1. Let \((K_n)\) be a compact exhaustion of a domain \(D \subset \mathbb{C}\). Show that a sequence of continuous functions \(f_n : D \to \mathbb{C}\) converge locally uniformly on \(D\) if and only if they converge for the metric
\[
d(f, g) = \sum_{n=1}^{\infty} 2^{-n} \min(1, \sup \{|f(z) - g(z)| : z \in K_n\}) .
\]

2. Let \(f : H^+ = \{x + iy : y > 0\} \to \mathbb{C}\) be a bounded analytic function on the upper half plane with \(f(iy) \to \ell\) as \(y \searrow 0\). Prove that \(f(z)\) converges uniformly to \(\ell\) in any cone of the form:
\[
\{x + iy \in H^+ : |x| \leq ky\}
\]

[Hint: Consider \(f_n(z) = f(z/n)\).]

3. Let \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) be a power series with radius of convergence \(R > 0\). Show that the partial sums converge locally uniformly to \(f\) on \(\{z \in \mathbb{C} : |z| < R\}\) but need not converge uniformly.

Give an example of a function \(f\) for which the partial sums do converge uniformly on the disc of convergence.

4. A power series \(f(z) = \sum a_n z^n\) has radius convergence \(R\) with \(0 < R < \infty\). Show that there is at least one singular point \(w\) with \(|w| = R\): that is a point \(w\) for which \(f\) can not be continued analytically to any neighbourhood of \(w\).

If \(a_n \geq 0\) for each \(n \in \mathbb{N}\), prove that \(R\) is a singular point. (Pringsheim’s theorem.)

Show that the (lacunary) power series
\[
\sum z^{2^n}
\]

has radius of convergence 1 and every point on the unit circle is a singular point.

5. Solve the differential equation:
\[
f'(z) = \frac{f(z) - z}{z^2}; \quad f(0) = 0 .
\]

[Write the answer as an integral.]

Explain why this can not be solved as a power series about 0.

6. Let \(T_n, T : M \to M\) be contraction mappings on a complete metric space \(M\), with fixed points \(w_n, w\) respectively. If \(T_n \to T\) uniformly, is it necessarily true that \(w_n \to w\)?

7. Let \(f : [0, 1] \to [0, \infty)\) be a continuous function with \(f(0) = 0\) and \(\lim_{t \searrow 0} \frac{f(t)}{t} = 0\). Show that, if \(f\) satisfies
\[
f(t) \leq \int_0^t \frac{f(u)}{u} \, du \quad \text{for all } t \in [0, 1]
\]
then \(f\) is identically 0.

8. Let \(f, g : [0, 1] \to [0, \infty)\) be continuous functions that satisfy
\[
f(t) \leq g(t) + K \int_0^t (t - u) f(u) \, du \quad \text{for all } t \in [0, 1] .
\]

Show that
\[
f(t) \leq g(t) + K^{1/2} \int_0^t \sinh \left( K^{1/2} (t - u) \right) g(u) \, du .
\]

9. Are there any non-trivial functions \(f : [0, 1] \to [0, \infty)\) that satisfy
\[
f'(t) \leq -1 - f(t)^2 \quad \text{for all } t \in [0, 1] ?
\]
10. Solve \( f'(z) = f(z); \ f(0) = 1 \) explicitly by finding successive approximations starting from the constant function 1.

Solve \( f'(z) = 1 + f(z)^2; \ f(0) = 0 \) explicitly by finding successive approximations starting from the identity function \( z \mapsto z \).

11. Find all of the solutions of \( f'(z) = 2f(z)^{1/2} \) when we take a branch of the square root. (Note that there is one exceptional solution with \( f(0) = 0 \).)

12. Let \( f_1, f_2 : D \to \mathbb{C} \) be two analytic functions on a domain \( D \subset \mathbb{C} \) that are linearly independent over \( \mathbb{C} \). Show that there is a (non-trivial) second order, linear differential equation

\[ f''(z) + a_1(z)f'(z) + a_0(z)f(z) = 0 \]

which has \( f_1 \) and \( f_2 \) as solutions. Where are the singular points of this differential equation?

13. Eisenstein series. Show that, for \( k \geq 2 \), the series

\[ \varepsilon_k(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z - n)^k} \]

converges locally uniformly on \( \mathbb{C} \) to give a meromorphic function. Prove the following properties of these functions.

(a) Each \( \varepsilon_k \) is periodic with period 1.
(b) Each \( \varepsilon_k \) has a pole of order \( k \) at each integer and nowhere else.
(c) \( \varepsilon_k(x + iy) \to 0 \) as \( y \to \pm\infty \) uniformly for \( x \in \mathbb{R} \).
(d) \( \varepsilon_k'(z) = -k\varepsilon_{k+1}(z) \).

Prove that a meromorphic function \( f : \mathbb{C} \to \mathbb{P} \) with period 1 can be written as a series:

\[ f(z) = \sum_{n \in \mathbb{Z}} f_n \exp 2\pi inz \]

that converges locally uniformly. Deduce that each \( \varepsilon_k \) is a rational function of \( \exp 2\pi iz \).

Prove that

\[ \varepsilon_2(z) = \frac{\pi^2}{\sin^2 \pi z} . \]

14. Eisenstein series (continued). Show that the function

\[ \varepsilon_1(z) = \frac{1}{z} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{z - n} + \frac{1}{n} \]

defines a meromorphic function on \( \mathbb{C} \) with \( \varepsilon_1'(z) = -\varepsilon_2(z) \). Solve this differential equation to find an explicit formula for \( \varepsilon_1 \).

Solve the equation

\[ f'(z) = \varepsilon_1(z)f(z) \]

and hence find an infinite product for \( \sin \pi z \).

15. Write \( 1/(z - n) \) as a Laurent series about 0. Hence find the Laurent series for \( \varepsilon_1 \) about 0. (Write the coefficients in terms of the Riemann \( \zeta \) function

\[ \zeta(s) = \sum_{n \in \mathbb{N}} n^{-s} . \]

What is its radius of convergence?

Find the Laurent series for each \( \varepsilon_k \) about 0.

Prove that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} . \]

Please send any comments or corrections to me at: t.k.carne@dpmms.cam.ac.uk.