analytic function on a domain $D \subset \mathbb{C}$. For any (piecewise continuously differentiable) path $\gamma : [0,1] \to D$ prove the *mean value theorem*:
$$|f(\gamma(1)) - f(\gamma(0))| \leqslant \sup\{|f'(\gamma(t))| : t \in [0,1]\}L(\gamma)$$
where $L(\gamma)$ is the length of γ.
Is it true that there must be a $t \in (0,1)$ with $f(\gamma(1)) - f(\gamma(0)) = f'(\gamma(t))\gamma'(t)$?

4. Give an example of a function $f : \mathbb{C} \to \mathbb{C}$ that is infinitely differentiable, when considered as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, but its integral around the unit circle is 1.

-5. Let D be the complement in $\mathbb{C}$ of the set $\{r(\cos\theta + i\sin\theta) : \ln r = c\theta\} \cup \{0\}$ for some constant $c > 0$. Show that D is a domain and that there is a branch of the logarithm defined on D.

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6. Show that any **real** linear map $T : \mathbb{C} = \mathbb{R}^2 \to \mathbb{C} = \mathbb{R}^2$ can be written as $T : z \mapsto Az + B\overline{z}$ for two complex numbers A and B. Then T is complex linear if and only if $B = 0$.
Suppose that $f : D \to \mathbb{C}$ is a **real** differentiable function at the point $z_0 \in D$. Show that we can write the derivative $f'(z_o)$ as $f'(z_o) : z \to Az + B\overline{z}$. We will write $\dfrac{\partial f}{\partial z}(z_o)$ for A and $\dfrac{\partial f}{\partial \overline{z}}(z_o)$ for B. (In spite of the notation, these are NOT partial derivatives.) Find a formula for $\dfrac{\partial f}{\partial z}(z_o)$ and $\dfrac{\partial f}{\partial \overline{z}}(z_o)$ in terms of the partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at z_o. Show that f is analytic if and only if $\dfrac{\partial f}{\partial \overline{z}} = 0$ at each point of D.

7. Let $s(z) = \sum a_n z^n$ be the sum of a power series with radius of convergence $R > 0$. For any continuously differentiable curve $\gamma : [0,1] \to \{z : |z| < R\}$ with $\gamma(0) = 0$ prove the following:
   (a) There is an $r < R$ with $|\gamma(t)| < r$ for each $t \in [0,1]$.
   (b)
$$\left|\int_\gamma s(z)\,dz\right| \leqslant \sup\{|s(z)| : |z| \leqslant r\}L(\gamma)\ .$$
   (c)
$$\int_\gamma s(z)\,dz = \sum_{n=0}^{\infty} \frac{a_n}{n+1}\gamma(1)^{n+1}\ .$$
Show also that the power series $\sum a_n w^{n+1}/(n+1)$ has radius of convergence R.

8. Show that the power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1 and so defines an analytic map $\lambda : \mathbb{D}(0,1) \to \mathbb{C}$. Prove that this satisfies the differential equation $\lambda'(z) = \frac{1}{1-z}$ and deduce that $\lambda(z) = -\log(1-z)$ for $|z| < 1$.

9. Define the complex trigonometric functions by
$$\cos z = \frac{\exp iz + \exp -iz}{2}\ ;\ \ \sin z = \frac{\exp iz - \exp -iz}{2i}\ .$$

Show that these are analytic functions on all of $\mathbb{C}$ and find their zeros. Prove that $\cos^2 z + \sin^2 z = 1$. Find all of the points $z \in \mathbb{C}$ for which $\cos z = 2$.

Let D be the domain $\{x + iy : 0 < x < \pi \text{ and } y > 0\}$. Show that cos is injective on D and describe the image of the boundary of D under cos. Prove that the restriction of cos maps the domain D conformally onto the half-plane $\{u + iv : v < 0\}$. Find the inverse of this conformal map in terms of branches of the logarithm and powers.

10. Show that the maps $z \mapsto \sqrt{\dfrac{i(1-z)}{1+z}}$ and $z \mapsto \dfrac{z-a}{1-\bar{a}z}$ for $a \in \mathbb{D}$ are both conformal maps on the unit disc. Here the square root denotes a branch of the square root, which you should specify. Find the image of the unit disc under each map.

11. Find conformal maps between the following pairs of domains, or show that they do not exist.
    (a) $\mathbb{D}$ and $\{x + iy : x > 0, y > 0\}$.
    (b) $\{x + iy : 0 < y < 1\}$ and $\{x + iy : x > 0, y > 0\}$.
    (c) $\{z \in \mathbb{C} : \Re z > 0\}$ and $\{z \in \mathbb{D} : \Re z > 0\}$.

12. Show, from first principles, that the curve $\phi : [0, 6\pi] \to \mathbb{C};\ t \mapsto 4e^{it}\cos\frac{2}{3}t$ has $n(\phi, 3) = 1$.

13. Let $f : D \to \mathbb{C}$ be an analytic map on a domain $D \subset \mathbb{C}$. Show that the following conditions are equivalent:
    (a) There is an antiderivative $F : D \to \mathbb{C}$, that is a function F with $F'(z) = f(z)$ for each $z \in D$.
    (b) For each closed curve γ in D we have $\displaystyle\int_\gamma f(z)\,dz = 0$. Give an example where both these conditions fail.

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14. Let $f : D \to \mathbb{C}$ be a function on a domain D that has continuous partial derivatives of any order but may not be complex differentiable. Suppose that the rectangle $R = \{x+iy : a_1 \leqslant x \leqslant a_2\ ,\ b_1 \leqslant y \leqslant b_2\}$ lies within D and that ∂R is the boundary curve of R, positively oriented.
    (a) Prove that
    $$\int_{a_1}^{a_2} \frac{\partial f}{\partial x}(x+iy)\,dx = f(a_2+iy) - f(a_1+iy)$$
    for $b_1 \leqslant y \leqslant b_2$.
    (b) Deduce that
    $$\int_R \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\,dx\,dy = -i\int_{\partial R} f(z)\,dz$$
    and hence that
    $$\int_R \frac{\partial f}{\partial \bar{z}}\,dx\,dy = \frac{-i}{2}\int_{\partial R} f(z)\,dz\ .$$
    (c) Show that $\int_{\partial R} f(z)\,dz = 0$ for all rectangles R within D if and only if f is complex differentiable at each point of D.

15. Let $\phi : [0,1] \to \mathbb{C}$ be a polygonal closed curve and z_o a point not on $[\phi]$. Suppose that the restriction of ϕ to $[0,1)$ is injective. Prove that the winding number of ϕ about z_o is -1 or 0 or $+1$. [*The assumption that ϕ be polygonal is not necessary but makes the question a little easier.*]

16. Give an example of a **real** differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves the angle between curves through a fixed point $\boldsymbol{x}$ but has derivative 0 at the point $\boldsymbol{x}$

17. Let $\gamma : [a,b] \to D$ be any continuous map into a domain $D \subset \mathbb{C}$. Show that there is an $r > 0$ with $B(\gamma(t), r) \subset D$ for each $t \in [a,b]$.
    Show how to construct a piecewise continuously differentiable curve $\beta; [a,b] \to D$ with $\beta(t) \in B(\gamma(t), r)$ for each $t \in [a,b]$ and $\beta(a) = \gamma(a)$, $\beta(b) = \gamma(b)$.
    Prove that if $\alpha, \beta : [a,b] \to D$ are any two such piecewise continuously differentiable curves, and $f : D \to \mathbb{C}$ is an analytic function, then $\int_\alpha f(z)\,dz = \int_\beta f(z)\,dz$.
    (*We could define* $\int_\gamma f(z)\,dz$ *to be the common value.*)

Please send any comment or corrections to t.k.carne@dpmms.cam.ac.uk .

Supervisors can obtain an annotated version of this example sheet from DPMMS.