COMPLEX DIFFERENTIAL EQUATIONS – Example Sheet 3 (For supervisors.)

TKC Lent 2008

1. Show that each of the following equations has a fixed singularity, where, along a suitable path approaching the singularity, the solutions have no limits.

\[ f'(z) = z^{-2} f(z) \]

\[ f'(z) = i(1 - z)^{-1} f(z) \]

\[ f'(z) = f(z) . \]

Solve by separation of variables.

\[ f(z) = A e^{1/z} \] singularity at 0 and \( f(iy) \) has no limit as \( y \to 0 \).

\[ f(z) = A(1 - z)^{-1} \] singularity at 1 and \( f(1 - e^{-t}) \) has no limit as \( t \to +\infty \).

\[ f(z) = A e^{z} \] singularity at \( \infty \) and \( f(iy) \) has no limit as \( y \to 0 \).

Note, in each case, that \( f \) is analytic at that point.

2. Give an example of a singular point of a differential equation where there is at least one solution that is analytic at that point.

See question 1. Alternatively, construct a second order linear differential equation with solutions, say, \( z \) and \( z^{1/2} \). \((2z^{2}f''(z) - zf'(z) + f(z) = 0)\)

3. Find all of the fixed singularities of

\[
(z + f(z))f'(z) - z + f(z) = 0
\]

and determine the character of the solutions near these points. Show that there are movable branch points of order 1.

Write \( w = f(z) \), so

\[
\frac{dw}{dz} = \frac{z - w}{z + w} = \frac{P(z,w)}{Q(z,w)} .
\]

Possible fixed singularities are at points \( z_{0} \) where:

(a) \( Q(z_{0},\cdot) \equiv 0 \).

(b) There exists \( w_{0} \) with \( P(z_{0},w_{0}) = Q(z_{0},w_{0}) = 0 \).

(c) Write \( \omega = 1/w \), so

\[
\frac{d\omega}{dz} = -\omega^{2} \frac{P(z,1/\omega)}{Q(z,1/\omega)} = \frac{P_{1}(z,\omega)}{Q_{1}(z,\omega)}
\]

for polynomials \( P_{1}, Q_{1} \). There exists \( \omega_{0} \) with \( P_{1}(z_{0},\omega_{0}) = Q_{1}(z_{0},\omega_{0}) = 0 \).

For this example, \( P(z,w) = z - w \), \( Q(z,w) = z + w \). So there are no fixed singularities of type (a). For (b) we have \( z_{0} = 0 \) or \( \infty \). For (c), \( P_{1}(z,\omega) = \omega^{2}(1 - z\omega) \), \( Q_{1}(z,\omega) = 1 + z\omega \). So there are no fixed singularities of type (c). In fact even 0 and \( \infty \) are not singularities.

The equation is homogeneous so we solve it by setting \( w = zv \). Then

\[
z \frac{dv}{dz} + v = \frac{1 - v}{1 + v}.
\]

Separating the variables gives \((v + 1)^{2} - 2 = Az^{-2}, \) so

\[
f(z) = z \left( (2 + Az^{-2})^{1/2} - 1 \right) .
\]

This is meromorphic at both 0 and \( \infty \).

There are movable branch points where \( Q(z_{0},w_{0}) = 0 \). Now \( Q(z_{0},w) = z_{0} + w \) has a simple zero at \(-z_{0}\) so the branch points are of order 1. These are the points where the square root \((2 + Az^{-2})^{1/2}\) is singular \((A = -2z_{0}^{2}).\)
4. Find the fixed singular points of

\[ f'(z) = P(z, f(z)) \]

where \( P \) is a polynomial in 2 variables.

There are no fixed singular points of type (a) or (b). Write \( P(z, w) = \sum_{n=0}^{N} P_n(z)w^n \) with \( P_N \neq 0 \).

Then

\[ \frac{dw}{dz} = -\sum_{n=0}^{N} P_n(z)w^{N-n} \omega^{N-2} \]

So there are fixed singular points at \( z_o \) where \( P_1(z_o, \cdot) \) and \( Q_1(z_o, \cdot) \) have a common zero. This is where \( P_N(z_o) = 0 \).

5. Find the singularities of

\[ f'(z) = z^{1/2} + z^{3/2}f(z) - f(z)^2. \]

The coefficients are algebraic so we need to add to the possible fixed singularities listed in the answer to question 3 the singularities of the coefficients. These are 0 and \( \infty \). By question 4, there are no other fixed singularities.

[We can convert this to a differential equation with holomorphic coefficients by setting \( x = z^{1/2} \).

Then

\[ \frac{dw}{dz} = 2x^2 + 2x^4w - 2xw^2 \]

is a Riccati equation and has local power series solutions. Thus the solutions of the original equation are power series in \( z^{1/2} \).]

6. Show that

\[ f'(z) = z^3 + f(z)^3 \; ; \; f(0) = w_0 \]

has movable branch points and find their order. If \( w_o > 0 \), the branch point \( b(w_o) \) nearest to the origin lies on the positive real axis. How does \( b(w_o) \) change as \( w_o \) increases? Where are the fixed singular points of the differential equation, if any?

By question 4 the differential equation

\[ \frac{dw}{dz} = z^3 + w^3 \]

has no fixed singularities. At points \( z_o \) where \( w(z_o) \) is finite, the solution is locally holomorphic.

Now consider those points \( z_o \) where \( w(z_o) = \infty \).

Write \( \omega = 1/w \) to get

\[ \frac{d\omega}{dz} = \frac{1 + z^3\omega^3}{-\omega} \]

Note the pole where \( \omega = 0 \). To solve this near \( z = z_o, w = \infty, \omega = 0 \), write it as

\[ \frac{dz}{d\omega} = \frac{-\omega}{1 + z^3\omega^3} \]

This has a power series solution:

\[ z = z_o - \frac{1}{2} \omega^2 \left(1 + a_1\omega + a_2\omega^2 + \ldots \right) = z_o - \frac{1}{2} (\omega h(\omega))^2. \]

So \( \omega h(\omega) = -2(z - z_o)^{1/2} \) and hence \( \omega \) is a power series in \( (z - z_o)^{1/2} \), say \( \omega = b_1(z - z_o) + b_2(z - z_o)^2 + \ldots \). Now it is clear that \( w = 1/\omega \) has a branch point at \( z_o \) of order 1

Consider only non-negative real values for \( z \) and \( w \). The graph of \( w \) against \( z \) is strictly increasing on \( [0, b(w_o)] \) and tends to \( +\infty \) at \( b(w_o) \). Now the solutions for different values of \( w_o \) can not intersect, so, as \( w_o \) increases, so the point \( b(w_o) \) where \( w \) becomes infinite must decrease.