If \((z_n)\) is an infinite sequence of points in \(\mathbb{C}\) which converges to \(\infty\) then the product

\[
\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)
\]

need not converge. However, if \(\sum |z_n|^{-1}\) converges, then the product will converge to an entire function with zeros precisely at the points \(z_n\). To deal with sequences \((z_n)\) which have \(\sum |z_n|^{-1}\) divergent we need to introduce exponential factors into the product.

**Theorem**  
Weierstrass products

Let \((z_n)\) be a sequence of points in \(\mathbb{C}\) which is either finite or else tends to \(\infty\). Then there is an entire function \(f\) which has a zero at each point \(\zeta\) in the sequence with order equal to the number of times that it occurs in the sequence, and no other zeros. If \(g\) is another such function then \(f(z) = g(z) \exp h(z)\) for some entire function \(h\).

**Proof:**

Choose positive numbers \(M_n\) for which \(\sum M_n\) converges. The function \(z \mapsto \log \left(1 - \frac{z}{z_n}\right)\) is analytic on \(\{z : |z| < |z_n|\}\) so its Taylor series

\[-\frac{z}{z_n} - \frac{1}{2} \left(\frac{z}{z_n}\right)^2 + \frac{1}{3} \left(\frac{z}{z_n}\right)^3 - \cdots\]

converges uniformly on \(\{z : |z| \leq \frac{1}{2} |z_n|\}\). Hence we can choose natural numbers \(N(n)\) so that

\[q_n(z) = \frac{z}{z_n} + \frac{1}{2} \left(\frac{z}{z_n}\right)^2 + \frac{1}{3} \left(\frac{z}{z_n}\right)^3 + \cdots + \frac{1}{N(n)} \left(\frac{z}{z_n}\right)^{N(n)}\]

satisfies

\[
\left|\log \left(1 - \frac{z}{z_n}\right) + q_n(z)\right| \leq M_n \quad \text{for} \quad |z| \leq \frac{1}{2} |z_n|.
\]

Therefore, the series

\[
\sum_{n=1}^{\infty} \left(\log \left(1 - \frac{z}{z_n}\right) + q_n(z)\right)
\]

will converge locally uniformly. Hence,

\[
f(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) \exp q_n(z)
\]

converges and gives an entire function \(f\) with the desired properties.

If \(g\) were another such function then \(g/f\) would be an entire function with no zeros and therefore equal to \(\exp h\) for some entire function \(h\).

**Corollary**

Every meromorphic function \(f : \mathbb{C} \to \mathbb{C}_\infty\) is the quotient \(a/b\) of two entire functions \(a\) and \(b\).
Proof:

The theorem enables us to construct an entire function $b$ whose zeros are poles of $f$. Then $a = b \cdot f$ is also entire. \hfill $\square$

As an example, let us try to construct a entire function with zeros at the integer points. The series $\sum n^{-2}$ converges so the proof of Weierstrass theorem shows that

$$f(z) = z \prod_{n \neq 0} \left(1 - \frac{z^2}{n^2}\right)^{e^{z/n}}$$

converges to the desired entire function. We can rewrite this series as

$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Because of the locally uniform convergence we can differentiate the product to obtain

$$f'(z) = f(z) \left\{ \frac{1}{z} + \sum_{n \neq 0} \left( \frac{1}{z - n} + \frac{1}{n} \right) \right\}$$

$$= f(z) \left\{ \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{2z}{z^2 - n^2} \right) \right\}$$

Hence $f'(z) = f(z) \frac{\pi \cot \pi z}{z}$. We also have $f'(0) = 1$ so we can solve this differential equation to obtain

$$z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = f(z) = \frac{\sin \pi z}{\pi}.$$

Exercises

Show that the product

$$g(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)^{e^{z/n}}$$

converges and satisfies

$$g'(z) = g(z) \sum_{n=1}^{\infty} \left( \frac{1}{z - n} + \frac{1}{n} \right).$$

Deduce that $g(z + 1) = -zg(z)e^\gamma$ for some constant $\gamma$ and prove that

$$\gamma = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n} - \log N.$$

(This is Euler's constant.)