Random walks and uniform spanning trees: Example Sheet 1

Tom Hutchcroft

Statslab, DPMMS, University of Cambridge.
Email: t.hutchcroft@maths.cam.ac.uk

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The following questions are all based on material from Section 2 of the notes. More difficult questions are marked with a ♠.

Exercise 1 (The Poisson equation)**.** Let G be a finite, connected network, and let A be a non-empty subset of V. Let $\varphi : A \to \mathbb{R}$ and $\psi : V \setminus A \to \mathbb{R}$ be arbitrary functions. Prove that there exists a unique function $F : V \to \mathbb{R}$ such that

$$F|_A \equiv \varphi \qquad \text{and} \qquad (\Delta F)|_{V\setminus A} \equiv \psi.$$

How is this related to hitting times?

Exercise 2. Prove that if $h : V \to \mathbb{R}$ is harmonic on $V \setminus A$ for some $A \subseteq V$ and $X = (X_n)_{n\geq 0}$ is a random walk on G started at some vertex v then $(h(X_{n\wedge\tau_A}))_{n\geq 0}$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ generated by X. Use the optional stopping theorem to give an alternative proof of the uniqueness of solutions to the Dirichlet problem on finite networks.

Exercise 3. Prove Lemma 2.9 from the notes.

Exercise 4. Prove Lemma 2.14 and Theorem 2.15 from the notes.

♠ **Exercise 5.** Prove or provide a counterexample to the *converse* of the Nash-Williams criterion: Is it true that if G is an infinite, connected, recurrent network and a is a vertex of G, then there exists a sequence of disjoint cutsets $(\Pi_i)_{i\geq 1}$ separating a from ∞ such that $\sum_{i=1}^{\infty} |\Pi_i|_c^{-1} = \infty$?

Exercise 6. Prove the reverse inequality in the method of random paths.

Exercise 7 (Triangle inequality)**.** Let G be a finite network. Prove that

$$\mathscr{R}_{\mathrm{eff}}(u \leftrightarrow w) \leq \mathscr{R}_{\mathrm{eff}}(u \leftrightarrow v) + \mathscr{R}_{\mathrm{eff}}(v \leftrightarrow w)$$

for every three distinct $u, v, w \in V$, and deduce that $\mathscr{R}_{\mathrm{eff}}$ defines a metric on V if we set $\mathscr{R}_{\mathrm{eff}}(v \leftrightarrow v) = 0$ for every $v \in V$. Is it true that

$$\mathscr{R}_{\mathrm{eff}}(A \leftrightarrow C) \leq \mathscr{R}_{\mathrm{eff}}(A \leftrightarrow B) + \mathscr{R}_{\mathrm{eff}}(B \leftrightarrow C)$$

for every three disjoint sets of vertices A, B, C?

Exercise 8. Let $f : \{0, 1, \ldots,\} \to \{0, 1, \ldots\}$ be an increasing function with $f(n+1)-f(n) \leq 1$ for every $n \geq 0$. Prove that the subgraph of $\mathbb{Z}^{d+1}$ induced by the set

$$\big\{(x, y_1, \ldots, y_d) \in \mathbb{Z}^{d+1} : x \geq 0,\, |y_i| \leq f(x)\big\}$$

in transient if and only if $\sum_{n \geq 1} f(n)^{-d} < \infty$.

Exercise 9. Fix $d \geq 2$, let $n \geq 1$, and consider the torus $\mathbb{Z}^d/n\mathbb{Z}^d$. Prove that if $u, v \in \mathbb{Z}^d/n\mathbb{Z}^d$ then $\mathscr{R}_{\text{eff}}(u \leftrightarrow v)$ is bounded if $d \geq 3$ and that $\mathscr{R}_{\text{eff}}(u \leftrightarrow v)$ is of order $\log(1 + d(u, v))$ if $d = 2$.

Exercise 10. Let G be a finite network. Prove that $\mathbf{E}_u\tau_v + \mathbf{E}_v\tau_u \geq 2d(u, v)^2$ for every $u, v \in V$.

Exercise 11. Let G be a finite network, and let $A, B \subseteq V$ be disjoint non-empty sets. Prove that

$$\mathscr{R}_{\text{eff}}(A \leftrightarrow B) = \min\left\{\sum_{u,v\in A} \frac{\mathscr{G}_B(u, v)}{c(v)}\mu(u)\mu(v) : \mu \text{ a probability measure on } A\right\}.$$

State and prove the analogous statement concerning $\mathscr{R}_{\text{eff}}(A \leftrightarrow \infty)$ in infinite networks.

♠ **Exercise 12.** Let $d \geq 3$, and consider the hypercubic lattice $\mathbb{Z}^d$. Prove that there exists a positive constant c such that

$$\mathscr{C}_{\text{eff}}(A \leftrightarrow \infty) \geq c|A|^{(d-2)/d}$$

for every finite set $A \subseteq \mathbb{Z}^d$. Show that the exponent $(d-2)/d$ cannot be improved.

♠ **Exercise 13** (Cover times)**.** Let G be a finite connected network, let X_n be a random walk on G started at some vertex v, and let the **cover time** $T_{\text{cov}} = \inf\{n \geq 0 : \{X_i : 0 \leq i \leq n\} = V\}$ be the first time that the walk has visited every vertex.

1. Prove that

$$\min\big\{\mathbf{E}_a\tau_b : a, b \in V \text{ distinct}\big\} \sum_{i=1}^{|V|-1} \frac{1}{i} \leq \mathbf{E}_v\left[T_{\text{cov}}\right] \leq \max\big\{\mathbf{E}_a\tau_b : a, b \in V\big\} \sum_{i=1}^{|V|-1} \frac{1}{i}.$$

   (Hint: The bound $\mathbf{E}_v\left[T_{\text{cov}}\right] \leq \max\big\{\mathbf{E}_a\tau_b : a, b \in V\big\}(|V| - 1)$ is very easy, but where on earth are these harmonic series supposed to come from...?)

2. Prove that if $d \geq 3$ and $G = \mathbb{Z}^d/n\mathbb{Z}^d$ is the d-dimensional torus with n^d vertices then there exist positive constants $c = c_d$ and $C = C_d$ such that $cn^d \log n \leq \mathbf{E}_v[T_{\text{cov}}] \leq Cn^d \log n$ for every $n \geq 2$.

3. Prove that if $d = 2$ and $G = \mathbb{Z}^2/n\mathbb{Z}^2$ is the two-dimensional torus with n^2 vertices then there exist positive constants c and C such that $cn^2(\log n)^2 \leq \mathbf{E}_v[T_{\text{cov}}] \leq Cn^2(\log n)^2$ for every $n \geq 2$. (Hint: The lower bound from part 1 will not be sufficient. How might we improve this bound?)

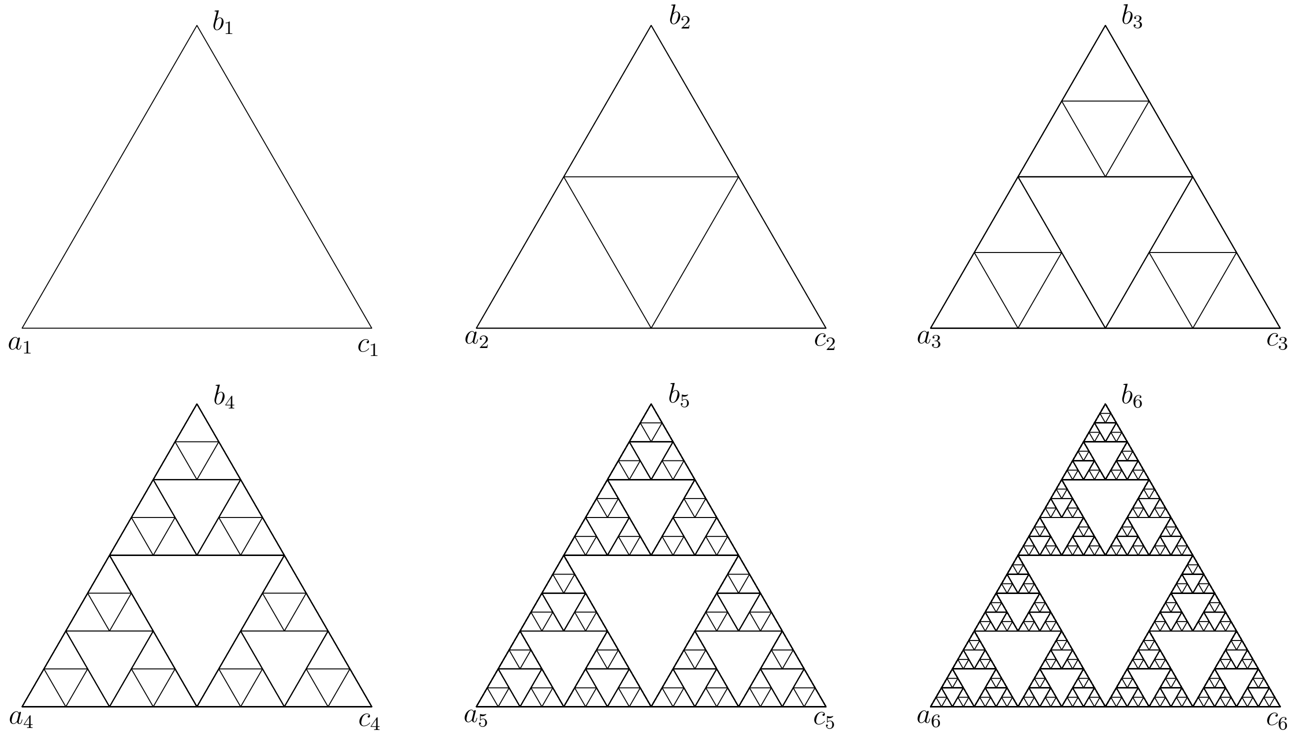

Figure 1: The first six graphical Sierpinski gaskets.

♠ **Exercise 14** (The Sierpinski Gasket)**.** In this question we will begin to analyse the simple random walk on a graphical version of the Sierpinski gasket. We define a sequence of graphs as follows: G_1 is a cycle of length three (i.e., a triangle), which has three vertices a_1, b_1, c_1. For each $n \geq 1$, we form G_{n+1} from three disjoint copies of G_n by identifying their distinguished corner vertices in the manner depicted in the figure. We will return to this example in later exercises.

1. (Volume growth) Let $n \geq 1$. Prove that there exist positive constants c and C such that

$$cm^{\log 3/\log 2} \leq \#B(v,m) \leq Cm^{\log 3/\log 2}$$

for every $v \in V_n$ and $1 \leq m \leq 2^n$. (This is related to the fact that the true Sierpinski gasket has Hausdorff dimension $\log 3/\log 2$.)

2. (Resistance growth) Let $n \geq 1$. Prove that there exist positive constants c and C such that

$$cm^{\log(5/3)/\log 2} \leq \mathscr{R}_{\mathrm{eff}}(v \leftrightarrow \partial B(v,m); G_n) \leq Cm^{\log(5/3)/\log 2}$$

for every $v \in V_n$ and $1 \leq m \leq 2^{n-1}$. Hint: first consider the effective resistance between a_n and b_n.