Random walks and uniform spanning trees: Example Sheet 1 Solutions

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Exercise 1 (The Poisson equation)**.** Let G be a finite, connected network, and let A be a non-empty subset of V. Let $\varphi : A \to \mathbb{R}$ and $\psi : V \setminus A \to \mathbb{R}$ be arbitrary functions. Prove that there exists a unique function $F : V \to \mathbb{R}$ such that

$$F|_A \equiv \varphi \qquad \text{and} \qquad (\Delta F)|_{V\setminus A} \equiv \psi.$$

How is this related to hitting times?

Solution. Consider the map

$$\begin{aligned} \mathbb{R}^V &\to \mathbb{R}^A \times \mathbb{R}^{V\setminus A} \\ F &\mapsto (F|_A, (\Delta F)|_{V\setminus A}). \end{aligned}$$

This is a linear map between spaces of the same dimension, so it is injective if and only if it is surjective by the rank-nullity theorem. Thus, it suffices to prove that this map is surjective. Let φ, ψ be as in the question, and let $F : V \to \mathbb{R}$ be defined by

$$F(v) = \mathbf{E}_v \left[\varphi(X^{\tau_A})\right] + \mathbf{E}_v \left[\sum_{n=0}^{\tau_A - 1} \frac{\psi(X_n)}{c(X_n)}\right].$$

We clearly have that $F|_A = \varphi$ as required. Meanwhile, if $v \in V \setminus A$ then we can compute that

$$\Delta F(v) = c(v)\left(F(v) - \mathbf{E}_v[F(X_1)]\right) = \psi(v)$$

as required. (Alternatively, one can prove injectivity by reduction to the Dirichlet problem.)

If $\varphi \equiv 0$ and $\psi(v) = c(v)$ for each $v \in V \setminus A$, then the solution F to the Poisson equation is given by the expected hitting times $F(v) = \mathbf{E}_v \tau_A$. □

Exercise 2. Prove that if $h : V \to \mathbb{R}$ is harmonic on $V \setminus A$ for some $A \subseteq V$ and $X = (X_n)_{n\geq 0}$ is a random walk on G started at some vertex v then $(h(X_{n\wedge\tau_A}))_{n\geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ generated by X. Use the optional stopping theorem to give an alternative proof of the uniqueness of solutions to the Dirichlet problem on finite networks.

Solution. The fact that $h(X_{n\wedge\tau_A})$ is a martingale is easily verified from the definitions and we omit the proof. Now suppose that G is finite and connected. Let $\varphi : A \to \mathbb{R}$ be given and let $h_1, h_2 : V \to \mathbb{R}$ be harmonic on $V \setminus A$ with $h_1|_A = h_2|_A = \varphi$. Then τ_A is almost surely finite, and the optional stopping theorem for bounded martingales implies that

$$h_1(v) = \mathbf{E}_v h_1(X_{\tau_A}) = \mathbf{E}_v \varphi(X_{\tau_A}) = \mathbf{E}_v h_2(X_{\tau_A}) = h_2(v)$$

for every $v \in V$ are required. □

Exercise 3. Prove Lemma 2.9 from the notes.

Solution. Lemma 2.9 is stated as follows:

> Let G be finite. The linear transformations $\nabla : \Omega_0^2 \to \Omega_1^2$ and $-\nabla\cdot : \Omega_1^2 \to \Omega_0^2$ are adjoints of each other in the sense that $(f, \nabla F)_r = -(\nabla \cdot f, F)$ for every 1-form $f \in \Omega_1^2$ and function $F \in \Omega_0^2$. In particular, $(F, \Delta F) = (\Delta F, F) = (\nabla F, \nabla F)_r$ for every function $F \in \Omega_0^2$.

Expanding out the definitions gives that

$$(f, \nabla F)_r = \frac{1}{2} \sum_{e \in E^{\rightarrow}} f(e)(F(e^+) - F(e^-)) \qquad \text{and} \qquad -(\nabla \cdot f, F) = -\sum_{v \in V} \sum_{e \in E_v^{\rightarrow}} f(e) F(v),$$

and it follows that

$$\begin{aligned}
(f, \nabla F)_r &= \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v^{\rightarrow}} f(e)(F(e^+) - F(v)) = \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v^{\rightarrow}} f(e) F(e^+) - \frac{1}{2}(\nabla \cdot f, F) \\
&= \frac{1}{2} \sum_{e \in E^{\rightarrow}} f(e) F(e^+) - \frac{1}{2}(\nabla \cdot f, F) = \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v^{\rightarrow}} f(e^{\leftarrow}) F(v) - \frac{1}{2}(\nabla \cdot f, F) \\
&= -(\nabla \cdot f, F)
\end{aligned}$$

as desired. The claimed equality $(F, \Delta F) = (\Delta F, F) = (\nabla F, \nabla F)_r$ follows from this together with the fact that $\nabla \cdot \nabla = -\Delta$. □

Exercise 4. Prove Lemma 2.14 and Theorem 2.15 from the notes.

Solution. We start with Lemma 2.14:

> Let G be a network, and let $(V_n)_{n \geq 1}$ be an increasing sequence of finite subsets of V with $\bigcup_n V_n = V$. Then
>
> $$\mathscr{C}_{\mathrm{eff}}(A \leftrightarrow \infty; G) = \lim_{n \to \infty} \mathscr{C}_{\mathrm{eff}}(A \cap V_n \leftrightarrow \partial_V^- V_n; G_n)$$
>
> for every non-empty finite subset A of V.

We may assume that $A \subseteq V_n$ for every $n \geq 1$. The claim is equivalent to the claim that

$$\sum_{a \in A} c(a) \mathbf{P}_a(\tau_A^+ = \infty) = \lim_{n \to \infty} \sum_{a \in A} c(a) \mathbf{P}_a(\tau_A^+ > \tau_{\partial_V^- V_n}).$$

This in turn follows from the fact that whenever X is a random walk started at $a \in A$ we have that the events $\{\tau_A^+ = \infty\}$ and $\bigcap_{n=1}^\infty \{\tau_A^+ > \tau_{\partial_V^- V_n}\}$ coincide up to an event of probability zero. (Indeed, the symmetric difference of these events is contained in the event that there exists n such that $\tau_{\partial_V^- V_n} = \infty$.)

Next we have Theorem 2.15:

Let G be a infinite network and let A be a non-empty finite subset of V. Then

$$\begin{aligned} \mathscr{C}_{\mathrm{eff}}(A \leftrightarrow \infty) &= \inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F \text{ finitely supported} \Big\} && (0.1)\\ &= \inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F(v) \to 0 \ \textit{as}\ v \to \infty \Big\}. && (0.2)\end{aligned}$$

Since $\mathscr{C}_{\mathrm{eff}}(A \cap V_n \leftrightarrow \partial_V^- V_n) \sum_{a \in A} c(a) \mathbf{P}_a(\tau_A^+ > \tau_{\partial_V^- V_n})$ is decreasing in n, we have by Dirichlet's principle that

$$\begin{aligned} \mathscr{C}_{\mathrm{eff}}(A \leftrightarrow \infty) &= \inf_{n \geq 0} \mathscr{C}_{\mathrm{eff}}(A \cap V_n \leftrightarrow \partial_V^- V_n) \\ &= \inf_{n\geq 1} \inf\left\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F \text{ supported on } V_n \setminus \partial_V^- V_n \right\} \\ &= \inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F \text{ finitely supported} \Big\} \end{aligned}$$

as claimed. For the second equality, we clearly have that

$$\begin{aligned} &\inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F(v) \to 0 \ \textit{as}\ v \to \infty \Big\} \\ &\qquad\qquad \leq \inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F \text{ finitely supported} \Big\}. \end{aligned}$$

For the reverse inequality, note that if $F(v) \to 0$ as $v \to \infty$ then for each $\varepsilon > 0$ the function F_ε defined by

$$F_\varepsilon(v) = \begin{cases} 1 & F(v) \geq \frac{2+\varepsilon}{2+2\varepsilon} \\ 0 & F(v) \leq \frac{\varepsilon}{2+2\varepsilon} \\ \frac{1}{2} + (1+\varepsilon)\left[F(v) - \frac{1}{2}\right] & \text{otherwise.} \end{cases}$$

is finitely supported and satisfies $F_\varepsilon | A = 1$ and $\|\nabla F_\varepsilon\|_r^2 \leq (1+\varepsilon)^2 \|\nabla F\|_r^2$. Since F and ε were arbitrary it follows that

$$\begin{aligned} &\inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F(v) \to 0 \ \textit{as}\ v \to \infty \Big\} \\ &\qquad\qquad \geq \inf\Big\{ \|\nabla F\|_r^2 : F \in \Omega_0^2,\ F|_A = 1,\ F \text{ finitely supported} \Big\}. \end{aligned}$$

as required. □

♠ **Exercise 5.** Prove or provide a counterexample to the *converse* of the Nash-Williams criterion: Is it true that if G is an infinite, connected, recurrent network and a is a vertex of G, then there exists a sequence of disjoint cutsets $(\Pi_i)_{i\geq 1}$ separating a from ∞ such that $\sum_{i=1}^\infty |\Pi_i|_c^{-1} = \infty$?

Solution. The converse to the Nash-Williams criterion is *not* true. Suppose that we attach an infinite path to each vertex of $\mathbb{Z}^2$. The resulting graph is clearly recurrent, since if T_n denotes the nth time the random walk on this graph visits $\mathbb{Z}^2$ then $(X_{T_n})_{n\geq 0}$ is a lazy random walk on $\mathbb{Z}^2$ and therefore visits the origin infinitely often almost surely. We claim that this graph does not satisfy the hypotheses of the Nash-Williams criterion. Indeed, suppose that $(\Pi_n)_{n\geq 0}$ is a sequence of disjoint cutsets separating 0 from ∞ in this graph. Let Π'_n be the intersection of Π_n with the edge set of $\mathbb{Z}^2$, so that (Π'_n) is a sequence of disjoint cutsets separating 0 from ∞ in $\mathbb{Z}^2$. Then we must have that

$$|\Pi_n| \geq |\{x \in \mathbb{Z}^2 : x \text{ is not separated from 0 by } \Pi'_n\}|.$$

By reordering if necessary, we may assume that $|\Pi_n|$ is an increasing function of n. Observe that if $d(0,x) = r$ then x is separated from 0 by at most r of the sets Π'_n. It follows that if $B(0,r)$ denotes the graph-distance ball of radius r around the origin in $\mathbb{Z}^2$ then for each $r \geq 1$ then

$$\sum_{n=1}^{2r} \mathbb{1}\left(x \text{ is not separated from 0 by } \Pi'_n\right) \geq r$$

for each $r \geq 1$. It follows that there exists a positive constant c such that

$$\sum_{n=1}^{2r} |\Pi_n| \geq \sum_{x\in B(0,r)} \sum_{n=1}^{2r} \mathbb{1}\left(x \text{ is not separated from 0 by } \Pi'_n\right) \geq cr^3$$

for every $r \geq 1$. Since $|\Pi_n|$ is increasing, it follows that

$$|\Pi_{2r}| \geq cr^2$$

for every $r \geq 1$ and hence that

$$\sum_{n=1}^{\infty} |\Pi_n|^{-1} \leq 1 + 2\sum_{n=1}^{\infty} |\Pi_{2n}|^{-1} \leq 1 + \frac{2}{c}\sum_{i=1}^{\infty} r^{-2} < \infty$$

as claimed. □

Exercise 6. Prove the reverse inequality in the method of random paths.

Solution. We may assume without loss of generality that $A = \{a\}$ is a singleton. Let X be a simple random walk started at a and stopped when it first visits B, and let Γ be the associated random path from a to B, so that Γ_i is equal to the ith edge crossed by X. It suffices to prove that

$$\frac{1}{2}\sum_{e\in E^{\rightarrow}} r(e)\mathbf{E}_a\left[\#\{1 \leq i \leq |\Gamma| : \Gamma_i = e\} - \#\{1 \leq i \leq |\Gamma| : \Gamma_i = e^{\leftarrow}\}\right]^2 = \mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B).$$

To prove this, note that

$$\mathbf{E}_a\left[\#\{1 \leq i \leq |\Gamma| : \Gamma_i = e\}\right] = \mathscr{G}_B(a, e^-)\frac{c(e)}{c(e^-)}$$

for every $e \in E$ and hence that

$$\mathbf{E}_a\left[\#\{1 \leq i \leq |\Gamma| : \Gamma_i = e\} - \#\{1 \leq i \leq |\Gamma| : \Gamma_i = e^{\leftarrow}\}\right]$$
$$= \mathscr{G}_B(a, e^-)\frac{c(e)}{c(e^-)} - \mathscr{G}_B(a, e^+)\frac{c(e)}{c(e^+)} = \nabla \frac{\mathscr{G}_B(a, \cdot)}{c(\cdot)}.$$

The claim therefore follows from Proposition 2.34 and Lemma 2.8. (There are also several other ways to do this question.) $\square$

Exercise 7 (Triangle inequality)**.** Let G be a finite network. Prove that

$$\mathscr{R}_{\text{eff}}(u \leftrightarrow w) \leq \mathscr{R}_{\text{eff}}(u \leftrightarrow v) + \mathscr{R}_{\text{eff}}(v \leftrightarrow w)$$

for every three distinct $u, v, w \in V$, and deduce that $\mathscr{R}_{\text{eff}}$ defines a metric on V if we set $\mathscr{R}_{\text{eff}}(v \leftrightarrow v) = 0$ for every $v \in V$. Is it true that

$$\mathscr{R}_{\text{eff}}(A \leftrightarrow C) \leq \mathscr{R}_{\text{eff}}(A \leftrightarrow B) + \mathscr{R}_{\text{eff}}(B \leftrightarrow C)$$

for every three disjoint sets of vertices A, B, C?

Solution. The strong Markov property implies that

$$\mathbf{E}_u\tau_w \leq \mathbf{E}_u\tau_v + \mathbf{E}_v\tau_w \qquad \text{and} \qquad \mathbf{E}_w\tau_u \leq \mathbf{E}_v\tau_u + \mathbf{E}_w\tau_v$$

and the claim follows from the commute time identity. The set version of the triangle inequality is not true: If $A = \{0\}$ and $C = \{n\}$ are the endpoints of the line graph with vertex set $\{0, \ldots, n\}$ while $B = \{1, n-1\}$ then $\mathscr{R}_{\text{eff}}(A \leftrightarrow B) = \mathscr{R}_{\text{eff}}(C \leftrightarrow B) = 1$ but $\mathscr{R}_{\text{eff}}(A \leftrightarrow C) = n$. $\square$

Exercise 8. Let $f : \{0, 1, \ldots, \} \to \{0, 1, \ldots\}$ be an increasing function with $f(n+1) - f(n) \leq 1$ for every $n \geq 0$. Prove that the subgraph of $\mathbb{Z}^{d+1}$ induced by the set

$$\left\{(x, y_1, \ldots, y_d) \in \mathbb{Z}^{d+1} : x \geq 0,\ |y_i| \leq f(x)\right\}$$

in transient if and only if $\sum_{n \geq 1}(1 + f(n))^{-d} < \infty$.

Solution. First suppose that $\sum_{n \geq 1}(1 + f(n))^{-d} = \infty$. The set Π_n of edges in the graph that have one endpoint with x-coordinate n and the other with x-coordinate $n+1$ has $(1 + 2f(n))^d$ elements for each $n \geq 0$. Since the sets $(\Pi_n)_{n \geq 0}$ are disjoint cutsets separating 0 from ∞ and

$$\sum_{i=0}^{n} |\Pi_n|^{-1} = \sum_{n \geq 1}(1 + 2f(n))^{-d} = \infty,$$

it follows from the Nash-Williams criterion that the graph is recurrent.

Now suppose that $\sum_{n \geq 1}(1 + f(n))^{-d} < \infty$. Let $U_1, \ldots, U_d$ be i.i.d. Uniform$[0, 1]$ random variables and consider the random set of vertices

$$W = \left\{(x, y_1, \ldots, y_d) \in \mathbb{Z}^{d+1} : x \geq 0 \text{ and } y_i \in [U_i f(x) - 2, U_i f(x)]\right\}.$$

We claim that W is connected almost surely. Indeed, the set $W_n = W \cap \{x = n\}$ is clearly connected for each $n \geq 0$, being a box in $\mathbb{Z}^d$, while the condition that $f(n) \leq f(n+1) \leq f(n)+1$

ensures that the the adjacent pair of vertices

$$(n, \lfloor U_1 f(n) \rfloor, \ldots, \lfloor U_d f(n) \rfloor) \text{ and } (n+1, \lfloor U_1 f(n) \rfloor, \ldots, \lfloor U_d f(n) \rfloor)$$

belong to W_n and W_{n+1} respectively for each $n \geq 0$, so that W is connected as claimed. To prove that the graph is transient, it suffices by Corollary 2.29 to prove that

$$\sum_{v \in \mathbb{Z}^{d+1}} \mathbb{P}(v \in W)^2 < \infty.$$

We do this by a direct computation:

$$\sum_{v \in \mathbb{Z}^{d+1}} \mathbb{P}(v \in W)^2 = \sum_{n \geq 0} \left(\sum_{y=0}^{f(n)} \mathbb{P}(y \in [U_1 f(n) - 2, U_1 f(n)])^2 \right)^d$$
$$\leq \sum_{n \geq 0} \left(\sum_{y=0}^{f(n)} \min\left\{1, \frac{2}{f(n)}\right\}^2 \right)^d = \sum_{n \geq 0} \min\left\{1 + f(n), \frac{4 + 4f(n)}{f(n)^2}\right\}^d,$$

and it is easy to see that this last sum converges if and only if $\sum_{n \geq 0}(1 + f(n))^{-d}$ does. □

Exercise 9. Fix $d \geq 2$, let $n \geq 1$, and consider the torus $\mathbb{Z}^d / n\mathbb{Z}^d$. Prove that if $u, v \in \mathbb{Z}^d / n\mathbb{Z}^d$ then $\mathscr{R}_{\text{eff}}(u \leftrightarrow v)$ is bounded if $d \geq 3$ and that $\mathscr{R}_{\text{eff}}(u \leftrightarrow v)$ is of order $\log(1 + d(u, v))$ if $d = 2$.

Solution. The lower bound is trivial when $d \geq 3$, and follows from the Nash-Williams inequality when $d = 2$: Just take the cut sets to be all the boxes centred on u that separate u and v.

We now give a sketch of how to prove the upper bounds. We will focus on the two-dimensional case, the high-dimensional case being similar. Let's first consider the same problem in $\mathbb{Z}^2$ rather than the torus. Take a random angle $\theta \in [-\pi/4, \pi/4]$ and consider the path in $\mathbb{R}^2$ consisting of two straight line segments, the first of which leaves u with an angle of θ from the line between u and v and ends when it meets the line bisecting the line between u and v, and ends when it meets the perpendicular bisector of this line, while the second segment is the reflection of the first through this perpendicular bisector.

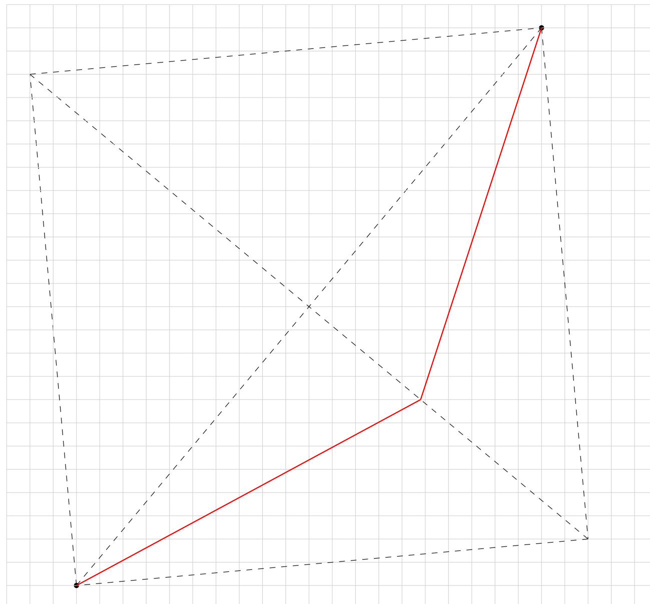
Approximating this random continuous path by a discrete path between u and v and performing a trig calculation similar to the one we did in Theorem 2.30, one can easily prove that $\mathscr{R}_{\text{eff}}(u \leftrightarrow v) = O(\log(1 + d(u, v)))$ as claimed. If you think about it for a minute you will see that there is no obstacle to perfoming this argument on the torus rather than the full plane, as for any u, v we can pick a fundamental domain so that the random path always lies inside this fundamental domain. □

Exercise 10. Let G be a finite network. Prove that $\mathbf{E}_u \tau_v + \mathbf{E}_v \tau_u \geq 2d(u, v)^2$ for every $u, v \in V$.

Solution. Let $d = d(u, v)$. The commute time identity gives that

$$\mathbf{E}_u \tau_v + \mathbf{E}_v \tau_u = 2|E|_c \mathscr{R}_{\text{eff}}(u \leftrightarrow b).$$

For each $1 \leq n \leq d$, let Π_n be the set of edges that have one endpoint at distance $n - 1$ from u and the other at distance n from u. Then $(\Pi_n)_{n=1}^d$ are disjoint cutsets separating u from v,



and it follows from the Nash-Williams inequality that

$$\mathscr{R}_{\mathrm{eff}}(u \leftrightarrow b) \geq \sum_{n=1}^{d} |\Pi_n|^{-1}.$$

Since x^{-1} is convex, Jensen's inequality implies that

$$\frac{1}{d}\sum_{n=1}^{d} |\Pi_n|^{-1} \geq \left[\frac{1}{d}\sum_{n=1}^{d} |\Pi_n|\right]^{-1} \geq d|E|_c^{-1}$$

and it follows that

$$\mathbf{E}_u \tau_v + \mathbf{E}_v \tau_u = 2|E|_c \mathscr{R}_{\mathrm{eff}}(u \leftrightarrow b) \geq 2|E|_c \sum_{n=1}^{d} |\Pi_n|^{-1} \geq 2d^2$$

as claimed. □

Exercise 11. Let G be a finite network, and let $A, B \subseteq V$ be disjoint non-empty sets. Prove that

$$\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B) = \min \left\{ \sum_{u,v \in A} \frac{\mathscr{G}_B(u,v)}{c(v)} \mu(u)\mu(v) : \mu \text{ a probability measure on } A \right\}.$$

State and prove the analogous statement concerning $\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow \infty)$ in infinite networks.

Solution. We will make repeated use of the equality $\mathscr{G}_B(u,v)/c(v) = \mathscr{G}_B(v,u)/c(u)$. For each finite measure μ on A, let

$$I(\mu) = \sum_{u,v \in A} \frac{\mathscr{G}_B(u,v)}{c(v)} \frac{\mu(u)}{\mu(A)} \frac{\mu(v)}{\mu(A)}.$$

We can compute the partial derivative

$$\frac{\partial}{\partial\mu(a)}I(\mu)=\frac{\mathscr{G}_B(a,a)}{c(a)}\frac{\partial}{\partial\mu(a)}\frac{\mu(a)^2}{\mu(A)^2}+2\sum_{v\in V\setminus\{a\}}\frac{\mu(v)\mathscr{G}_B(v,a)}{c(a)}\frac{\partial}{\partial\mu(a)}\frac{\mu(a)}{\mu(A)^2}$$
$$+\sum_{u,v\in V\setminus\{a\}}\frac{\mathscr{G}_B(u,v)}{c(v)}\mu(u)\mu(v)\frac{\partial}{\partial\mu(a)}\frac{1}{\mu(A)^2}$$

and since we have that

$$\frac{\partial}{\partial\mu(a)}\frac{\mu(a)^2}{\mu(A)^2}=\frac{2\mu(a)\mu(A)^2-2\mu(a)^2\mu(A)}{\mu(A)^4},$$
$$\frac{\partial}{\partial\mu(a)}\frac{\mu(a)}{\mu(A)^2}=\frac{\mu(A)^2-2\mu(a)\mu(A)}{\mu(A)^4},\qquad\text{and}$$
$$\frac{\partial}{\partial\mu(a)}\frac{1}{\mu(A)^2}=-\frac{2}{\mu(A)^3},$$

it follows that if μ is a probability measure then

$$\frac{\partial}{\partial\mu(a)}I(\mu)=\frac{\mathscr{G}_B(a,a)}{c(a)}(2\mu(a)-\mu(a)^2)+2\sum_{v\in V\setminus\{a\}}\frac{\mu(v)\mathscr{G}_B(v,a)}{c(a)}(1-2\mu(a))$$
$$-2\sum_{u,v\in V\setminus\{a\}}\frac{\mathscr{G}_B(u,v)}{c(v)}\mu(u)\mu(v).$$

This can also be expressed as

$$\frac{\partial}{\partial\mu(a)}I(\mu)=\frac{\mathscr{G}_B(a,a)}{c(a)}2\mu(a)+2\sum_{v\in V\setminus\{a\}}\frac{\mu(v)\mathscr{G}_B(v,a)}{c(a)}-I(\mu)$$
$$=2\sum_{v\in V}\frac{\mu(v)\mathscr{G}_B(v,a)}{c(a)}-2I(\mu).$$

Thus, if μ_0 is a probability measure such that $I(\mu_0)=\inf_\mu I(\mu)$ then $\frac{\partial}{\partial\mu(a)}I(\mu_0)=0$ for every $a\in A$ with $\mu_0(a)\neq 0$, and we deduce that

$$\sum_{v\in V}\frac{\mu_0(v)\mathscr{G}_B(v,a)}{c(a)}=\sum_{v\in V}\frac{\mu_0(v)\mathscr{G}_B(a,v)}{c(v)}=I(\mu_0)$$

for every $a\in A$ with $\mu_0(a)\neq 0$. Using the equality

$$\mathscr{G}_B(a,v)=\mathbb{1}(v=a)+\mathbf{P}_a(\tau_A^+<\tau_B)\sum_{u\in V}\mathbf{P}_a(X_{\tau_A^+}=u\mid\tau_A^+<\tau_B)\mathscr{G}_B(u,v)$$

we deduce that

$$I(\mu_0) = \sum_{v\in V} \frac{\mu_0(v)\mathscr{G}_B(a,v)}{c(v)} = \frac{\mu_0(a)}{c(a)} + \mathbf{P}_a(\tau_A^+ < \tau_B) \sum_{u\in V} \mathbf{P}_a(X_{\tau_A^+} = u \mid \tau_A^+ < \tau_B) \sum_{v\in V} \frac{\mu_0(v)\mathscr{G}_B(u,v)}{c(v)}$$
$$= \frac{\mu_0(a)}{c(a)} + \mathbf{P}_a(\tau_A^+ < \tau_B) I(\mu_0)$$

for every $a \in A$ with $\mu_0(a) \neq 0$ and hence that

$$\mu_0(a) = c(a)\mathbf{P}_a(\tau_A^+ > \tau_B) I(\mu_0)$$

for every $a \in A$ with $\mu_0(a) \neq 0$. Letting $A' = \{a \in A : \mu_0(a) \neq 0\}$, it follows by summing over A' that

$$I(\mu_0)^{-1} = \sum_{a\in A'} c(a)\mathbf{P}_a(\tau_A^+ > \tau_B) \leq \mathscr{C}_{\mathrm{eff}}(A \leftrightarrow B),$$

and our choice of μ_0 implies that

$$\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B) \leq \min\left\{ \sum_{u,v\in A} \frac{\mathscr{G}_B(u,v)}{c(v)} \mu(u)\mu(v) : \mu \text{ a probability measure on } A \right\}.$$

On the other hand, a straightforward calculation shows that if we define a probability measure on A by

$$\mu_e(a) = \frac{c(a)\mathbf{P}_a(\tau_A^+ > \tau_B)}{\sum_{v\in A} c(v)\mathbf{P}_v(\tau_A^+ > \tau_B)}$$

then $I(\mu_e) = \mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B)$, yielding the reverse inequality

$$\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B) \geq \min\left\{ \sum_{u,v\in A} \frac{\mathscr{G}_B(u,v)}{c(v)} \mu(u)\mu(v) : \mu \text{ a probability measure on } A \right\}.$$

In the infinite case we have that

$$\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow \infty) = \min\left\{ \sum_{u,v\in A} \frac{\mathscr{G}(u,v)}{c(v)} \mu(u)\mu(v) : \mu \text{ a probability measure on } A \right\}.$$

The proof is similar to the finite case. (Alternatively, one may deduce the infinite case from the finite case via a limiting argument.) □