This on-line version contains a proof of the extended omitting types theorem which is omitted (Ha!) from the published version.

We start by recapitulating Yablo’s paradox from [1].

We have infinitely many assertions \( \{p_i : i \in \mathbb{N}\} \) and each \( p_i \) is equivalent to the assertion that all subsequent \( p_j \) are false. A contradiction follows.

There is a wealth of literature on this delightful puzzle, and I have been guilty of a minor contribution to it myself. This literature places Yablo’s paradox in the semantical column of Ramsey’s division of the paradoxes into semantical versus logical paradoxes. However—as I hope to show below—there is merit to be gained by regarding it as a purely logical puzzle.

Yablo’s Paradox in Propositional Logic

If we are to treat Yablo’s paradox as a purely logical puzzle we should try to capture it entirely within a first-order language with no special predicates. In fact we can even make progress while using nothing more than a propositional language; the obvious language \( \mathcal{L} \) to use has infinitely many propositional letters \( \{p_i : i \in \mathbb{N}\} \). Next we want a propositional theory with axioms

\[
p_i \leftrightarrow \bigwedge_{j > i} \neg p_j
\]

for each \( i \in \mathbb{N} \),

...but of course we cannot do this in a finitary language. However, one thing we can do in a finitary language is capture the left-to-right direction of these biconditionals, and we do that with the simple scheme

\[
p_i \rightarrow \neg p_j
\]

for all \( i < j \in \mathbb{N} \).

It can be seen that this is equivalent to the even simpler scheme
\[ \neg p_i \lor \neg p_j \quad (3) \]

for all \( i \neq j \in \mathbb{N} \).

Let us call this theory \( Y \). \( Y \) says that at most one \( p_i \) can be true.

It is the right-to-left direction of the biconditionals that gives us trouble . . .

\[
(\bigwedge_{j>i} \neg p_j) \rightarrow p_i \quad (4)
\]

for each \( i \in \mathbb{N} \).

For each \( i \) the right-to-left direction of the \( i \)th biconditional (4) asserts that at least one of the formulæ in the set \( \Sigma(i) \) is false:

\[
\{ \neg p_j : j \geq i \} \quad (\Sigma(i))
\]

\( \Sigma(i) \) is an example of what model theorists call a 0-type, a type being nothing more than a set of formulae\(^1\). The ‘0’ means that the formulæ in the type have no free variables. Our desire that at least one thing in a type should be false is—in the terminology of model theory—a desire to omit that type. What we need is a theorem that tells us that a theory can have models that omit a specified type. There is a such a theorem, and it is known as the Omitting Types Theorem. We say a theory \( T \) in a language \( \mathcal{L} \) locally omits a type \( \Sigma \) if, whenever \( \phi \in \mathcal{L} \) is a formula such that \( T \) proves \( \phi \rightarrow \sigma \) for every \( \sigma \in \Sigma \), then \( T \vdash \neg \phi \). The omitting types theorem for propositional languages now says:

**Theorem 1:**

Let \( T \) be a consistent theory in a propositional language \( \mathcal{L} \). If \( T \) locally omits a type \( \Sigma \) then there is an \( \mathcal{L} \)-valuation \( v \) that satisfies every theorem of \( T \) but falsifies at least one \( \sigma \in \Sigma \).

We say in these circumstances that \( v \) omits \( \Sigma \).

However, what we need here is the slightly stronger:

**Theorem 2:** (Extended Omitting Types Theorem)

Let \( T \) be a consistent theory in a propositional language \( \mathcal{L} \). If \( T \) locally omits each type \( \Sigma \) in a countable class \( \mathcal{G} \) of types then there is an \( \mathcal{L} \)-valuation that satisfies every theorem of \( T \) but, for each \( \Sigma \in \mathcal{G} \), falsifies at least one \( \sigma \in \Sigma \).

I will omit a proof of this result, since it is standard in the model-theoretic literature.

\( \quad 1 \) A countably infinite set unless otherwise specified.
In asserting the right-to-left directions (4) of the biconditionals we are restricting ourselves to \( L \)-valuations that omit all the types \( \Sigma(i) \). There are countably many of these types so it would be natural to reach for the extended omitting types theorem, theorem 2. Now if we are to exploit theorem 2 we want our theory \( Y \) to locally omit each \( \Sigma(i) \). But it doesn’t. The formula \( p_0 \), in conjunction with the axioms of \( Y \), implies \( \neg p_i \) for every \( i > 0 \) and thereby implies everything in \( \Sigma(1) \). If \( Y \) were to locally omit \( \Sigma(1) \) as we desire then we would have to have \( Y \vdash \neg p_0 \). But \( Y \) clearly does not prove \( \neg p_0 \). If we were to add \( \neg p_0 \) as part of a project of adding axioms to \( Y \) to obtain a theory that did omit \( \Sigma(1) \) we would find by the same token that we would have to add \( \neg p_i \) for all other \( i \in \mathbb{N} \) as well, and then we end up realising all the \( \Sigma(i) \).

Thus \( Y \) does not locally omit even one of the \( \Sigma(i) \), let alone all of them. So we cannot invoke theorem 2. However, for each \( i \) the valuation that makes \( p_i \) true and everything else false satisfies \( Y \) all right, and it omits all \( \Sigma(j) \) for all \( j < i \). This illustrates how a theory \( T \) can sometimes have a model that omits a type \( \Sigma \) even though \( T \) does not locally omit \( \Sigma \).

Very well: for each \( i \) there is an \( L \)-valuation that satisfies \( Y \) and omits \( \Sigma(j) \) for all \( j < i \). Can we find a \( L \)-valuation that satisfies \( Y \) and omits all the \( \Sigma(i) \)? No! Such a valuation would satisfy all the right-to-left directions of the biconditionals in (1), namely the conditionals in (4) and thereby manifest Yablo’s paradox!

**Conclusion**

Yablo’s paradox provides us with an illustration of a setting where there is a theory \( Y \) and an infinite family \( \{ \Sigma(i) : i \in \mathbb{N} \} \) of types where, although \( Y \) does not locally omit any of the \( \Sigma(i) \), it nevertheless has valuations that omit any finite set of them. Further, it has no valuation that omits them all. That last fact illustrates how the condition in theorem 2—namely that \( T \) locally omit every \( \Sigma \in \mathcal{S} \)—really is necessary, so the extended omitting types theorem for propositional logic really is best possible.

For \( T \) to have a model omitting all the \( \Sigma_i \) it is not sufficient for it to have models omitting any given finite family of them; we really do need the stronger condition that \( T \) should locally omit every finite subset of \( \Sigma_i \).

It illustrates that for \( T \) to have a model that omits a type \( \Sigma \) is is sufficient but not necessary for \( T \) to locally omit \( \Sigma \).

This is pedagogically quite instructive!
Appendix: A Proof of the Extended Omitting Types theorem for Propositional Logic

I supply a proof of this fact in this on-line version of the paper because—despite the cheerful observation above that the result is standard in the literature—I cannot actually find a proof anywhere!

A 0-type (Hereafter merely ‘type’: the ‘0’ means that the formulæ in the type have no free variables) in a propositional language $L$ is a set of formulæ (a countably infinite set unless otherwise specified).

For $T$ an $L$-theory a $T$-valuation is an $L$-valuation that satisfies $T$. A valuation $v$ realises a type $\Sigma$ if $v(\sigma) = \text{true}$ for every $\sigma \in \Sigma$. Otherwise $v$ omits $\Sigma$.

We say a theory $T$ locally omits a type $\Sigma$ if, whenever $\phi$ is a formula such that $T$ proves $\phi \rightarrow \sigma$ for every $\sigma \in \Sigma$, then $T \vdash \neg \phi$.

**Theorem 1** The Omitting Types Theorem for Propositional Logic

Let $T$ be a propositional theory, and $\Sigma \subseteq L(T)$ a type. If $T$ locally omits $\Sigma$ then there is a $T$-valuation omitting $\Sigma$.

**Proof:**

By contraposition. Suppose there is no $T$-valuation omitting $\Sigma$. Then every formula in $\Sigma$ is a theorem of $T$ so there is an expression $\phi$ (namely ‘$\top$’) such that $T \vdash \phi \rightarrow \sigma$ for every $\sigma \in \Sigma$ but $T \nvdash \neg \phi$. Contraposing, we infer that if $T \vdash \neg \phi$ for every $\phi$ such that $T \vdash \phi \rightarrow \sigma$ for every $\sigma \in \Sigma$ then there is a $T$-valuation omitting $\Sigma$.

However, we can prove something stronger.

**Theorem 2** The Extended Omitting Types Theorem for Propositional Logic

Let $T$ be a propositional theory and, for each $i \in \mathbb{N}$, let $\Sigma_i \subseteq L(T)$ be a type. If $T$ locally omits every $\Sigma_i$ then there is a $T$-valuation omitting all of the $\Sigma_i$.

**Proof:**

We will show that whenever $T \cup \{\neg \phi_1, \ldots, \neg \phi_i\}$ is consistent, where $\phi_n \in \Sigma_n$ for each $n \leq i$, then we can find $\phi_{i+1} \in \Sigma_{i+1}$ such that $T \cup \{\neg \phi_1, \ldots, \neg \phi_i, \neg \phi_{i+1}\}$ is consistent.

Suppose not, then $T \vdash \bigwedge_{1 \leq j \leq i} \neg \phi_j \rightarrow \phi_{i+1}$ for every $\phi_{i+1} \in \Sigma_{i+1}$. But, by assumption, $T$ locally omits $\Sigma_{i+1}$, so we would have $T \vdash \neg \bigwedge_{1 \leq j \leq i} \neg \phi_j$ contradicting the assumption that $T \cup \{\neg \phi_1, \ldots, \neg \phi_i\}$ is consistent.

Now, as long as there is an enumeration of the formulæ in $L(T)$, we can run an iterative process where at each stage we pick for $\phi_{i+1}$ the first formula in $\Sigma_{i+1}$ such that $T \cup \{\neg \phi_1, \ldots, \neg \phi_i, \neg \phi_{i+1}\}$ is consistent. This gives us a theory $T \cup \{\neg \phi_i : i \in \mathbb{N}\}$ which is consistent by compactness. Any model of $T \cup \{\neg \phi_i : i \in \mathbb{N}\}$ is a model of $T$ that omits each $\Sigma_i$. 

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Observe that in this above result we do not construct an actual valuation of $T$. What we construct is an extension $T'$ of $T$ with the property that any valuation makes $T'$ true must omit all the types $\Sigma_i$. This sounds less useful but it is actually more, for it means that we have actually proved the omitting types theorem for 1-types in LPC as well. The enhancements that follow (using the concept of “locally-$\infty$-omits) are not known to generalise to first-order logic.

If $\Sigma = \{\sigma_i : i \in \mathbb{N}\}$ is a [countable] type, then, for each $n \in \mathbb{N}$, the type $\{\sigma_j : j > i\}$ is a terminal segment of $\Sigma$.

We say a theory $T$ locally-$\infty$-omits a type $\Sigma$ if, whenever $\phi$ is a formula such that $T$ proves $\phi \to \sigma$ for all but finitely many $\sigma \in \Sigma$, then $T \vdash \neg \phi$. If a valuation $v$ omits every terminal segment of $\Sigma$ we say $v$ inf-omits $\Sigma$.

**Corollary 1** Let $T$ be a propositional theory, and $\Sigma \subseteq \mathcal{L}(T)$ a type. If $T$ locally-$\infty$-omits $\Sigma$ then there is a $T$-valuation $\infty$-omitting $\Sigma$.

**Proof:**
We obtain this from the extended omitting types theorem by thinking of the family of terminal segments of $\Sigma$ as the family $\Sigma_n$ in the statement of the extended omitting types theorem.

I am skiving out of proving this since there is something stronger that I really am going to prove...
How can we be sure that such a $\phi_i$ can always be found? Here we argue much as in the case of the original extended omitting types theorem. Suppose at stage $i$ we could not find a $\phi_i$ such that $T \cup \{\neg \phi_0, \ldots, \neg \phi_i\}$ is consistent, and let $i$ be minimal with this undesirable feature. Then $T \vdash (\bigwedge_{1 \leq j \leq i-1} \neg \phi_j) \rightarrow \phi_i$ for every candidate $\phi_i \in \Delta_i$. Now $\Delta_i$ is a cofinite subset of $\Sigma_i$ and, by assumption, $T$ inf-locally omits $\Sigma_i$, so we would have $T \vdash \neg \bigwedge_{1 \leq j \leq i} \neg \phi_j$ contradicting the assumption that $T \cup \{\neg \phi_1, \ldots, \neg \phi_i\}$ is consistent.

This gives us a theory $T \cup \{\neg \phi_i : i \in \mathbb{N}\}$ which is consistent by compactness. Any model of $T \cup \{\neg \phi_i : i \in \mathbb{N}\}$ is a model of $T$ that $\infty$-omits each $\Sigma_i$.

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Now, as long as there is an enumeration of the formulae in $\mathcal{L}(T)$, we can run an iterative process where at each stage we pick for $\phi_{i+1}$ the first formula in $\Sigma_{i+1}$ such that $T \cup \{\neg \phi_1, \ldots, \neg \phi_i, \neg \phi_{i+1}\}$ is consistent.

Coda

Since I wrote out this proof some of my enquiries have borne fruit, and I am much indebted to Oren Kolmen for directing me to pp 118–9 of [?].

References


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