Random walks and uniform spanning trees: Example Sheet 2

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The following questions are all based on material from Section 2 of the notes. More difficult questions are marked with a ♠.

Exercise 1. Let T and T' be two infinite, bounded degree trees all of whose degrees are at least three. Prove that T and T' are rough-isometric.

Exercise 2. Let T be a k-regular tree (i.e., the unique tree in which every vertex has degree k). Prove that

$$\Phi_E(T) = \frac{k-2}{k} \qquad \text{and} \qquad \rho(T) = \frac{2\sqrt{k-1}}{k} = \sqrt{1-\Phi_E(T)^2}.$$

Exercise 3. Let G be an infinite planar graph all of whose vertex degrees are at least 7. Use Euler's formula to prove that G is nonamenable.

Exercise 4. Let G and G' be connected, bounded-degree, rough-isometric graphs with isoperimetric profiles Φ_E and Φ'_E respectively. Prove that there exist positive constants c and C such that

$$c\Phi_E(Ct) \leq \Phi'_E(t) \leq C\Phi_E(ct)$$

for every $t > 0$. Prove directly that $\rho(G) < 1$ if and only if $\rho(G') < 1$ by using Lemma 3.10.

Exercise 5. Fill in the details in the proof of Theorem 3.21 by completing Exercises 27 and 28 from the notes. (Hint: For Exercise 27, reduce to the finite case and use the spectral theorem.)

Exercise 6. Without appealing to Gromov's theorem, prove that a Cayley graph is recurrent if and only if it satisfies

$$\sum_{r\geq 1} \frac{r}{\mathrm{Gr(r)}} = \infty.$$

Exercise 7. Construct a connected, locally finite, nonamenable graph for which the walk does not have positive speed. (Such a graph must have unbounded degrees.)

Exercise 8. Construct an infinite, connected, bounded degree graph G with a vertex v such that the random walk $(X_n)_{n\geq 0}$ started from v satisfies $d(v, X_n) \leq C \log n$ with high probability for some constant C as $n \to \infty$. Show that no slower rate of growth of the typical displacement is possible in a bounded degree graph.

Exercise 9. Apply the Varopoulos-Carne inequality to prove Corollaries 3.43-3.45 in the notes.

Exercise 10. Construct a connected, locally finite graph G such that the invariant σ-algebra $\mathcal{I}$ is trivial but the tail σ-algebra $\mathcal{T}$ is not.

Exercise 11 (The lamplighter graph.)**.** Let G be a connected, locally finite, simple graph. Let $\mathrm{Lamps}(V)$ be the set of finitely supported functions $\psi : V \to \{0, 1\}$. We define the lamplighter graph $\mathrm{LampLighter}(G)$ to be the graph with vertex set $V \times \mathrm{Lamps}(V)$, and where two vertices (u, ϕ) and (v, ψ) are adjacent if and only if $u \sim v$ or $u = v$ and ϕ and ψ differ only at u. We interpret (u, ϕ) as describing the configuration of a collection of lamps, one at each vertex, together with the location of a lamplighter, who is at u. At each time step, the lamplighter may either move to a location adjacent to their current location or change the status of the lamp at its current location.

1. Prove that if G is transitive, then $\mathrm{LampLighter}(G)$ is transitive.
2. Prove that if G is infinite then $\mathrm{LampLighter}(G)$ has exponential growth.
3. Prove that if G is transient then $\mathrm{LampLighter}(G)$ has non-trivial invariant σ-algebra.
4. Prove that $\mathrm{LampLighter}(\mathbb{Z}^d)$ is Liouville if and only if $d \leq 2$.

♠ **Exercise 12.** Construct an example to prove that the Liouville property is *not* preserved by rough isometry between connected, bounded degree graphs.

♠♠ **Exercise 13.** Construct a connected, bounded degree, nonamenable graph that has the Liouville property.

Exercise 14. We say that a function $\phi : \mathbb{Z}^d \to \mathbb{R}$ has **sublinear growth** if

$$\limsup_{x\to\infty} \frac{|\phi(x)|}{\|x\|} = 0.$$

Prove that if x and y are two vertices of $\mathbb{Z}^d$, then there exists a random variable $(Z_n)_{n\geq 0} = ((X_n, Y_n))_{n\geq 0}$ and a random time T such that $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ are both lazy simple random walks on $\mathbb{Z}^d$, $X_n = Y_n$ for every $n \geq T$, and there exists a constant C_{xy} such that $\mathbb{P}(T \geq n) \leq C_{xy} n^{-1}$ for every $n \geq 1$. Deduce that if $\phi : \mathbb{Z}^d \to \mathbb{R}$ is harmonic and has sublinear growth then it is constant.