The Painlevé Transcendents

Painlevé considered second order diferential equations of the form

$$f''(z) = A(z, f(z), f'(z)) \qquad f(z_o) = w_o,\ f'(z_o) = w'_o$$

where A is a rational function of three variables. If this differential equation has only fixed singular points, then Painlevé determined all of the possible equations. There are 50 different types, up to conjugation by Möbius transformations of these six are the *Painlevé transcendents*:

(1 : *Painlevé*) $\quad f''(z) = 6f(z)^2 + z$

(2 : *Painlevé*) $\quad f''(z) = 2f(z)^3 + zf(z) + a$

(3 : *Painlevé*) $\quad f''(z) = \frac{f'(z)^2}{f(z)} - \frac{f'(z)}{f(z)} + \frac{af(z)^2+b}{z} + cf(z)^3 + \frac{d}{f(z)}$

(4 : *Gambier*) $\quad f''(z) = \frac{f'(z)^2}{2f(z)} + \frac{3f(z)^3}{2} + 4zf(z)^2 + 2(z^2-a)f(z) + \frac{b}{f(z)}$

(5 : *Gambier*) $\quad f''(z) = \left(\frac{1}{2f(z)} + \frac{1}{f(z)-1}\right) f'(z)^2 - \frac{f'(z)}{z} + \frac{(f(z)-1)^2}{z^2}\left(af(z) + \frac{b}{f(z)}\right) + c\frac{f(z)}{z} + d\frac{f(z)(f(z)+1)}{f(z)-1}$

(6 : *R.Fuchs*) $\quad f''(z) = \frac{1}{2}\left(\frac{1}{f(z)} + \frac{1}{f(z)-1} + \frac{1}{f(z)-z}\right) f'(z)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{f(z)-z}\right) f'(z) +$

$$+\frac{f(z)(f(z)-1)(f(z)-z)}{z^2(z-1)^2}\left(a + b\frac{z}{f(z)^2} + c\frac{(z-1)}{(f(z)-1)^2} + d\frac{z(z-1)}{(f(z)-z)^2}\right)$$

Here a, b, c, d are arbitrary constants.

The others 44 equations can all be solved in terms of elementary equations or in terms of the transcendents.

The other 5 equations can all be derived from (5) by allowing the fixed singular points to coalesce or taking a limit.

Exercise: Find the fixed singular points for each of these 6 equations.