Yablo’s Paradox and the Omitting Types
Theorem for Propositional Languages

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This on-line version contains a proof of the extended omitting types theorem which is omitted (Ha!) from the published version.

We start by recapitulating Yablo’s paradox from [1]. We have infinitely many assertions \( \{p_i : i \in \mathbb{N}\} \) and each \( p_i \) is equivalent to the assertion that all subsequent \( p_j \) are false. A contradiction follows.

There is a wealth of literature on this delightful puzzle, and I have been guilty of a minor contribution to it myself. This literature places Yablo’s paradox in the semantical column of Ramsey’s division of the paradoxes into semantical versus logical paradoxes. However—as I hope to show below—there is merit to be gained by regarding it as a purely logical puzzle.

Yablo’s Paradox in Propositional Logic

If we are to treat Yablo’s paradox as a purely logical puzzle we should try to capture it entirely within a first-order language with no special predicates. In fact we can even make progress while using nothing more than a propositional language; the obvious language \( L \) to use has infinitely many propositional letters \( \{p_i : i \in \mathbb{N}\} \). Next we want a propositional theory with axioms

\[
p_i \leftrightarrow \bigwedge_{j>i} \neg p_j \tag{1}
\]

for each \( i \in \mathbb{N} \),

... but of course we cannot do this in a finitary language. However, one thing we can do in a finitary language is capture the left-to-right direction of these biconditionals, and we do that with the simple scheme

\[
p_i \rightarrow \neg p_j \tag{2}
\]

for all \( i < j \in \mathbb{N} \).

It can be seen that this is equivalent to the even simpler scheme.
\[ \neg p_i \vee \neg p_j \] (3)

for all \( i \neq j \in \mathbb{N} \).

Let us call this theory \( Y \). \( Y \) says that at most one \( p_i \) can be true.

It is the right-to-left direction of the biconditionals that gives us trouble \( \ldots \)

\[ (\bigwedge_{j>i} \neg p_j) \rightarrow p_i \] (4)

for each \( i \in \mathbb{N} \).

For each \( i \) the right-to-left direction of the \( i \)th biconditional (4) asserts that at least one of the formulæ in the set \( \Sigma(i) \) is false:

\[ \{ \neg p_j : j \geq i \} \] (\( \Sigma(i) \))

\( \Sigma(i) \) is an example of what model theorists call a 0-type, a type being nothing more than a set of formulae\(^1\). The ‘0’ means that the formulæ in the type have no free variables. Our desire that at least one thing in a type should be false is—in the terminology of model theory—a desire to omit that type. What we need is a theorem that tells us that a theory can have models that omit a specified type. There is a such a theorem, and it is known as the Omitting Types Theorem. We say a theory \( T \) in a language \( \mathcal{L} \) locally omits a type \( \Sigma \) if, whenever \( \phi \in \mathcal{L} \) is a formula such that \( T \) proves \( \phi \rightarrow \sigma \) for every \( \sigma \in \Sigma \), then \( T \vdash \neg \phi \). The omitting types theorem for propositional languages now says:

**Theorem 1:**

Let \( T \) be a consistent theory in a propositional language \( \mathcal{L} \). If \( T \) locally omits a type \( \Sigma \) then there is an \( \mathcal{L} \)-valuation \( v \) that satisfies every theorem of \( T \) but falsifies at least one \( \sigma \) in \( \Sigma \).

We say in these circumstances that \( v \) omits \( \Sigma \).

However, what we need here is the slightly stronger:

**Theorem 2:** (Extended Omitting Types Theorem)

Let \( T \) be a consistent theory in a propositional language \( \mathcal{L} \). If \( T \) locally omits each type \( \Sigma \) in a countable class \( \mathcal{S} \) of types then there is an \( \mathcal{L} \)-valuation that satisfies every theorem of \( T \) but, for each \( \Sigma \in \mathcal{S} \), falsifies at least one \( \sigma \) in \( \Sigma \).

I will omit a proof of this result, since it is standard in the model-theoretic literature.

\(^1\)A countably infinite set unless otherwise specified.
In asserting the right-to-left directions (4) of the biconditionals we are restricting ourselves to $L$-valuations that omit all the types $\Sigma(i)$. There are countably many of these types so it would be natural to reach for the extended omitting types theorem, theorem 2. Now if we are to exploit theorem 2 we want our theory $Y$ to locally omit each $\Sigma(i)$. But it doesn’t. The formula $p_0$, in conjunction with the axioms of $Y$, implies $\neg p_i$ for every $i > 0$ and thereby implies everything in $\Sigma(1)$. If $Y$ were to locally omit $\Sigma(1)$ as we desire then we would have to have $Y \vdash \neg p_0$. But $Y$ clearly does not prove $\neg p_0$. If we were to add $\neg p_0$ as part of a project of adding axioms to $Y$ to obtain a theory that did omit $\Sigma(1)$ we would find by the same token that we would have to add $\neg p_i$ for all other $i \in \mathbb{N}$ as well, and then we end up realising all the $\Sigma(i)$.

Thus $Y$ does not locally omit even one of the $\Sigma(i)$, let alone all of them. So we cannot invoke theorem 2. However, for each $i$ the valuation that makes $p_i$ true and everything else false satisfies $Y$ all right, and it omits all $\Sigma(j)$ for all $j < i$. This illustrates how a theory $T$ can sometimes have a model that omits a type $\Sigma$ even though $T$ does not locally omit $\Sigma$.

Very well: for each $i$ there is an $L$-valuation that satisfies $Y$ and omits $\Sigma(j)$ for all $j < i$. Can we find a $L$-valuation that satisfies $Y$ and omits all the $\Sigma(i)$? No! Such a valuation would satisfy all the right-to-left directions of the biconditionals in (1), namely the conditionals in (4) and thereby manifest Yablo’s paradox!

**Conclusion**

Yablo’s paradox provides us with an illustration of a setting where there is a theory $Y$ and an infinite family $\{\Sigma(i) : i \in \mathbb{N}\}$ of types where, although $Y$ does not locally omit any of the $\Sigma(i)$, it nevertheless has valuations that omit any finite set of them. Further, it has no valuation that omits them all. That last fact illustrates how the condition in theorem 2—namely that $T$ locally omit every $\Sigma \in \mathcal{S}$—really is necessary, so the extended omitting types theorem for propositional logic really is best possible.

For $T$ to have a model omitting all the $\Sigma_i$ it is not sufficient for it to have models omitting any given finite family of them; we really do need the stronger condition that $T$ should locally omit every finite subset of $\Sigma_i$.

It illustrates that for $T$ to have a model that omits a type $\Sigma$ is is sufficient but not necessary for $T$ to locally omit $\Sigma$.

This is pedagogically quite instructive!
Appendix: A Proof of the Extended Omitting Types theorem for Propositional Logic

I supply a proof of this fact in this on-line version of the paper because—despite the cheerful observation above that the result is standard in the literature—I cannot actually find a proof anywhere!

A 0-type (Hereafter merely 'type': the '0' means that the formulæ in the type have no free variables) in a propositional language \( \mathcal{L} \) is a set of formulæ (a countably infinite set unless otherwise specified).

For \( T \) an \( \mathcal{L} \)-theory a \( T \)-valuation is an \( \mathcal{L} \)-valuation that satisfies \( T \). A valuation \( v \) realises a type \( \Sigma \) if \( v(\sigma) = \text{true} \) for every \( \sigma \in \Sigma \). Otherwise \( v \) omits \( \Sigma \). We say a theory \( T \) locally omits a type \( \Sigma \) if, whenever \( \phi \) is a formula such that \( T \) proves \( \phi \rightarrow \sigma \) for every \( \sigma \in \Sigma \), then \( T \vdash \neg \phi \).

**Theorem 1** The Omitting Types Theorem for Propositional Logic

Let \( T \) be a propositional theory, and \( \Sigma \subseteq \mathcal{L}(T) \) a type. If \( T \) locally omits \( \Sigma \) then there is a \( T \)-valuation omitting \( \Sigma \).

**Proof:**

By contraposition. Suppose there is no \( T \)-valuation omitting \( \Sigma \). Then every formula in \( \Sigma \) is a theorem of \( T \) so there is an expression \( \phi \) (namely ‘\( \top \)’) such that \( T \vdash \phi \rightarrow \sigma \) for every \( \sigma \in \Sigma \) but \( T \nvdash \neg \phi \). Contraposing, we infer that if \( T \vdash \neg \phi \) for every \( \phi \) such that \( T \vdash \phi \rightarrow \sigma \) for every \( \sigma \in \Sigma \) then there is a \( T \)-valuation omitting \( \Sigma \).

However, we can prove something stronger. **Theorem 2** The Extended Omitting Types Theorem for Propositional Logic

Let \( T \) be a propositional theory and, for each \( i \in \mathbb{N} \), let \( \Sigma_i \subseteq \mathcal{L}(T) \) be a type. If \( T \) locally omits every \( \Sigma_i \) then there is a \( T \)-valuation omitting all of the \( \Sigma_i \).

**Proof:**

We will show that whenever \( T \cup \{\neg \phi_1, \ldots, \neg \phi_i\} \) is consistent, where \( \phi_n \in \Sigma_n \) for each \( n \leq i \), then we can find \( \phi_{i+1} \in \Sigma_{i+1} \) such that \( T \cup \{\neg \phi_1, \ldots, \neg \phi_i, \neg \phi_{i+1}\} \) is consistent.

Suppose not, then \( T \vdash (\bigwedge_{1 \leq j \leq i} \neg \phi_j) \rightarrow \phi_{i+1} \) for every \( \phi_{i+1} \in \Sigma_{i+1} \). But, by assumption, \( T \) locally omits \( \Sigma_{i+1} \), so we would have \( T \vdash \neg (\bigwedge_{1 \leq j \leq i} \neg \phi_j) \) contradicting the assumption that \( T \cup \{\neg \phi_1, \ldots, \neg \phi_i\} \) is consistent.

Now, as long as there is an enumeration of the formulæ in \( \mathcal{L}(T) \), we can run an iterative process where at each stage we pick for \( \phi_{i+1} \) the first formula in \( \Sigma_{i+1} \) such that \( T \cup \{\neg \phi_1, \ldots, \neg \phi_i, \neg \phi_{i+1}\} \) is consistent. This gives us a theory \( T \cup \{\neg \phi_i : i \in \mathbb{N}\} \) which is consistent by compactness. Any model of \( T \cup \{\neg \phi_i : i \in \mathbb{N}\} \) is a model of \( T \) that omits each \( \Sigma_i \).

\( \blacksquare \)
References