Random walks and uniform spanning trees: Example Sheet 2 Solutions

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Harder questions are marked by a ♠. Note that solutions given here often have less detail than you would ideally have included in your own solutions. For some of the harder questions we have not written full solutions but instead have provided pointers to where a solution can be found.

Exercise 1. Let T and T' be two infinite, bounded degree trees all of whose degrees are at least three. Prove that T and T' are rough-isometric.

Solution. We will give a very explicit construction of a rough isometry; if you were writing something down by hand you would probably describe a similar construction in a less formal way. Let T be a bounded degree tree all of whose vertices have degree between 3 and M for some M, and let S be the infinite binary tree. Since rough isometry is an equivalence relation, it suffices to prove that T and S are rough-isometric. The vertices of S may be identified with the set $\{0,1\}^* = \bigcup_{n\geq 0}\{0,1\}^n$ of finite strings of 0s and 1s so that the root is represented by the empty string, its two children are 0 and 1, 0 has children 00 and 01, and so on. Similarly, the vertex set of T can be identified with a non-empty set $A \subseteq \{1,\ldots,M\}^*$ with the properties that:

1. The empty string belongs to A as do the strings 1, 2, and 3.
2. If a string $a_1 \cdots a_n$ belongs to A then so do all of its initial segments $a_1 \cdots a_m$ with $m \leq n$.
3. If a string $a_1 \cdots a_n$ belongs to A then so do the strings $a_1 \cdots a_n 1$ and $a_1 \cdots a_n 2$.

(Conversely, such a set A defines a rooted tree in which all vertices have degree between 3 and M.) For each string $a = a_1 \cdots a_n$ let $\mathrm{ch}(a)$ be the number of children of the corresponding vertex of T, i.e., the maximal k such that $a_1 \cdots a_n k$ belongs to A. We define a map from $\phi : A \to \{0,1\}^*$ sending recursively as follows: ϕ sends the empty string to the empty string. Suppose we have defined ϕ for all strings of length at most n. Then we define ϕ for strings of length $n+1$ by

$$\phi(a_1 \cdots a_n a_{n+1}) = \begin{cases} \phi(a_1 \cdots a_n)1^{a_{n+1}-1}0 & \text{if } a_{n+1} < \mathrm{ch}(a_1 \cdots a_n) \\ \phi(a_1 \cdots a_n)1^{a_{n+1}-1} & \text{if } a_{n+1} = \mathrm{ch}(a_1 \cdots a_n). \end{cases}$$

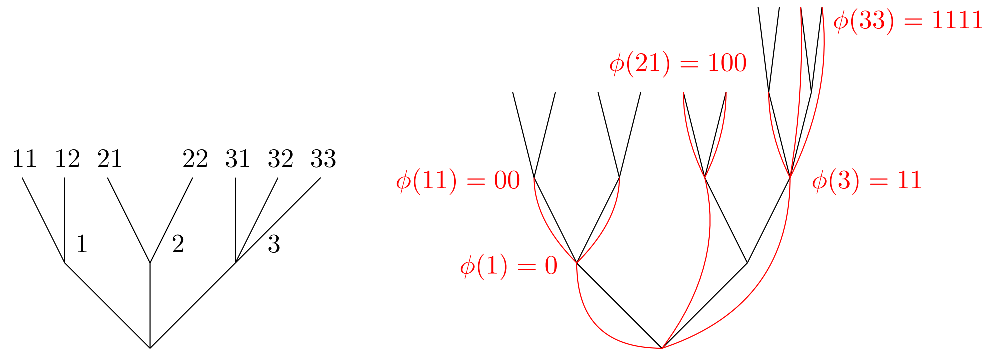

Figure 1: Example of recursively mapping (the first two generations of) a tree into a binary tree as in the solution to Exercise 1. Note that the red tree on the right is a copy of the tree on the left.

(This procedure becomes much clearer if you work through an example graphically as in Figure 1 below.) Where 1^k means the string of length k consisting entirely of 1s. It is easily verified that this map satisfies

$$d_T(a,b) \leq d_S(\phi(a),\phi(b)) \leq (M-2)d_T(a,b)$$

for every two vertices a and b of T and that every vertex of S has at least one child belonging to the image of ϕ, so that ϕ is a rough isometry as claimed. □

Exercise 2. Let T be a k-regular tree (i.e., the unique tree in which every vertex has degree k). Prove that

$$\Phi_E(T) = \frac{k-2}{k} \qquad \text{and} \qquad \rho(T) = \frac{2\sqrt{k-1}}{k} = \sqrt{1-\Phi_E(T)^2}.$$

Solution. Every tree with n vertices has $n-1$ edges. Let W is a finite set of vertices in T and let $E(W)$ be the set of edges with both endpoints in W. Then we have that

$$k|W| = \sum_{w\in W} \deg(w) = 2|E(W)| + |\partial_E W|$$

and

$$2|E(W)| = 2|W| - 2\#\{\text{connected components of } W\}$$

so that

$$|\partial_E W| = \frac{k-2}{k}\sum_{w\in W} \deg(w) + 2\#\{\text{connected components of } W\}$$

for every finite set of vertices W in T. This clearly implies that $\Phi_E(t) \geq \frac{k-2}{k}$. On the other hand, the ball of radius r around the origin has

$$1 + \sum_{i=0}^{r} k(k-1)^i \sim k(k-1)^r\frac{k-1}{k-2}$$

vertices and $k(k-1)^{r+1}$ boundary edges, so that

$$\Phi_E \leq \lim_{k\to\infty} \frac{k(k-1)^{r+1}}{k + k^2\sum_{i=0}^r (k-1)^i} = \frac{k-2}{k}$$

as claimed.

We now turn to the spectral radius. Cheeger's inequality gives that

$$\rho(G) \leq \frac{2\sqrt{k-1}}{k},$$

so it suffices to prove the matching lower bound. We have the intuition that the functions that are close to attaining the supremum in the definition of the spectral radius are likely to be highly symmetric, and will try to analyze one of the simplest highly symmetric functions we can think of. Fix a root vertex v_0 of T, let $\lambda < 1/\sqrt{k-1}$ and consider the radially symmetric function

$$f(v) = \lambda^{d(v_0,v)}.$$

Then we can compute that

$$\frac{1}{k}\|f\|_c^2 = 1 + \sum_{r=1}^\infty k(k-1)^{r-1}\lambda^{2r} = 1 + \frac{k}{k-1}\frac{\lambda^2(k-1)}{1-\lambda^2(k-1)} = 1 + \frac{\lambda^2 k}{1+\lambda^2-\lambda^2 k} = \frac{1+\lambda^2}{1+\lambda^2-\lambda^2 k}$$

and

$$\|\nabla f\|_r^2 = \sum_{r=1}^\infty k(k-1)^{r-1}(\lambda^{r-1}-\lambda^r)^2 = k(1-\lambda)^2\sum_{r=1}^\infty (k-1)^{r-1}\lambda^{2(r-1)} = \frac{k(1-\lambda)^2}{1+\lambda^2-\lambda^2 k}$$

so that

$$\frac{\|\nabla f\|_r^2}{\|f\|_c^2} = \frac{(1-\lambda)^2}{1+\lambda^2}$$

and hence that

$$\rho(T) = 1 - \inf\left\{\frac{\|\nabla f\|_r^2}{\|f\|_c^2} : f \in L(V,c),\ \|f\|_c > 0\right\} \geq 1 - \frac{(1-\lambda)^2}{1+\lambda^2}$$

for every $\lambda < 1/\sqrt{k-1}$. Taking the limit as $\lambda \uparrow 1/\sqrt{k-1}$ gives that

$$\rho(T) \geq 1 - \frac{(\sqrt{k-1}-1)^2}{k} = \frac{2\sqrt{k-1}}{k}$$

as claimed. □

Exercise 3. Let G be an infinite simple planar graph all of whose vertex degrees are at least 7. Use Euler's formula to prove that G is nonamenable.

Correction to this question: One needs to assume that every face in the drawing of G has at least three sides. This is clearly true if G is simple.

Solution. Let H be a *finite* simple planar graph. When we draw H in the plane, every face of H must have at least three sides. Thus, there is a map from oriented edges to faces, sending each oriented edge to the face lying to its right, in which each face has at least 3 oriented edges in its preimage. It follows that

$$\#\{\text{Faces}\} \leq \frac{2}{3}\#\{\text{Edges}\}$$

and hence by Euler's formula that

$$2 = \#\{\text{Vertices}\} - \#\{\text{Edges}\} + \#\{\text{Faces}\} \leq \#\{\text{Vertices}\} - \frac{1}{3}\#\{\text{Edges}\}$$

and hence that

$$\#\{\text{Edges}\} \leq 3\#\{\text{Vertices}\} - 6$$

(This is an equality when every face has three sides.)

If G is an *infinite* simple planar graph in which all degrees are at least 7, we can apply this inequality to the connected components of the subgraphs induced by a finite subset W of G to get that

$$\begin{aligned}\sum_{w \in W} \deg(w) &= |\partial_E W| + 2|E(W)| \\ &\leq |\partial_E W| + 6|W| - 12\#\{\text{connected components of } W\}\end{aligned}$$

and hence that

$$\frac{|\partial_E W|}{\sum_{w \in W} \deg(w)} \geq 1 - \frac{6|W|}{\sum_{w \in W} \deg(w)} \geq \frac{1}{7}.$$

Since W was arbitrary, $\Phi_E(G) \geq 1/7 > 0$ as claimed. □

Exercise 4. Let G and G' be connected, bounded-degree, rough-isometric graphs with isoperimetric profiles Φ_E and Φ'_E respectively. Prove that there exist positive constants c and C such that

$$c\Phi_E(Ct) \leq \Phi'_E(t) \leq C\Phi_E(ct)$$

for every $t > 0$. Prove directly that $\rho(G) < 1$ if and only if $\rho(G') < 1$ by using Lemma 3.10.

Solution. Since rough isometry is an equivalence relation, it suffices to prove one of these two inequalities. Let $\phi : V \to V'$ be an (α, β)-rough isometry between G and G'. Let $r_0 = \lceil \alpha(1+\beta) \rceil$, so that if $u, v \in V$ have $d(u,v) \geq r_0$ then $\phi(u) \neq \phi(v)$. We write $\succeq$, $\preceq$, and $\asymp$ for equalities inequalities that hold up to multiplication by positive constants depending only on α, β, and the maximum degrees of G and G'. Since G has degrees bounded by some constant M, we have that $|\phi^{-1}(u)| \leq M^{r_0} \preceq 1$ for every $u \in V$.

Given a set A in V or V' and $r \geq 0$, write $N_r(A)$ for the set of vertices at distance at most r from A in V or V' as appropriate. Let W be a finite set of vertices in G'. We claim that if W has a small boundary-to-volume-ratio then so does

$$K := N_{r_0}(\phi^{-1}(N_\beta(W))).$$

The definition of a rough isometry ensures that $\phi^{-1}(N_\beta(v)) \neq \emptyset$ for every $v \in V'$, and it follows easily that

$$|K|_c \asymp |K| \asymp |\phi^{-1}(N_\beta(W))| \asymp |W| \asymp |W|_c.$$

On the other hand, let $\partial_V^+ K = N_1(K) \setminus K$ be the outer vertex boundary of K. If $v \in \partial_V^+ K$ then the definition of r_0 ensures that $\phi(v) \notin W$ but that $d(\phi(v), \partial_V^- W) = d(\phi(v), W) \leq \alpha(r_0 + 1) + \beta =: r_1$. It follows easily that

$$|\partial_E K| \preceq |\partial_V^+ K| \leq M^{r_0} |N_{r_1}(\partial_V^- W)| \preceq |\partial_V^- W| \leq |\partial_E W|$$

and hence that

$$|K|_c \asymp |W|_c \text{ and } \frac{|\partial_E K|}{|K|_c} \preceq \frac{|\partial_E W|}{|W|_c}.$$

Since W was arbitrary, it follows that there exist constants C and c such that

$$c\Phi_E(Ct) \leq \Phi'_E(t)$$

as claimed.

The second part of the question asks us to prove directly that $\rho(G) < 1$ is invariant under rough isometry without using Cheeger's inequality. (This is more involved than I realised when I set the question.) We use the formulas

$$((I-P)f, f)_c = \|\nabla f\|_r^2$$

and

$$\rho(G) = 1 - \inf\left\{ \frac{\|\nabla f\|_r^2}{\|f\|_c^2} : f \in L^0(V,c),\ \|f\|_c > 0 \right\}.$$

We also use Lemma 3.10, which states that if $\phi : V \to V'$ is a rough isometry between bounded degree graphs then there exists a constant C such that $\|f \circ \phi\|_c^2 \leq C\|f\|_c^2$ for every function $f : V' \to \mathbb{R}$.

Let $G = (V, E)$ and $G' = (V', E')$ be two connected, bounded degree graphs and let $\phi : V \to V'$ be an (α, β)-rough isometry. It suffices to prove that if $\rho(G') = 1$ then $\rho(G) = 1$ also. Since $\rho(G') = 1$, for every $\varepsilon > 0$ there exists $f : V' \to \mathbb{R}$ such that $\|\nabla f\|_r^2 / \|f\|_c^2 \leq \varepsilon$. Since $\|\nabla |f|\|_r^2 \leq \|\nabla f\|_r^2$, we may take f to be non-negative. Naively we would like to compose such a function f with the rough isometry ϕ to get an analogous function making this ratio small on G. Unfortunately this does not quite work as we don't have an obvious way to lower bound $\|f \circ \phi\|_c^2$. Instead, we will 'smear out' f using the random walk on G' before composing it with ϕ: Take $k = \lceil \beta \rceil$. Since the k-step random walk on G' corresponds to the simple random walk on another bounded degree graph with the same vertex set as G' for which the identity function is a rough isometry, we have by Lemma 3.10 that

$$\|\nabla(P^k f)\|_c^2 \preceq \|\nabla f\|_c^2.$$

and hence also that

$$\|\nabla(P^k f) \circ \phi\|_c^2 \preceq \|\nabla(P^k f)\|_c^2 \preceq \|\nabla f\|_c^2.$$

Note that there exists a positive constant c such that $P^k(u, v) \geq c$ whenever $d(u, v) \leq k$. For every $v \in V'$, there exists $w(v) \in \phi(V)$ with $d(v, w(v)) \leq k$. Since $|\{v \in V' : w(v) = w\}|$ is bounded as a function of $w \in V'$, we deduce that

$$\|(P^k f) \circ \phi\|_c^2 \asymp \sum_{v \in V} P^k f(\phi(v))^2 \geq \sum_{u \in \phi(V)} P^k f(u)^2 \succeq \sum_{u \in V'} f(v)^2 \asymp \|f\|_c^2.$$

It follows that

$$\frac{\|\nabla (P^k f) \circ \phi\|_c^2}{\|(P^k f) \circ \phi\|_c^2} \preceq \frac{\|\nabla f\|_c^2}{\|f\|_c^2} \leq \varepsilon,$$

and the claim follows since $\varepsilon > 0$ was arbitrary. □

Exercise 5. Fill in the details in the proof of Theorem 3.21 by completing Exercises 27 and 28 from the notes. (Hint: For Exercise 27, reduce to the finite case and use the spectral theorem.)

Solution. □

Exercise 6. Without appealing to Gromov's theorem, prove that a Cayley graph is recurrent if and only if it satisfies

$$\sum_{r \geq 1} \frac{r}{\mathrm{Gr(r)}} = \infty.$$

Solution. First suppose that $\sum_{r \geq 1} \frac{r}{\mathrm{Gr(r)}} = \infty$. We will apply a small variation on the proof of Theorem 2.21. Let Π_i be the set of edges with one endpoint at distance $i-1$ from v and the other with distance i from v. Jensen's inequality then implies that

$$\frac{1}{r} \sum_{i=r+1}^{2r} |\Pi_i|_c^{-1} \geq \left[\frac{1}{r} \sum_{i=r+1}^{2r} |\Pi_i|_c \right]^{-1} \geq \left[\frac{1}{r} \sum_{i=1}^{2r} |\Pi_i|_c \right]^{-1} \geq \frac{r}{\mathrm{Gr}(2r)}.$$

We deduce that

$$\begin{aligned} \sum_{i=1}^{\infty} |\Pi_i|_c^{-1} &= \sum_{k=0}^{\infty} \sum_{i=2^k+1}^{2^k} |\Pi_i|_c^{-1} \geq \sum_{k=0}^{\infty} \frac{2^{2k}}{\mathrm{Gr}(2^{k+1})} \\ &\geq \sum_{k=0}^{\infty} \sum_{r=2^{k+1}}^{2^{k+2}-1} \frac{2^{k-1}}{\mathrm{Gr}(r)} \geq \frac{1}{4} \sum_{r=0}^{\infty} \frac{r}{\mathrm{Gr}(r)} = \infty, \end{aligned}$$

so that G is recurrent by the Nash-Williams criterion. (Note that this argument did not really require transitivity.)

Now suppose that $\sum_{r \geq 1} \frac{r}{\mathrm{Gr(r)}} < \infty$. Let $d = \deg(\mathrm{id})$, and extend $\mathrm{Gr}(n)$ to a smooth, strictly increasing function $\mathrm{Gr} : (0, \infty) \to (0, \infty)$. We have by Theorem 3.18 that

$$\Phi_E(t) \geq \frac{1}{2d \, \mathrm{Gr}^{-1}(2t)} \qquad \text{for every integer } t \geq 1.$$

Since $\mathrm{Gr}(n+1) \leq d \, \mathrm{Gr}(n)$ for every $n \geq 0$, it follows that

$$\Phi_E(t) \geq \frac{1}{2d^2 \, \mathrm{Gr}^{-1}(2t)} \qquad \text{for every } t \geq 1.$$

Thus, if we define $\Psi : [0,\infty) \to [s,\infty)$ by

$$\Psi(x) = \sup\left\{ t \geq d : \int_d^t \frac{32d^4}{y} \operatorname{Gr}^{-1}(2y)^2 \, \mathrm{d}y \leq x \right\}.$$

Then we have by Theorem 3.21 that

$$p_n(u,v) \leq \frac{2c(v)}{\Psi(n)}$$

for every $n \geq 1$ and $u, v \in V$. To prove transience, it suffices to prove that

$$\int \frac{1}{\Psi(x)} \, \mathrm{d}x < \infty.$$

Using the substitution $t = \Phi(x)$, we have that

$$x = \int_d^t \frac{32d^4}{y} \operatorname{Gr}^{-1}(2y)^2 \, \mathrm{d}y \qquad \text{so that} \qquad \frac{dx}{dt} = \frac{32d^4}{t} \operatorname{Gr}^{-1}(2t)^2.$$

This gives (after a little bit of work) that

$$\int_1^\infty \frac{1}{\Psi(x)} \, \mathrm{d}x < \infty \iff \int_1^\infty \frac{\operatorname{Gr}^{-1}(2t)^2}{t^2} \, \mathrm{d}t < \infty.$$

Using the substitution $s = \operatorname{Gr}^{-1}(2t)$ we have that

$$\frac{dt}{ds} = \frac{1}{2} \operatorname{Gr}'(s)$$

and hence that

$$\int_1^\infty \frac{1}{\Psi(x)} \, \mathrm{d}x < \infty \iff \int_1^\infty \frac{s^2 \operatorname{Gr}'(s)}{\operatorname{Gr}(s)^2} \, \mathrm{d}s < \infty \iff \int_1^\infty \frac{s}{\operatorname{Gr}(s)} \, \mathrm{d}s < \infty$$

where the second implication follows by integration by parts. This is easily seen to imply the claim. □

Exercise 7. Construct a connected, locally finite, nonamenable graph for which the walk does not have positive speed. (Such a graph must have unbounded degrees.)

Solution. Consider the graph with vertex set $\{1, 2, \ldots\}$ in which for each $n \geq 1$ there are 2^n edges connecting n and $n+1$ and a single edge connecting n to $2n$. It is easy to check that this graph is nonamenable. It is also easy to verify using Borel-Cantelli that the random walk on this graph satisfies $|X_n|/n \to 1/3$ as $n \to \infty$ and only crosses at most finitely many of the 'long' edges connecting n to $2n$ almost surely. In the graph metric we have that $d(0, 2^n) = n$, and it follows that $d(0, X_n)$ is of order $\log n$ for infinitely many n almost surely. (By playing around with the example we can replace $\log n$ with any function tending to infinity as n tends to infinity.) □

Exercise 8. Construct an infinite, connected, bounded degree graph G with a vertex v such that the random walk $(X_n)_{n\geq 0}$ started from v satisfies $d(v, X_n) \leq C \log n$ with high probability for some constant C as $n \to \infty$. Show that no slower rate of growth of the typical displacement is possible in a bounded degree graph.

Solution. One example is given by the *canopy tree*, which can be thought of as 'an infinite binary tree viewed from a leaf', and is pictured above. One way to construct this tree T formally is as the graph whose vertices are the dyadic intervals

$$\{[k2^i, (k+1)2^i] : k, i \geq 0\}$$

where two such intervals $I_1 = [k2^i, (k+1)2^i]$ and $I_1 = [\ell 2^j, (\ell+1)2^j]$ are adjacent if either $I_1 \subseteq I_2$ and $i = j-1$ or $I_2 \subseteq I_1$ and $j = i-1$. Note that the vertex set of T is naturally decomposed into levels

$$L_i = \{[k2^i, (k+1)2^i] : k \geq 0\}$$

and that any two vertices in the same level can be mapped to each other by an automorphism of T. For each vertex v of T, we define $|v|$ to be the unique integer with $v \in L_{|v|}$. Observe that if $X_{n\geq 0}$ is a random walk on T started from v_0 then $(|X_n|)_{n\geq 0}$ is a random walk with drift on $\{0, 1, \ldots\}$. Using this description, it is not too hard to prove that

$$\limsup_{n\to\infty} \frac{\max_{0\leq m\leq n} |X_m|}{\log_2 n} = 1$$

almost surely; we leave the details to the reader. The fact that the typical displacement of the random walk on T is logarithmic follows from the fact that

$$d(v_0, X_n) \leq 2 \max\{i : \tau_{v_i} \leq n\}.$$

Now suppose that G is an infinite connected, bounded degree graph. We have from lectures that there exists a constant C such that $p_n(v,v) \leq Cn^{-1/2}$ for every vertex v and $n \geq 1$. We also trivially have that if M denotes the maximum degree of G then $|B(v,r)| \preceq M^r$ for every $r \geq 1$. Then we have by Cauchy-Schwarz that

$$p_{2n}(v,v) \geq \sum_{u\in B(v,r)} p_n(v,u)p_n(u,v) \asymp \sum_{u\in B(v,r)} p_n(v,u)^2 \geq \frac{\mathbf{P}_v(X_n \in B(v,r))^{1/2}}{|B(v,r)|^{1/2}}$$

for every $n, r \geq 1$ and $v \in V$ and hence that

$$\mathbf{P}_v(X_n \in B(v,r)) \preceq M^{r/2} n^{-1/2}$$

for every $n, r \geq 1$ and $v \in V$. It follows that there exists a constant c such that

$$\mathbf{P}_v(d(v, X_n) \leq c \log n) \to 0 \text{ as } n \to \infty$$

as required. □

Exercise 9. Apply the Varopoulos-Carne inequality to prove Corollaries 3.43-3.45 in the notes.

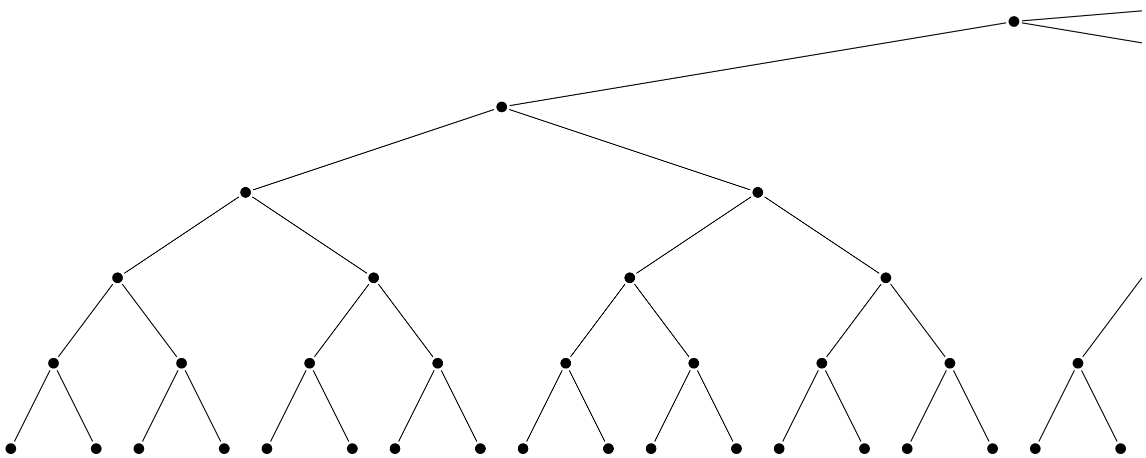

Figure 2: The canopy tree is a bounded degree graph in which the typical displacement of the random walk grows logarithmically.

Solution. Corollary 3.43:

> Let G be a connected, locally finite graph, let $v \in V$, and suppose that G has subexponential volume growth in the sense that $\limsup_{r\to\infty} |B(v,r)|_c^{1/r} = 1$. If $(X_n)_{n\geq 0}$ is a random walk on G, then
>
> $$\limsup_{n\to\infty} \frac{d(X_0,X_n)}{n} = 0$$
>
> almost surely.

Since G is a connected graph, all vertex conductances are at least one and hence satisfy $\sqrt{c(v)} \leq c(v)$. Varopoulos-Carne implies that there exists a constant C such that

$$\mathbf{P}_v(d(v,X_n) \geq r) = \sum_{u\in B(v,n)\setminus B(v,r-1)} p_n(v,u) \leq 2e^{-r^2/2n}|B(v,n)|_c.$$

If G has subexponential growth then we deduce that

$$\mathbf{P}_v(d(v,X_n) \geq \varepsilon n) \leq 2\exp\left[-\frac{\varepsilon^2}{2}n + o(n)\right]$$

as $n \to \infty$ for every $\varepsilon > 0$. Borel-Cantelli therefore implies that at most finitely many of the events $\{d(v,X_n) \geq \varepsilon n\}$ occur for each $\varepsilon > 0$ almost surely, which is equivalent to the claim.

Corollary 3.44:

> Let G be a connected, locally finite graph, let $v \in V$, and suppose that G has polynomial volume growth in the sense that there exist constants C and d such that $|B(v,r)|_c \leq Cr^d$ for every $r \geq 1$. If $(X_n)_{n\geq 0}$ is a random walk on G, then
>
> $$\limsup_{n\to\infty} \frac{d(X_0,X_n)}{\sqrt{n\log n}} < \infty$$

almost surely.

The same calculation as above gives in this situation that

$$\mathbf{P}_v(d(v,X_n) \geq \lambda\sqrt{n\log n}) \leq 2C\exp\left[-\frac{\lambda^2}{2}\log n + d\log(\lambda\sqrt{n\log n})\right].$$

If $\lambda > \sqrt{d}$ then the right hand side is summable in n, and the claim follows by Borel-Cantelli.

Corollary 3.45:

> Let G be a connected, locally finite graph and let $v \in V$. Let $d \geq 1$ and suppose that G has polynomial volume growth of degree at most d in the sense that there exist constants C and d such that $|B(v,r)|_c \leq Cr^d$ for every $r \geq 1$. Prove that there exists a positive constant c such that
>
> $$p_{2n}(v,v) \geq c(n\log n)^{-d/2}$$
>
> for every $n \geq 0$.

We have that

$$p_{2n}(v,v) = \sum_u p_n(v,u)p_n(u,v) = \sum_u \frac{c(v)}{c(u)}p_n(v,u)^2.$$

Cauchy-Schwarz gives that

$$\mathbf{P}_v(X_n \in B(v,r)) = \sum_{u\in B(v,r)} p_n(v,u) \leq \sqrt{\sum_{u\in B(v,r)} c(u)}\sqrt{\sum_{u\in B(v,r)} \frac{1}{c(u)}p_n(v,u)^2}$$

and it follows that

$$p_{2n}(v,v) \geq \frac{c(v)\mathbf{P}_v(X_n \in B(v,r))^2}{|B(v,r)|_c}.$$

Taking $r = C\sqrt{n\log n}$ for C sufficiently large and using Corollary 3.44 completes the proof. □

Exercise 10. Construct a connected, locally finite graph G such that the invariant σ-algebra $\mathcal{I}$ is trivial but the tail σ-algebra $\mathcal{T}$ is not.

Solution. Take $\mathbb{Z}^3$ and attach a single loop to the origin to get a graph G' with vertex set $\mathbb{Z}^3$. The resulting graph is Liouville since a function $h : \mathbb{Z}^3 \to \mathbb{R}$ is harmonic on this graph if and only if it is harmonic on $\mathbb{Z}^3$. The tail σ-algebra of G' is non-trivial, however: Decompose $\mathbb{Z}^3 = \text{Odd} \cup \text{Even}$ into the sets of vertices with odd and even coordinate sum respectively. Since $\mathbb{Z}^3$ is transient, the random walk on G' will go around the loop at the origin an a.s. finite random number of times, the distribution of the number of times being geometric. This implies that the tail event

$$\{X_{2n} \in \text{Odd for infinitely many } n\}$$

has probability strictly between 0 and 1 as required. □

Exercise 11 (The lamplighter graph.)**.** Let G be a connected, locally finite, simple graph. Let $\mathrm{Lamps}(V)$ be the set of finitely supported functions $\psi : V \to \{0,1\}$. We define the lamplighter graph $\mathrm{LampLighter}(G)$ to be the graph with vertex set $V \times \mathrm{Lamps}(V)$, and where two vertices (u,ϕ) and (v,ψ) are adjacent if and only if either $u \sim v$ and $\phi = \psi$ or $u = v$ and ϕ and ψ differ only at u. We interpret (u,ϕ) as describing the configuration of a collection of lamps, one at each vertex, together with the location of a lamplighter, who is at u. At each time step, the lamplighter may either move to a location adjacent to their current location or change the status of the lamp at their current location.

1. Prove that G is amenable if and only if $\mathrm{LampLighter}(G)$ is amenable.
2. Prove that if G is transitive, then $\mathrm{LampLighter}(G)$ is transitive.
3. Prove that if G is infinite then $\mathrm{LampLighter}(G)$ has exponential growth.
4. Prove that if G is transient then $\mathrm{LampLighter}(G)$ has non-trivial invariant σ-algebra.
5. Prove that $\mathrm{LampLighter}(\mathbb{Z}^d)$ is Liouville if and only if $d \le 2$.

(There was a mistake in the definition of the lamplighter graph as originally written. The first part of the question was erroneously omitted from the example sheet.)

Solution.

1. For each finite $W \subseteq V$, let $\mathrm{Lamps}(W) \subseteq \mathrm{Lamps}(V)$ be the set of lamp configurations supported on W. Observe that

$$|\partial_E(W \times \mathrm{Lamps}(W))| = |\partial_E W| \cdot |\,\mathrm{Lamps}(W)|$$

and that

$$|W \times \mathrm{Lamps}(W)|_c = (|W|_c + 1) \cdot |\,\mathrm{Lamps}(W)|$$

so that

$$\frac{|\partial_E(W \times \mathrm{Lamps}(W))|}{|W \times \mathrm{Lamps}(W)|_c} \le \frac{|\partial_E W|}{|W|_c}$$

for every finite set $W \subseteq V$. It follows in particular that if G is amenable then $\mathrm{LampLighter}(G)$ is amenable also.

For each $\phi \in \mathrm{Lamps}(V)$, the subgraph G_ϕ of $\mathrm{LampLighter}(G)$ induced by $V \times \{\phi\}$ is isomorphic to G. Let W be a finite set of vertices in $\mathrm{LampLighter}(G)$. For each $\phi \in \mathrm{Lamps}(V)$, let $W_\phi = W \cap (V \times \{\phi\})$ and let $\partial_E^\phi W_\phi$ be the boundary of W_ϕ in G_ϕ. We clearly have that

$$|\partial_E W| \ge \sum_\phi |\partial_E^\phi W_\phi|,$$

and hence that if G is nonamenable with Cheeger constant $\Phi_E(G) > 0$ then

$$|\partial_E W| \ge \Phi_E(G) \sum_\phi \sum_{(v,\phi)\in W_\phi} (\deg((v,\phi)) - 1) = \Phi_E(G)\left(|W|_c - |W|\right),$$

where we used the fact that the degree of (v,ϕ) in G_ϕ is one less than its degree in LampLighter(G). Now observe that if W is connected then $|W|_c \geq 2(|W|-1)$ and hence that

$$|\partial_E W| \geq \frac{1}{2}\Phi_E(G)\left(|W|_c - 1\right).$$

This is easily seen to imply that LampLighter(G) is nonamenable.

2. For each automorphism γ of G, it is easily verified that

$$(v,\phi) \mapsto (\gamma(v), \phi \circ \gamma^{-1})$$

defines an automorphism of LampLighter(G). Moreover, if $\psi \in$ Lamps(V) then

$$(v,\phi) \mapsto (v, \phi + \psi \mod 2)$$

also defines an automorphism of LampLighter(G). If G is transitive, then we can take any element of $V \times$ Lamps(G) to any other by composing one of each of these two types of automorphisms.

3. Since G is infinite, connected, and locally finite, it must contain an infinite path of distinct vertices $v_0 \sim v_1 \sim v_2 \cdots$. (You should prove this fact if you haven't seen it before.) In $2n$ steps, the lamplighter started at v_0 can create any lamp configuration on the n vertices $v_0, \ldots, v_{n-1}$, and it follows that $|B(v,2n)| \geq 2^n$ for every $n \geq 0$.

4. Let (X_n, Φ_n) be the random walk on LampLighter(G). If we define stopping times $(T_k)_{k\geq 0}$ recursively by $T_0 = 0$ and $T_k = \min\{n \geq T_{k-1} : X_n \neq X_{T_{k-1}}\}$ then $(X_{T_k})_{k\geq 0}$ is a random walk on G, and therefore visits each vertex at most finitely often almost surely since G is transient. It follows that $\lim_{n\to\infty}\Phi_n(v)$ exists almost surely for each $v \in V$ and moreover that $\{\lim_{n\to\infty}\Phi_n(v) = 0\}$ is a non-trivial invariant event for each $v \in V$.

5. (This part is harder than the other parts.) Part (4) gives that LampLighter$(\mathbb{Z}^d)$ is non-Liouville when $d \geq 3$, so it suffices to prove that it is Liouville when $d \in \{1,2\}$. We will use the entropy characterization of the Liouville property for transitive graphs. Let $d \in \{1,2\}$ and let (X_n, Φ_n) be the random walk on LampLighter$(\mathbb{Z}^d)$. We first claim that

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{E}\left[\#\{X_i : 0 \leq i \leq n\}\right] = 0$$

almost surely. In the case $d = 1$, this is an immediate consequence of the central limit theorem. The case $d = 2$ is a little more subtle. One way to prove the claim is to use Donsker's invariance principle together with the fact that 2d Brownian motion has zero Lebesgue measure almost surely. Here is a direct proof that works for any recurrent transitive graph: We have by reversibility that

$$\mathbf{E}_0\left[\#\{X_i : 0 \leq i \leq n\}\right] = \sum_{i=0}^{n} \mathbf{P}_0(X_i \notin \{X_0, \ldots, X_j\}) = \sum_{i=0}^{n} \mathbf{P}_0(\tau_0^+ > i)$$

and hence that

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{E}\left[\#\{X_i : 0 \leq i \leq n\}\right] = \mathbf{P}_p(\tau_0^+ = \infty) = 0$$

when $d \in \{1,2\}$.

Let A_n be the set of connected subsets of $\mathbb{Z}^d$ containing 0 that have at most n elements, and let A'_n be the set of all *subsets* of sets belonging to A_n. One can prove that there exists a constant $C = C_d$ such that $|A_n| \leq C^n$ and hence that $|A'_n| \leq 2^n C^n$ for every $n \geq 1$. Let $R_n = \{X_i : 0 \leq i \leq n\}$ and fix $\delta > 0$. Then we have that

$$\begin{aligned} -\sum_{(v,\phi)} \mathbf{P}_p(X_k = (v,\phi)) \log \mathbf{P}_p(X_k = (v,\phi)) \\ = -\sum_{(v,\phi),\phi \in A'_{\delta k}} \mathbf{P}_p(X_k = (v,\phi)) \log \mathbf{P}_p(X_k = (v,\phi)) \\ - \sum_{(v,\phi),\phi \notin A'_{\delta k}} \mathbf{P}_p(X_k = (v,\phi)) \log \mathbf{P}_p(X_k = (v,\phi)). \end{aligned}$$

Since there are at most $k^d (2C)^{\delta k}$ points (v,ϕ) with $\phi \in A'_{\delta k}$ for which the transition probability is positive, it follows (by a little calculation) that

$$-\sum_{(v,\phi)} \mathbf{P}_p(X_k = (v,\phi)) \log \mathbf{P}_p(X_k = (v,\phi)) \preceq \delta k + \mathbf{P}_p(|R_n| \geq \delta k) k.$$

(The details are similar to the proof of the direction ($\Leftarrow$) in Proposition 3.51.) The probability $\mathbf{P}_p(|R_n| \geq \delta k)$ tends to zero as $k \to \infty$ by the above calculation, and since $\delta > 0$ was arbitrary it follows that the Avez entropy is zero. □

♠ **Exercise 12.** Construct an example to prove that the Liouville property is *not* preserved by rough isometry between connected, bounded degree graphs.

Solution. See *Instability of the Liouville Property for Quasi-Isometric Graphs and Manifolds of Polynomial Volume Growth* by Itai Benjamini. □

♠♠ **Exercise 13.** Construct a connected, bounded degree, nonamenable graph that has the Liouville property.

Solution. See Chapter 13 of Itai Benjamini's book *Coarse Geometry and Randomness.* □

Exercise 14. We say that a function $\phi : \mathbb{Z}^d \to \mathbb{R}$ has **sublinear growth** if

$$\limsup_{x \to \infty} \frac{|\phi(x)|}{\|x\|} = 0.$$

Prove that if x and y are two vertices of $\mathbb{Z}^d$, then there exists a random variable $(Z_n)_{n \geq 0} = ((X_n, Y_n))_{n \geq 0}$ and a random time T such that $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ are both lazy simple random walks on $\mathbb{Z}^d$, $X_n = Y_n$ for every $n \geq T$, and there exists a constant C_{xy} such that $\mathbb{P}(T \geq n) \leq C_{xy} n^{-1/2}$ for every $n \geq 1$. Deduce that if $\phi : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and has sublinear growth then it is constant.

The exponent was incorrectly written as -1 instead of $-1/2$. In fact, it seems more useful to prove a similar estimate on the maximum distance reached before coupling, which is what we do below. This question is more difficult than I realised and should have been given a ♠. The solution below is quite sketchy.

Solution. We couple the two walks X_n and Y_n together as follows: Let A_n and B_n be independent lazy random walks on $\mathbb{Z}^d$. We use superscripts to denote choices of coordinates. Let $(T_i)_{i=0}^d$ be defined recursively by $T_0 = 0$ and

$$T_i = \min\{n \geq T_{i-1} : A_n^i - A_{T_{i-1}}^i + x^i = B_n^i - B_{T_{i-1}}^i + y^i\}.$$

Define the two random walks X and Y in terms of the above as

$$X_{n+1}^i - X_n^i = A_{n+1}^i - A_n^i$$

and

$$Y_{n+1}^i - Y_n^i = \begin{cases} B_{n+1}^i - B_n^i & T_i \leq n < T_{i+1} \\ A_{n+1}^i - A_n^i & \text{otherwise.} \end{cases}$$

It is easily verified that X and Y are both distributed as lazy random walks on $\mathbb{Z}^d$ and that $X_n^i = Y_n^i$ for every $n \geq T_i$. In particular, $X_n = Y_n$ for every $n \geq T_d$. Moreover, conditional on $(Z_n)_{m=0}^{T_{i-1}}$, the process $(X_{T_{i-1}+n}^i - Y_{T_{i-1}+n}^i)_{n\geq 0}$ is conditionally distributed as a lazy random walk on $\mathbb{Z}$ started at $x^i - y^i$ and stopped when it first hits zero. Using the fact that random walk on $\mathbb{Z}$ satisfies

$$\mathbb{P}_a(\tau_0 \geq n) \preceq \frac{|a|}{n^{1/2}}$$

for every $a \in \mathbb{Z}$ and $n \geq 1$, we obtain that

$$\mathbb{P}(T_i - T_{i-1} \geq n) \preceq |x^i - y^i| n^{-1/2}$$

for every $n \geq 1$ and hence that

$$\mathbb{P}(T_d \geq n) \leq \sum_{i=1}^{n} \mathbb{P}(T_i - T_{i-1} \geq n/d) \preceq \left[\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |x^i - y^i|\right] n^{-1/2} \preceq \|x - y\| n^{-1/2}$$

as required.

For each $k \geq 1$, let τ_k and τ_k' be the first times that X_n and Y_n satisfy $d(0, X_n) \geq k$ or $d(0, Y_n) \geq k$ as appropriate. Let's estimate the probability that $T_d \geq \min\{\tau_k, \tau_k'\}$. From now on we will allow the implicit constants in our $\preceq$ notation to depend on x and y. We have by Gambler's ruin that

$$\mathbb{P}(\max_{T_{i-1} \leq n \leq T_i} |X_n^i - Y_n^i| \geq k \mid \mathcal{F}_{T_{i-1}}) \leq \frac{|X_{T_{i-1}}^i - Y_{T_{i-1}}^i|}{k}.$$

Observe that conditional on $\mathcal{F}_{T_{i-1}}$ and the sequence of differences $(X_n^i - Y_n^i)_{n=T_{i-1}}^{T_i}$, the sequence of sums $(X_n^i + Y_n^i)_{n=T_{i-1}}^{T_i}$ is a martingale with bounded increments, so that Azuma-Hoeffding gives that

$$\mathbb{P}(\max_{T_{i-1}\le n\le T_i} |X_n^i - x^i - (Y_n^i - y^i)| \ge k \mid \mathcal{F}_{T_{i-1}}, T_i) \le e^{-ck^2/(T_i - T_{i-1})}.$$

Taking expectations we get that

$$\begin{aligned}\mathbb{P}(\max_{T_{i-1}\le n\le T_i} |X_n^i - x^i + Y_n^i - y^i| \ge k \mid \mathcal{F}_{T_{i-1}}) &\le \mathbb{E}\left[e^{-ck^2/(T_i-T_{i-1})} \mid \mathcal{F}_{T_{i-1}}\right] \\ &= \int_0^\infty \frac{ck^2}{x^2} e^{-ck^2/x} \mathbb{P}(T_i - T_{i-1} \ge x \mid \mathcal{F}_i)\,\mathrm{d}x \\ &\preceq \int_0^\infty \frac{k^2}{x^{5/2}} e^{-ck^2/x}\,\mathrm{d}x \\ &= \frac{1}{c^{3/2}k}\int_0^\infty y^{1/2}e^{-y}\,\mathrm{d}y \preceq \frac{1}{k}.\end{aligned}$$

Together, these two estimates allow us to conclude that

$$\mathbb{P}(\max_{T_{i-1}\le n\le T_i} |X_n^i| + |Y_n^i| \ge k \mid \mathcal{F}_{T_{i-1}}) \preceq \frac{1}{k}.$$

Similar considerations allow us to control the random walks X^j and Y^j on intervals other than $[T_{j-1}, T_j]$, in the end allowing us to conclude that

$$\mathbb{P}(T_d \ge \min\{\tau_k', \tau_k\}) \preceq \frac{1}{k}$$

Now take $h : \mathbb{Z}^d \to \mathbb{R}$ to be a harmonic function of sublinear growth, and let $Z = (X, Y)$ be as above. For each $k \ge 1$, let τ_k and τ_k' be the first times that X_n and Y_n satisfy $d(0, X_n) \ge k$ or $d(0, Y_n) \ge k$ as appropriate. Then we have that

$$\begin{aligned}h(x) - h(y) = \mathbf{E}_x h(X_{\tau_k}) - \mathbf{E}_y h(Y_{\tau_k'}) &\le \mathbb{E}\left[|h(X_{\tau_k}) - h(Y_{\tau_k'})|\mathbb{1}(T_d \ge \min\{\tau_k, \tau_k'\})\right] \\ &\le 2\sup\{|h(z)| : d(0,z) \le k\}\mathbb{P}(T_d \ge \min\{\tau_k, \tau_k'\}).\end{aligned}$$

The right hand side tends to zero as $k \to \infty$, so that $h(x) = h(y)$. Since x and y were arbitrary, h is constant. □
