Random walks and uniform spanning trees: Example Sheet 2

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March 17, 2020

The following questions are all based on material from Section 2 of the notes. More difficult questions are marked with a ♠.

Exercise 1. Let G be a finite connected graph drawn in the plane and let $G^\dagger$ be its planar dual. Every edge e of G is crossed by a unique edge $e^\dagger$ of $G^\dagger$, and the map $e \mapsto e^\dagger$ is a bijection between the two edge sets. Let T be a uniform spanning tree of G. Prove that $T^\dagger = \{e^\dagger : e \notin T\}$ is a uniform spanning tree of $G^\dagger$.

Exercise 2. Let G be a finite connected network. Prove that the UST chain is irreducible.

Exercise 3. Complete the proof of Theorem 4.8 by solving exercises 45, 46, and 47 in the notes.

Exercise 4. Prove the strong spatial Markov property for the UST.

Exercise 5 (Dirichlet's principle in infinite volume.)**.** Let G be an infinite, locally finite, connected network, and let A, B be two disjoint, finite sets of vertices. Prove that

$$\mathscr{C}^F_{\text{eff}}(A \leftrightarrow B) = \inf\left\{\|\nabla F\|_r^2 : F|_A = 1, F|_B = 0\right\}$$

and that

$$\mathscr{C}^W_{\text{eff}}(A \leftrightarrow B) = \inf\left\{\|\nabla F\|_r^2 : F|_A = 1, F|_B = 0, F \text{ is finitely supported}\right\}.$$

Exercise 6 (Thompson's principle in infinite volume.)**.** Let G be an infinite, locally finite, connected network, and let A, B be two disjoint, finite sets of vertices. Prove that

$$\mathscr{R}^F_{\text{eff}}(A \leftrightarrow B) = \inf\left\{\|\theta\|_r^2 : \theta \text{ is a finitely-supported unit flow from } A \text{ to } B\right\}$$

and that

$$\mathscr{R}^W_{\text{eff}}(A \leftrightarrow B) = \inf\left\{\|\theta\|_r^2 : \theta \text{ is a unit flow from } A \text{ to } B\right\}.$$

Exercise 7. Give an example of a connected, locally finite graph that admits bounded harmonic functions but does not admit any bounded harmonic functions of finite energy.

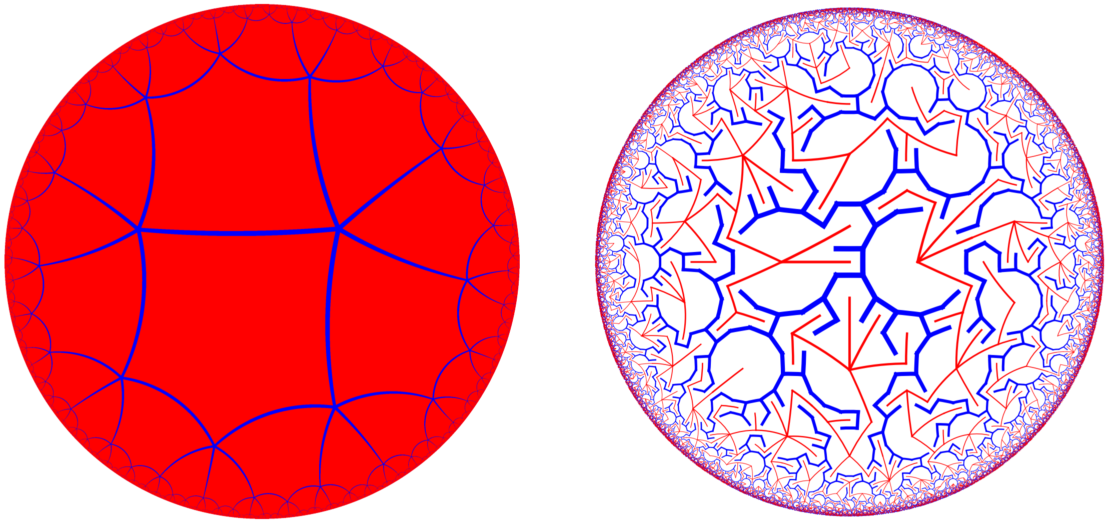

Figure 1: Left: The $(5,5)$ tesselation of the hyperbolic plane in which five pentagons meet at every point. Right: The dual pair of free (blue) and wired (red) uniform spanning forests on another tesselation of the hyperbolic plane, the $(2,3,7)$ tesselation.

Exercise 8. Let G be a transitive amenable graph, and let v be a vertex of G. Prove that the expected degree of v in either the free or wired spanning forest of G is equal to 2, and deduce that G does not admit any non-constant harmonic Dirichlet functions. (Hint: use the fact that every finite tree with n vertices has $n-1$ edges.)

Exercise 9. Prove Corollary 4.23 in the notes. *Hint: (Use Levy's 0-1 law.)*

Exercise 10. Prove that if T is a transient tree then the wired uniform spanning forest of T is not connected almost surely, and deduce that the free and wired uniform spanning forests of T are distinct.

Exercise 11. Let G be an infinite, connected, locally finite network. Let u and v be vertices of G and let X and Y be independent random walks started at u and v. If G is Liouville then

$$\mathbb{P}(|\{X_n : n \geq 0\} \cap \{Y_n : n \geq 0\}| = \infty) \in \{0,1\}$$

and does not depend on the choice of $u, v \in V$.

♠ **Exercise 12.** Let A be an infinite, connected subset of $\mathbb{Z}^3$. Prove that random walk on $\mathbb{Z}^3$ visits A infinitely often almost surely. Deduce that the wired uniform spanning forest of $\mathbb{Z}^3$ is connected almost surely.

Exercise 13. Let G be an infinite, locally finite, connected graph drawn in the plane in such a way that every bounded region of the plane intersects at most finitely many edges of G and such that every face of G contains finitely many edges, so that the planar dual $G^\dagger$ of G is also locally finite. Prove that if F is a wired uniform spanning forest of G then $F^\dagger = \{e^\dagger : e \notin F\}$ is a *free* uniform spanning forest of $G^\dagger$.

Deduce that if G is the graph corresponding of the tiling of the hyperbolic by pentagons in which five pentagons meet at every vertex then the free and wired uniform spanning forests of G are distinct (pictured above).