Stratification mod n

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1 Introduction and Summary

Recently Zuhair Abdul Ghafoor Al-Johar [12] has directed our attention to a syntactic constraint that is—on the face of it—*tighter* than NF's device of stratification[1]; in this little essay I consider a *weakening*, namely the generalisation of stratification to *stratification mod n*. So far the coterie of NFistes has considered neither the possibility that the class of unstratified formulæ in the language of set theory might admit any structure or gradation, nor the possibility that failure-of-stratification (which perhaps we can call *dysstratification*) might come in degrees, nor the possibility that recognition of such degrees might allow one to gain understanding and prove useful facts.

So stratification-mod-n opens a new vein, but not one i've been able to get anything really substantial out of. Not so far, anyway . . . mostly just simple-minded generalisations of the standard stratified case—not that those are without merit, since

[1]Tho' recent work of Nathan Bowler seems to establish that every stratifiable formula is equivalent (modulo some very minor set-theoretic assumptions) to an acyclic formula.

they prepare the ground for subsequent work. It has to be admitted that stratification-mod-n comes across as a highly artificial notion, of interest only to those whose critical faculties have been weakened by prior exposure to the idea of stratification. However there is a nontrivial result that makes essential use of this notion, and we will see it in section 6 where i show (theorem 1) that—for NF—duality for formulæ that are stratifiable-mod-2 is consistent relative to AC_2. Altho' I do not believe that this result is best possible it is nevertheless worth mentioning beco's it is a significant improvement on what has so far been known about duality. I still believe that duality for *all* formulæ is consistent relative to NF—and that we do not need AC_2. If we achieve that, stratification-mod-n can perhaps go back to the shades whence it came. But perhaps by then the idea will have thrown useful light on other ideas: we shall see.

2 Stratification

Even readers who are familiar with the idea of stratification should probably read this section, since the treatment here is slightly more abstract than the usual one, and is tailored to the developments that follow.

Let $\mathcal{L} = \mathcal{L}(\in, =)$ be the language of set theory. We associate to every formula $\phi \in \mathcal{L}$ a digraph as follows. First we identify two variables 'v' and 'v'' if ϕ contains either of the atomic subformulæ '$v = v'$' or '$v' = v$', and so on, recursively. The vertices of the digraph are the equivalence classes of variables in ϕ, and we place a directed edge from one vertex v to another vertex v' if the atomic formula '$v \in v'$' is a subformula of ϕ.

We call this graph the *derived graph* of ϕ, and write it G_ϕ.

Our digraphs are allowed to have loops at vertices, and may have multiple edges in the restricted sense that there could be a directed edge from v to v' as well as a directed edge from v' to v—but only one in each direction. In a digraph we can have a special notion of a path from v_1 to v_2 which allows us to "go the wrong way". The **length** of such a path is computed by adding 1 every time you follow an arrow the right way, and subtracting 1 every time you go the wrong way.

For $n \leq \aleph_0$ the n-**gon** G_n is the unique connected digraph with precisely n vertices where every vertex has indegree 1 and outdegree 1. It is a reduct of the integers mod n, in that it has successor-mod-n but does not have addition or multiplication. If we are to sensibly describe the circular stratification that is of interest to us here then it is the n-gon G_n that we need rather than $\mathbb{Z}/n\mathbb{Z}$, beco's the additive and multiplicative structures of $\mathbb{Z}/n\mathbb{Z}$ do nothing for us when computing stratifications; they are merely distractions.

Unlike the integers-mod-n the n-gon G_n is not rigid: its automorphism group is the cyclic group C_n. This matters beco's the set of stratifications-mod-n of a formula ϕ are "closed under rotation" so that if there is one there are n.

There is a slight problem when $n = 2$, since digraphs cannot normally have multiple edges, but we will tough this one out. And i still entertain hopes that the $\aleph_0$-gon will turn out to have a name already. For the moment let's call it the $\mathbb{Z}$-gon.

The theory of n-gons is Horn so the class of n-gons is closed under products and homomorphisms. In particular there is a homomorphism $G_m \twoheadrightarrow G_n$ whenever n divides m, and we will exploit this fact, for example in the proof of remark 1.

Definition 1
A **stratification graph** *is one where*

$$(\forall v_1)(\forall v_2)(\textit{all paths from } v_1 \textit{ to } v_2 \textit{ are the same length}).$$

A **stratification-mod-n** *graph is one with a homomorphism onto the n-gon.*

If we don't want to mention the 'n' we will say that a graph that is stratified-mod-n is **circularly stratified**.

A formula is **(Crabbé)-elementary** iff all its variables are related by the ancestral of the relation "v and v' occur in an atomic subformula together". We will tacitly assume in what follows that all our formulæ are Crabbé-elementary. Classically (though not constructively) every first-order formula is equivalent to a boolean combination of elementary formulæ (and every *closed* first-order formula is equivalent to a boolean combination of *closed* elementary formulæ) so there is little cost in making this simplifying assumption. Without it, some of the proofs below would become snarled up in annoying minor details, so i plead for the reader's indulgence.

Definition 2
A formula is **stratifiable** *iff its derived digraph is a stratification graph.*
A **stratification** *of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ to the $\mathbb{Z}$-gon;*
A **stratification-mod**-*n of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ onto the n-gon.*
A formula is **stratifiable mod** *n iff its derived digraph is a stratification-mod-n graph.*
Again, if we do not want to mention the 'n' we will say of a formula that is stratifiable-mod-n that it is **circularly stratifiable**.

Equivalently a stratification graph is one where, for all vertices v, all paths from v to v are of length 0; a stratification-mod-n graph is one where, for all vertices v and v', all paths from v to v' are of the same length mod n, or—equivalently—for all vertices v, all paths from v to v are of length 0 mod n.

Remark 1
(i) A formula that can be stratified both mod-n and mod-m can be stratified mod-LCM(m, n), and conversely.
(ii) A formula that is stratifiable-mod-n for arbitrarily large n is just plain stratifiable, and a stratifiable formula is stratifiable-mod-n for all n.

Proof:

(i) Let ϕ be such a formula, and G_ϕ its derived graph. ϕ is both stratifiable-mod-n and stratifiable-mod-m which is to say that there are homomorphisms $f : G_\phi \twoheadrightarrow G_n$ and $g : G_\phi \twoheadrightarrow G_m$. Consider now the graph $G = \{\langle f(v), g(v)\rangle : v \in G_\phi\}$ with the obvious edge relation. We want to show that G is the LCM(m, n)-gon. It is a graph of size at most $n{\cdot}m$. There is a homomorphism $\lambda v.\langle f(v), g(v)\rangle : G_\phi \twoheadrightarrow G$. Clearly every vertex in G has indegree 1 and outdegree 1, so it is either a gon (if it is connected) or a union of gons (o/w). It is also clear that if we apply the edge operation of the graph G n times to an ordered pair we reach an ordered pair with the same first component, and if we apply the edge operation m times to an ordered pair we reach an ordered pair with the same second component, so if we apply the edge operation LCM(m, n) times to an ordered pair we get back to that same ordered pair. And LCM(m, n) is the smallest number of times we can apply the edge operation of G to secure this effect. Therefore one of the connected components of G is the LCM(m, n)-gon, so G is the LCM(m, n)-gon as long as it is connected.

To establish that it is, indeed, connected we show that, for all vertices v, v' in G, there is a path from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$. Recall that G_ϕ is a stratification graph, so there is a well-defined distance, d, from v to v'. We can now see that the distance from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$ is precisely d, so G is connected.

For the converse, if ϕ is stratifiable-mod-LCM(m, n) then there is a homomorphism $f : G_\phi \twoheadrightarrow G_{LCM(m,n)}$. We compose f with the homomorphism from $G_{LCM(m,n)}$ onto G_n, thereby showing that ϕ is stratifiable-mod-n; similarly ϕ is also stratifiable-mod-m.

(ii) If $n >$ length(ϕ), then any stratification-mod-n of ϕ is (or, more correctly, can be easily modified into) a stratification. For the other direction, observe that, for every n, the $\mathbb{Z}$-gon maps onto the n-gon G_n. ■

So the picture is: we only have to worry about stratifiability-mod-p for p prime, and the various stratifiabilities-mod-p are the weakest conditions; stratifiability-mod-mn is stronger than stratifiability-mod-n, and all these are weaker than stratifiability *tout court*, which is their conjunction. The various stratifiabilities-mod-p with p prime all seem to be equally weak, and they are all of minimal strength.

It may be worth noting that we cannot strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* both to a formula that is stratifiable-mod-n and to a formula that is stratifiable-mod-m, because of the axiom of counting. For every n, the axiom of counting is equivalent (modulo NF) to a formula that is stratifiable mod n[2], so the analogue of remark 1 (ii) would tell us that it is equivalent to a stratifiable formula. However it is known that it is not equivalent (modulo NF) to any stratifiable formula.

[2] We will see a proof of this on p 9.

3 Preservation Results for Stratification-mod-n

We start with a definition from [4].

Definition 3 $H(0,\tau) =: \mathbb{1}_V;\quad H(n+1,\tau) =: (j^n\tau)\cdot H(n,\tau)$.

This H notation will only ever be used with concrete naturals in first argument place.[3]

The effect of this notation is that, for any τ and any concrete n, $(\forall xy)(x \in \tau(y) \longleftrightarrow H(n,\tau)(x) \in H(n+1,\tau)(y))$. The intention behind the design of this family of permutations derived from a single τ is to prove that, when ϕ is stratifiable, ϕ^τ is equivalent to the result of replacing every occurrence of each free variable 'v' with '$H(n_v,\tau)(v)$' where n_v is the concrete natural number associated to the variable 'v' in a fixed stratification of ϕ. In the treatment here, our stratifications are functions from $\mathit{vbls}(\phi)$ to the $\mathbb{Z}$-gon or the n-gon and do not take numbers as values. This can be remedied by composing a stratification with a decoration-by-numbers (satisfying the obvious adjacency condition) of the gon in question.

It might be worth minuting other facts about the family of permutations engendered in this way from a permutation σ. For example $H(n+m,\sigma) = j^m(H(n,\sigma))\cdot H(m,\sigma)$. I don't think there is a nice formula for $H(n\cdot m,\sigma)$. This is another manifestation of the fact that there is no natural arithmétic structure on the set of type indices.

We have a theorem of Scott that stratifiable formulæ are preserved under the Rieger-Bernays permutation construction. This is an assertion of the form

$$(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi))) \tag{A}$$

or equivalently

$$(\forall\phi)(\phi \in \Gamma \to (\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)))$$

Assertions like (A) have converses of the form

$$(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)] \tag{B}$$

and

$$(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma] \tag{C}$$

In this section we consider the project of proving assertions like these where Γ is the set of formulæ that are stratifiable-mod-n. This will involve us in identifying interesting properties of permutations to serve as the 'F' in the statement of the results

[3] so we shouldn't use these purely concrete chaps as arguments; they should be hidden in the syntax? The trouble with this policy is that we don't want footnotesized things like '$LCM(n,m)$'.

3.1 Instances of (A): $(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)))$

PROPOSITION 1 *If ϕ is stratifiable-mod-n then it is preserved under all Rieger-Bernays constructions using setlike permutations π s.t. $H(n, \pi) = \mathbb{1}$.*

Proof:

The proof is a straightforward adaptation of the proof given by Henson.

In Henson's treatment of the stratified case we fix a stratification s for ϕ. [In that treatment stratifications take values in $\mathbb{Z}$, not in the $\mathbb{Z}$-gon.] Then, whenever we look at a subformula '$x \in \sigma(y)$' in ϕ^σ we replace it by '$H(n, \sigma)(x) \in H(n+1, \sigma)(y)$' where n is the type given to 'x' by the stratification s. We then observe that, for every variable, all occurrences of that variable in the rewritten version of ϕ^σ are prefixed by a '$H(n, \sigma)$' where n is the type given to 'x' by the stratification s. Then we appeal to the fact that $H(n, \sigma)$ is a permutation, so we can reletter '$H(n, \sigma)(x)$' as 'x', and this manipulation turns ϕ^σ back into ϕ. The difference here, in this case, is that our subscripts are no longer integers but are integers-mod-n, so that if $i \equiv j \pmod{n}$ we must have $H(i, \sigma) = H(j, \sigma)$. This is equivalent to requiring that $H(n, \sigma)$ be the identity. ■

3.2 Instances of (C): $(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma]$

There is a theorem, proved by Pétry and the author ([6], [10], [11]) to the effect that: if a formula is preserved under all Rieger-Bernays constructions using setlike permutations then it is equivalent to a stratified formula.

Is there an analogous result to the effect that if a formula is preserved under all Rieger-Bernays constructions using setlike permutations $\sigma = H(n, \sigma)$ then it is equivalent to a formula that is stratifiable-mod-n? Something like that ought to be true, and it's probably worth proving.

3.3 Instances of (B): $(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)]$

We start with a very easy example:

REMARK 2 *If $f : V \to V$ (possibly a proper class) satisfying $\phi \longleftrightarrow \phi^f$ for all stratified expressions then f must be a setlike permutation.*

Proof: The axiom of extensionality is stratified, and any f that preserves it must be onto. If f preserves an $(n+1)$-stratified formula then $H(n, f)$ has to be defined, so f has to be n-setlike. ■

One might expect that if π is a permutation that preserves all formulæ that are stratifiable-mod-n then $H(n, \pi) = \mathbb{1}$. Something with that sort of flavour should be true. The following is a straw in the wind.

REMARK 3 *If $H(n, \sigma) = \mathbb{1}$ and $H(k, \sigma) = \mathbb{1}$ then $H(HCF(n, k), \sigma) = \mathbb{1}$.*

Proof: This is because, for every σ, the class of naturals n s.t. $H(n, \sigma) = \mathbb{1}$ is closed under subtraction[4] so we can, as it were, perform Euclid's algorithm. If $H(n, \sigma) = \mathbb{1}$ and $H(k, \sigma) = \mathbb{1}$, with $n > k$ then reflect that $H(n, \sigma)$ is $(j^k H(n - k, \sigma)) \cdot H(k, \sigma)$. So $j^k H(n - k, \sigma) = H(n, \sigma) \cdot H(k, \sigma)^{-1} = \mathbb{1} \cdot \mathbb{1} = \mathbb{1}$. But then $H(n - k, \sigma) = \mathbb{1}$ as well. ■

This doesn't actually say that if σ both preserves formulæ that are stratifiable-mod-n and preserves preserves formulæ that are stratifiable-mod-k) then it preserves formulæ that are stratifiable-mod-$HCF(n, k)$, but it has that flavour.

One wants to say that a permutation that preserves *all* closed formulæ must be an $\in$-automorphism, but that doesn't seem to be strictly true. At any rate i don't know how to prove it! Perhaps we can prove it by reasoning about Ehrenfeucht games. What i *do* know how to prove is that, if $V \simeq V^\sigma$, then σ is skew-conjugate to the identity. The only permutation that preserves *all* expressions (i.e., including open formulæ) is $\mathbb{1}$.

And, once we have identified predicates F that appear in theorems of flavour (B), one wants to find a structure for the set of all permutations on V such that, for each F, the class of permutations that are F is a sub*structure* not a mere sub*class*.

One thing one might have hoped to prove is that if ϕ is stratifiable-mod-n and is logically equivalent to a formula that is stratifiable-mod-m then it is logically equivalent to a formula that is stratifiable-mod-nm, but his possibility is denied us by the axiom of counting, as noted above (p 2).

Definitely work to be done in section 3!

4 Cylindrical Types

We should note that stratification-mod-n is not a useful notion from the point of view of comprehension principles, since there are paradoxical objects that are the extension of formulæ that are stratifiable-mod-n; one thinks of the n-fold Russell class $\{x : x \notin^n x\}$—being the extension of the formula '$x \notin^n x$' (which is stratifiable-mod-n) which is a paradoxical object even in mere first-order logic. This is discussed in section 4 of [3].

So *that*'s a dead end, but there is an obvious link from formulæ that are stratifiable-mod-n to the theory T$\mathbb{Z}$T+ Amb^n. The usual Specker equiconsistency analysis leads one thence to type theories whose levels are indexed by the n-gon. One could perhaps call these theories "type theory mod n", and that is what i shall do here; the proper name will be "TC$_n$T" ("theory of n cylindrical types").

Let's be formal about it.

[4] And it is *prima facie* a class not a set, since it is defined by an unstratified expression

Definition 4 *The language $\mathcal{L}(TC_nT)$, where n is a concrete natural number, has two binary relation symbols: '$=$' and '$\in$'. Its variables each have a type index as an integral part, and those type indices are precisely the elements of the n-gon.*

The axioms of TC_nT are extensionality at each type, as with $T\mathbb{Z}T$, but there is a subtlety with the set comprehension axioms. One cannot allow $(\exists x)(\forall y)(y \in x \longleftrightarrow y \not\in^n y)$ to be an axiom (for obvious reasons) even tho' this formula is a wff of $\mathcal{L}(TC_nT)$ and has the syntactic form of a comprehension axiom. One allows set comprehension only for the old $T\mathbb{Z}T$ axioms. To be formal about it, a wff that looks like a comprehension axiom is adopted as an axiom only if it is possible to rejig the type indices in it so that the resulting formula is an axiom of $T\mathbb{Z}T$.

Thus the axioms of $\mathrm{TC}_n\mathrm{T}$ are "closed under rotation", or *ambiguous* in traditional parlance. The fact that the existence of $\{x : x \not\in^n x\}$ is not a comprehension axiom does not mean that $\{x : x \not\in^n x\}$ cannot exist at *any* of the n levels; it might exist at some. However it cannot exist at *all* of them, and that's why we cannot have $(\exists x)(\forall y)(y \in x \longleftrightarrow y \not\in^n y)$ as an axiom (scheme).

Now that we know that NF is consistent (see Holmes [9], in preparation) we also know that $\mathrm{TC}_2\mathrm{T}$ is consistent: any model $\mathfrak{M}$ of NF straightforwardly gives rise to a model $\mathfrak{M}^{(n)}$ of $\mathrm{TC}_n\mathrm{T}$ (for any concrete n we please) and all such models are typically ambiguous. Altho' no model of $\mathrm{TC}_2\mathrm{T}$ can contain the double Russell class $\{x : (\forall y)(x \in y \to y \not\in x)\}$ at *both* levels, we don't know whether or not there can be a model of $\mathrm{TC}_2\mathrm{T}$ that contains this object at *one* of its two levels ... and there are of course more complicated analogues of this question for larger values of 2.

It's an old result (it was in my Ph.D. thesis, with a much improved proof by Crabbé [1] subsequently) that $\mathrm{T}\mathbb{Z}\mathrm{T}+\ \mathrm{Amb}^n$ refutes AC, and by essentially the same mechanism as does $\mathrm{T}\mathbb{Z}\mathrm{T}+$ Amb.

5 Modulo-n analogues of *strongly cantorian*

In this section we consider the property "$\iota^n \restriction x$ exists" which is stratifiable-mod-n.

It's an analogue of *strongly cantorian*. Lots of things to be said about it. Is this generalisation of strong cantorian-ness a good notion of small set? In the categorial sense, that is?

I noticed years ago the fact that altho' the existence of $\iota \restriction x$ clearly implies the existence of $\iota^n \restriction x$, the converse does not seem to hold. If $\iota^2 \restriction x$ exists then certainly $x \sqcup \iota``x$ is cantorian but that (and its analogues for $n > 2$) seem to be as far as one can go. It would appear that, in principle, there might be sets x s.t. $\iota^n \restriction x$ exists for some n but which are nevertheless not strongly cantorian. [I'm guessing that the assertion that such sets exist is invariant; it might be an idea to write out a proof]. The property "$\iota^n \restriction x$ exists" is inherited by subsets in the same way that strong-cantorianness is, so it it an *analogue* of 'strong cantorian' rather than a mere *weakening* of it, like 'cantorian'.

The possible existence of such sets is worth noting in the present context, since for them one can prove an analogue of *subversion of stratification* for formulæ that are stratifiable-mod-n.

Subversion of stratification says that, if M is a strongly cantorian set, and ϕ an arbitrary formula, then $\{x \in M : \phi^M(x)\}$ exists. (ϕ^M is the result of restricting all quantifiers in ϕ to M.) The analogue here would say that, if $\iota^n \restriction M$ exists and ϕ is stratifiable-mod-n, then $\{x \in M : \phi^M(x)\}$ exists.

Should really write out a proof

Just as subversion for strongly cantorian sets gives us interpretations into (extensions of) NF of fully unstratified set theories, subversion for sets x for which $\iota^n \restriction x$ exists will give us interpretations into (extensions of) NF of set theories satisfying syntactic contraints correspondingly less onerous than full stratification. Does this open up a vein of novel, more delicate, relative consistency proofs? Possibly, but not if we are adopting an axiom of infinity: the assumption that there is an (infinite) x s.t. $\iota^n \restriction x$ exists is as strong as the assumption that there is an infinite strongly cantorian set. This triviality is worth minuting because we will make use of it elsewhere (see p. 4).

Remark 4

(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then x is strongly cantorian.

(ii) If there is an infinite x and a concrete n such that $\iota^n \restriction x$ exists then the axiom of counting holds.

Proof:

(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then the order type of any worder of x is certainly going to be less than Ω, Ω_1 ...[5], so we can assume without loss of generality that x is an initial segment X of the ordinals. This means that $\iota^n \restriction X$ exists, and that in turn means that $T^n \restriction X$ exists, and that in turn means that we can prove by induction on the ordinals that $T^n \restriction X$ is the identity. So, for every $\alpha \in X$, $T^n\alpha = \alpha$. For every ordinal α (and so in particular for every $\alpha \in X$) we have $\alpha = T\alpha \vee \alpha < T\alpha \vee \alpha > T\alpha$. The second disjunct implies (apply T to both sides) $T\alpha <^2 \alpha$ giving $\alpha < T\alpha < \ldots T^n\alpha$ contradicting $T^n\alpha = \alpha$; the third disjunct is refuted similarly. So $T \restriction X$ exists beco's it is the identity, so $\iota \restriction X$ exists as well.

(ii) This property "$\iota^n \restriction x$ exists" is preserved by power set as well as by subset, so if there is even one infinite set which has it then $\mathbb{N}$ will have it as well. (Just as: $\mathbb{N}$ is strongly cantorian if there is even one infinite strongly cantorian set). But $\mathbb{N}$ is wellordered, so we can apply part (i). ■

The other direction (inferring "$\iota^n \restriction \mathbb{N}$ exists" for any concrete n from the axiom of counting) is easy. Thus, for every (concrete) n, the axiom of counting is equivalent modulo NF to a formula that is stratifiable-mod-n.

However if the axiom of infinity is not assumed we do get some play.

[5] Ω is the order type of the set of ordinals; $\Omega_1 = T\Omega$, and so on.

Let Mac_n be Mac with separation restricted to formulæ that are Δ_0 and stratifiable-mod-n. Analogues of the result in [8] to the effect that Mac + Tcl can be interpreted can be obtained, saying that Mac_n + TCl can be interpreted into KF, but these results are weaker than the result in [8]. However these refined constructions could turn out to be useful should there turn out to be theories of the form $\mathrm{Mac}_n \cup \{A\}$ (where A is some formula not a theorem of Mac ... but no such examples leap to mind. Not to the author's mind anyway: $\exists$NO might have sounded like a starter but is is inconsistent with the existence of $\iota^n \restriction x$ for all x. (This last follows from remark 4 part (i).)

It might be worth writing out a proof that if x is a wellorderable set s.t. $\iota^n \restriction x$ exists then $stcan(x)$. It goes as follows...

The upshot of this is that $\exists$NO is incompatible with Mac_n, the point being that $\iota^n \restriction$the representative set of wellorderings would exist and that the quotient would be strongly cantorian.

Reflect that if $\iota^n \restriction x$ exists then $\iota^{n \cdot k} \restriction x$ exists for all concrete k, for the following reason. RUSC(R) always exists, so $\mathrm{RUSC}^k(R)$ exists for all R and all concrete k, so $\mathrm{RUSC}^k(\iota^n \restriction x)$ exists and so $\iota^n \restriction x$ composed with $\mathrm{RUSC}^n(\iota^n \restriction x)$ exists, and that is $\iota^{n \cdot 2} \restriction x$. And so on for all the other multiples of n.

5.0.1 Finitising the restriction of the scheme of Δ_0 separation to formulæ that are stratifiable-mod-n

We know how to finitely axiomatise stratified Δ_0 separation, and we can get full Δ_0 separation from that axiomatisation simply by adding the existence of $\iota \restriction x$ for all x. It seems fairly clear that the way to modify the collection of rudimentary functions to obtain separation for Δ_0 formulæ that are stratifiable-mod-n is to replace the function giving $\{\langle \iota(x), y\rangle : x \in y \in A\}$ by the function giving $\{\langle \iota^{n+1}(x), y\rangle : x \in y \in A\}$. It seems clear, but it might be an idea to write out the details; all it would involve is a simple modification of the proof in the second edition of the monograph [5].

6 Applications to Duality

The special case of stratification-mod-n which will concern us here is $n = 2$. The context throughout this section is NF.

Definition 5 *The dual $\widehat{\phi}$ of a formula ϕ is the formula obtained from ϕ by replacing all occurrences of '$\in$' in ϕ by '$\not\in$'.*

It is known that $\phi \longleftrightarrow \widehat{\phi}$ is a theorem of NF whenever ϕ is a closed stratified formula. Permutation models can be found in which $\phi \longleftrightarrow \widehat{\phi}$ *fails* for some unstratified ϕ, but it remains an open question whether or not there are models in which $\phi \longleftrightarrow \widehat{\phi}$ *holds* for all ϕ.

It turns out that if we have AC_2 then we can prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ for all ϕ that are stratifiable-mod-2. This will be theorem 1 below, and it is the principal aim of this section to prove it.

We consider the sequence of permutations: $\mathbb{1}$, c, $jc \cdot c$, $j^2c \cdot jc \cdot c$, where c is the complementation permutation. The subscripts are all small (are all numerals, in fact), so we will be using the (original) notation of Henson, in which these permutations are written 'c_i', thus: $c_1 := c$; $c_{i+1} := j(c_i) \cdot c$ (rather than the $H(c, i)$ notation used above).

We will need some lemmas:

Lemma 1 *AC_2 implies that, for all permutations τ, $j\tau \cdot c$ has fixed points iff τ has no odd cycles.*

Proof:

R $\rightarrow$ L

Suppose X is a fixed point for $j\tau \cdot c$. Then, for each τ-cycle C, we must have $\tau``(X \cap C) = C \setminus X$ and that means that $|C|$ must be even (or infinite). This direction does not need AC_2.

L $\rightarrow$ R

This direction needs AC_2. Suppose τ has no odd cycles. Each τ-cycle splits into precisely two τ^2 cycles. Use AC_2 to pick, for each τ-cycle one of the two τ^2-cycles into which it splits. The union of the set of chosen τ^2-cycles is a fixed point for $j\tau \cdot c$.

■

Lemma 2
(i) All the c_i are involutions;
(ii) All the c_i commute with each other;
(iii) Assuming AC_2 the c_{2i} have fixed points and the c_{2i+1} have none.

Proof:

We start by noting a key triviality: c commutes with $j\tau$ for all τ.

(i) We prove this by induction on i. Suppose c_i is an involution. $c_{i+1} = jc_i \cdot c$. So $(c_{i+1})^2 = (jc_i \cdot c)^2 = jc_i \cdot c \cdot jc_i \cdot c$. Now by the key triviality we can rearrange to $jc_i \cdot jc_i \cdot c \cdot c = \mathbb{1}$. In fact this even shows that all products of the c_i are involutions.

(ii) The key triviality implies that jc commutes with $j^{n+1}c$ for all n, and so on by induction on 'n'. This means that the various permutations that we multiply together to obtain c_i can be multiplied together *in any order* and we still get c_i.

(iii) We prove by induction using lemma 1.

■

Some of the cases of (iii) we can establish without any use of AC_2. Clearly $c_1 = c$ has no fixed points. Also c_2 does have fixed points, since any ultrafilter on V is a fixed point, and—although we need choice to create nonprincipal ultrafilters—there are always principal ulltrafilters around. c_3 now cannot have fixed points, because the proof that if τ has fixed points then $j\tau \cdot c$ has none (this was the first part, the L $\rightarrow$ R direction, of lemma 1) does not need AC_2.

We will need the concept of a *transversal* for a disjoint family; it is a set that meets every member of the family on a singleton.

We will make much use of the fact that an involution without fixed points can be thought of as a partition of V into pairs.

Lemma 3 *Any two involutions-without-fixed-points whose corresponding partitions-of-V-into-pairs have transversals are conjugate.*

Proof: We do not need AC_2 for this.

First we establish that if P is a transversal for a partition Π of V into pairs then its cardinality is $|V|$. Clearly $|\Pi| = T|P|$, since we can send each piece of Π to the unique singleton $\subset P$ that meets it. Observe that there is a bijection between $\iota``V$ and $\Pi \times \{0, 1\}$, as follows. For each x there is a unique $p_x \in \Pi$ with $x \in p_x$. If $x \in P$ we send $\{x\}$ to $\langle p_x, 0\rangle$; if $x \notin P$ we send $\{x\}$ to $\langle p_x, 1\rangle$.

Finally if π_1 and π_2 are two involutions-without-fixed-points then not only are their transversals both of size $|V|$ but the two involutions are conjugate, as follows.

Let the transversals be P_1 and P_2. These two transversals are in 1-1 correspondence, by a map π^*, say. Any such π^* can be extended to a permutation π of the universe by adding all the ordered pairs $\langle \pi_1(x), \pi_2(\pi^*(x))\rangle$ for $x \in P_1$.

■

This proof tells us nothing about the cycle type of permutations that conjugate π_1 and π_2. Fortunately we do not need any such information in what follows.

Lemma 4 *(AC_2) c and c_3 are conjugate.*

Proof:

We established in the rider to part (iii) of lemma 2 that both c and c_3 are involutions without fixed points, so they can be tho'rt of as partitions of V into pairs. The partition-of-V-into-pairs that corresponds to the permutation c has a definable transversal: simply pick from each pair $\{x, V \setminus x\}$ that element that contains the empty set. There does not appear to be a definable transversal for the partition-of-V-into-pairs that corresponds to the permutation c_3, but AC_2 will provide one. But now we can apply lemma 3 to conclude that c and c_3 are conjugate.

■

As remarked above there is a definable transversal for the partition-into-pairs corresponding to the permutation c. However i see no way of producing a definable transversal for the set of pairs corresponding to c_3, and we do seem to need AC_2 at this point. However it is clear that AC_2 is not needed anywhere else, so it may be a worthwhile exercise to see if a definable transversal can be found for [the partition-of-V-into-pairs corresponding to] c_3, and thereby eliminate all use of AC.

Lemma 5 *If AC_2 then there is a permutation model containing two permutations σ and τ satisfying*

$$(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y)) \quad \textit{and} \quad (\forall xy)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y)).$$

which is to say: $\sigma = j\tau \cdot c$ and $\tau = j\sigma \cdot c$.

Proof: Consider what happens in the model V^π, where π is the permutation whose existence is promised in lemma 4.

π conjugates c to c_3, which is to say

$$\pi \cdot c \cdot \pi^{-1} = j^2c \cdot jc \cdot c$$

Lift by j:

$$j\pi \cdot jc \cdot j\pi^{-1} = j^3c \cdot j^2c \cdot jc$$

compose both sides with c on the right:

$$j\pi \cdot jc \cdot j\pi^{-1} \cdot c = j^3c \cdot j^2c \cdot jc \cdot c$$

But c commutes with $j\pi^{-1}$ giving

$$j\pi \cdot jc \cdot c \cdot j\pi^{-1} = j^3c \cdot j^2c \cdot jc \cdot c$$

which says that $j\pi$ conjugates c_2 with c_4.

We now, in V^π, have two permutations of the universe, namely: σ (which was c) and τ (which was c_2) with $\sigma = j\tau \cdot c$ and $\tau = j\sigma \cdot c$.

■

Theorem 1 *Con(NF + AC_2)$\to$ Con(NF + AC_2 + Duality for formulæ that are stratifiable-mod-2)*

Proof: It will suffice to establish that from the existence of these two functions σ and τ whose existence-in-a-permutation-model was proved in lemma 5 in V^π it follows that duality will hold in V^π for formulæ that are stratifiable-mod-2.

If a formula ϕ is stratifiable-mod-2 then its variables can be assigned to two types `yin` and `yang` in such a way that in subformulæ like '$x = y$' the two variables receive the same type and in subformulæ like '$x \in y$' the two variables receive different types. Let us associate σ to variables given type `yin` in the assignment and associate τ to variables given type `yang` in the assignment. '$x \in y$' is equivalent to '$\sigma(x) \notin \tau(y)$' and if x is of type `yin` we make this replacement. '$x \in y$' is also equivalent to '$\tau(x) \notin \sigma(y)$' and if x is of type `yang` we make this replacement. We deal with equations analogously. In the rewritten version of ϕ we find that every variable 'x' of type `yin` now appears only as '$\sigma(x)$' and that every variable 'y' of type `yang` now appears only as '$\tau(y)$'. So we can reletter '$\sigma(x)$' as 'x', and '$\tau(y)$' as 'y' and the result is $\widehat{\phi}$. ■

It's worth bearing in mind that σ and τ retain in V^π all the stratified properties they had in their previous life in V, where they were c and c_2. Thus they commute, and $\sigma^2 = \tau^2 = \mathbb{1}$. Observe also that $j(\sigma\tau) = j\sigma \cdot j\tau = \tau \cdot c \cdot c \cdot \sigma = \tau\sigma = \sigma\tau$, so $\sigma\tau$ is actually an $\in$-automorphism of V^π. It is a nontrivial automorphism beco's σ and τ are not inverse to each other: τ has fixed points and σ does not. By the remark in the proof of part (i) of lemma 2 it's an involution.

Can we use this technique to obtain models in which duality holds for formulæ that are stratifiable-mod-p for other primes? No. If we were to attempt to rejig the above development to obtain a proof for formulæ that are stratifiable-mod-3 then we would be looking for an i such that c_i and c_{i+3} are conjugate. To show that two involutions are conjugate we are likely to need AC_2, but unfortunately AC_2 will ensure that if we have two c_i whose suffices are of different parity then precisely one of them will have fixed points.

We see this most starkly in the case of formulæ which are stratifiable-mod-1, which is to say *all* formulae. To find—by this method—a permutation model in which duality held for *all* formulæ we would want the model to contain an antimorphism: a permutation τ such that $\tau = j\tau \cdot c$. This would involve finding a permutation τ in our home model such that τ and $j\tau \cdot c$ were conjugate. Unfortunately, as lemma 1 tells us, AC_2 implies, for all permutations τ, that $j\tau \cdot c$ has fixed points iff τ has no odd cycles. So, in particular, τ and $j\tau \cdot c$ cannot be conjugate.

Very well, so we drop AC_2, in the hope that this might open up the possibility of an involution τ such that τ and $j\tau \cdot c$ have the same cycle type. Such a τ would not be definable. But then we would need AC_2, after all, to show that τ and $j\tau \cdot c$ are conjugate.

Clearly if we are to prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ for *all* ϕ we need a new idea.

I mentioned earlier that duality for sentences that are stratifiable-mod-2 is much weaker than the conjectured duality for all sentences. In one respect, however, the result we have just shown does more: the existence of the τ and σ combining as above would appear to be more than is needed to establish duality for sentences that are stratifiable-mod-2; The existence of the τ and σ stand to duality for sentences that are stratifiable-mod-2 in the same way that the existence of an antimorphism stands to full duality. In both cases the first party to the relation seems to be on the face of it much stronger than the second. The existence of an antimorphism certainly implies duality but the converse looks most unlikely, since the existence of an antimorphism strongly contradicts AC. It ought to be possible to obtain models of duality for sentences that are stratifiable-mod-2 without actually exhibiting functions that witness it.

6.1 Full Duality?

It may be that the set of things fixed by $\sigma\tau$ is a model of NF + full Duality. Something to check!

First we check that $\sigma\tau$ (which is the same as $\tau\sigma$) is an $\in$-automorphism. For all x and y we have $x \in y \longleftrightarrow \sigma(x) \not\in \tau(y)$ so $\sigma(x) \not\in \tau(y) \longleftrightarrow \tau\sigma(x) \in \sigma\tau(y) = \tau\sigma(y)$ so $\tau\sigma$ is an $\in$-automorphism as desired.

Next we check that if π is an $\in$-automorphism the the set of fixed points is a model of NF. The big gap here is extensionality. We would have to show that every nonempty fixed set has a fixed member.

Finally we check that the set of fixed points of $\sigma\tau$ is additionally a model of duality. Observe that, for all such fixed x we have $x = \sigma(\tau(x))$ whence $\sigma^{-1}(x) = \tau(x)$. But $\sigma^2 = \mathbb{1}$ so $\sigma(x) = \tau(x)$.

Now suppose x and y both fixed. Then $x \in y \longleftrightarrow \sigma(x) \not\in \tau(y) = \sigma(y)$. So σ is an antimorphism of the fixed points.

But this relies on the set of fixed points being extensional. It may be that we can ensure this by a judicious choice of the permutation in lemma 5. The current proof of that lemma just appeals to AC_2 and it may be that a more refined analysis is possible.

We seek a π that conjugates c to $j^2c \cdot jc \cdot c$ and moreover has the extra feature that in V^π the set $\{x : \sigma(x) = \tau(x)\}$ is extensional. Must turn this into a condition on π.... I think

$$V^\pi \models (\forall x)(x \neq \emptyset \wedge \sigma\tau(x) = x \to (\exists y \in x)(\sigma\tau(y) = y))$$

is

$$(\forall x)(\pi(x) \neq \emptyset \wedge \sigma\tau(x) = x \to (\exists y \in \pi(x)(\sigma\tau(y) = y))$$

which becomes

$$(\forall x)(x \neq \emptyset \wedge j^2c \cdot jc(x) = x \to (\exists y \in \pi(x))(j^2c \cdot jc(y) = y))$$

where π conjugates c and $j^2c \cdot jc \cdot c$.

Let us write 'F' for $\{x : x = jc \cdot j^2c(x)\}$ to keep things readable. The π we seek has got to inject F into $\{y : y \cap F \neq \emptyset\}$—a set i elsewhere notate "$\rotatebox[origin=c]{180}{$\mathcal{P}$}(F)$". '$\rotatebox[origin=c]{180}{$\mathcal{P}$}$' is an upside-down '$\mathcal{P}$' since $\rotatebox[origin=c]{180}{$\mathcal{P}$}(x)$ is $V \setminus (\mathcal{P}(V \setminus x))$ and is thus dual to $\mathcal{P}$). Observe that $\rotatebox[origin=c]{180}{$\mathcal{P}$}(x)$ is always a moiety, since it is $V \setminus (\mathcal{P}(V \setminus x))$, and the complement of a power set (of anything other than V) is always the same size as V. This is beco's every set (other than V itself) is included in the complement of a singleton, and the power set of a complement of a singleton is a principal prime ideal and therefore a moiety.

So there's no problem on *that* score.

It's not blindingly obvious to me that it cannot be done.

7 Work still to do

Are there anywhere in the world embeddings that are elementary for formulæ that are stratifiable-mod-n? Between iterated CO models perhaps...?

There remains of course the challenge of proving consistency of duality for all sentences, not merely those that are stratifiable-mod-2. But more to the point are the possibilities of extending to formulæ that are stratifiable-mod-n things known about the rather more restricted class of stratified formulæ—and these I haven't started thinking about. Here are some, in no particular order.

Is there any interest in versions of Forti-Honsell Antifoundation along the lines "Every set picture that is a n-stratification graph is a picture of a set"?

The axiom of counting is unstratified and not equivalent modulo NF to any stratified formula but is, for each concrete n, equivalent modulo NF to a formula that is stratifiable-mod-n. It's also invariant. The same goes for $\mathrm{AxCount}_{\leq}$ (with a bit more work) since—for any concrete k—$\mathrm{AxCount}_{\leq}$ can be written as $(\forall n \in \mathbb{N})(n \leq T^k n)$.

André Pétry suggests a generalisation of a result of his-and-mine alluded to earlier ([6], [10], and [11]) to the effect that if two structures are elementarily equivalent for formulæ that are stratifiable-mod-n then they have stratimorphic (as it were) ultrapowers.

One could investigate whether the construction of [7] could be modified to encompass expressions that are stratifiable-mod-n. That looks messy.

There are natural settings where one encounters embeddings that are elementary for stratifiable formulæ, and where one might hope to get embeddings that are elementary for some of these larger classes of formulæ. CO models is one setting: the embedding from the ground model into the hereditary low sets is elementary for stratifiable formulæ. (That particular example is probably not a good one, because if the inclusion embedding is elementary for formulæ that are stratifiable-mod-n for even one n then the hereditarily low sets cannot contain any Quine atoms). For another, let $\mathfrak{M}$ be a structure for $\mathcal{L}$. Consider the class of those $m \in M$ s.t. m is fixed by all permutations of M that, for all n, are j^n of something. It's an elementary substructure as long as it's extensional. Now use instead those permutations π of M s.t. $j^m \pi = \mathbb{1}$. Now the class of fixed things is a substructure elementary for expressions that are stratifiable mod m (again, assuming extensionality).

Str(ZF) is the theory axiomatised by the stratifiable axioms of ZF; by analogy $\mathrm{str}_n(\mathrm{ZF})$ will be the theory axiomatised by those axioms of ZF that are stratifiable-mod-n. ZF can be interpreted in str(ZF) + IO. (IO is the axiom "every set is the same size as a set of singletons"). Observe that IO is a theorem of $\mathrm{str}_n(\mathrm{ZF})$, since it proves that $\iota^n {\restriction} x$ exists for all x, so ZF can be interpreted in $\mathrm{str}_n(\mathrm{ZF})$. At this stage i cannot see how to prove that $\mathrm{str}_n(\mathrm{ZF}) = \mathrm{ZF}$. There are parallel questions about the fragments of Mac.

Stratified parameter-free induction seems to prove no more that the nonexistence of a universal set. How about stratifiable-mod-n parameter-free induction…what does that do?

Every weakly stratifiable theorem of first-order logic has a cut-free weakly stratifiable proof; every stratifiable theorem of first-order logic has a stratifiable proof (Crabbé, [2]); are there analogues for stratification-mod-n?

Stratifiable parameter-free $\in$-induction implies the nonexistence of the universal set. (If none of your members are the universal set, you can't be either). It's not known if the converse holds. However the strengthening of the converse one would consider in this context, namely "the non-existence of the universal set implies $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" clearly does not go through: $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" implies $(\forall x)(x \notin^2 x)$, and that clearly doesn't follow from the nonexistence of V.

In stratifiable set theories one has to have for one's pairing function something that gives 'x' and 'y' the same type in '$z = \langle x, y \rangle$'. This is to ensure that the composition of two relations always exists. Of course if we have separation for formulæ that are stratifiable mod n then we can allow the types of 'x' and 'y' in '$z = \langle x, y \rangle$' to differ by any integer multiple of 'n.

There is a conneection here with the universal-existential conjecture for T$\mathbb{Z}$T. The corresponding conjecture for $\mathrm{TC_2T}$ does not hold: the expression '$(\forall x)(\exists y)(x \in y \longleftrightarrow y \notin x)$' is universal-existential and is a wff of $\mathcal{L}\mathrm{TC_2T}$) but its truth-value is not constant on all models of $\mathrm{TC_2T}$. It says that there is no $x = \overline{Bx}$. Now this last is a theorem of NF so it must be true at at least one of the two types—if both types contain an $x = \overline{Bx}$ we obtain a contradiction by enquiring about membership between them. It can [apparently[6]] be true at both, so it's universal-existential but its truth-value is not constant on all models of $\mathrm{TC_2T}$.

This formula also crops up in attempts to prove the consistency of duality by means of the Barwise approximants, in my notes in universal3.tex . . . but that may be mere coincidence.

There is an old question about whether the atoms of a model of NFU can be indiscernible. We know that they are indiscernible wrt stratifiable formulæ; now that we've started looking into stratification-mod-n it is natural to wonder whether one might be able to show that the atoms of a model of NFU must be indiscernible wrt expressions that are stratifiable-mod-2. At this stage it's not looking hopeful.

Consider "$\Box$(Duality for sentences that are stratifiable-mod-2)"

Is this consistent? Does it imply $\mathrm{AC_2}$?

ZF + Foundation and ZF + antifoundation are alike extensions of ZF + Coret's axiom "every set is the same size as a wellfounded set" conservative for stratifiable sentences. Does this hold also for sentences that are stratifiable-mod-n?

[6]To be honest, this is guesswork on my part. To find a model of $\mathrm{TC_2C}$ in which this holds at one type and not at the other will need at least as much work as finding a model of NF.

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