Stratification mod n and the Cylindrical Theory of Types

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1 Introduction and Summary

Recently Zuhair Abdul Ghafoor Al-Johar [16] has directed our attention to a syntactic constraint that is—on the face of it—*tighter* than NF's device of stratification[1]; in this little essay I consider a *weakening*, namely the generalisation of stratification to *stratification modulo* n. So far the coterie of NFistes has considered neither the possibility that the class of unstratified formulæ in the language of set theory might admit any structure or gradation, nor the possibility that failure-of-stratification (which perhaps we can call *dysstratification*) might come in degrees, let alone the possibility that recognition of such degrees might allow one to gain understanding and prove useful facts.

So stratification-mod-n opens a new vein, but not one i've been able to get anything really substantial out of. Not so far, anyway ... mostly just simple-minded generalisations of the standard stratified case—not that those are without merit, since they prepare the ground for subsequent work. It has to be admitted that stratification-mod-n comes across as a highly artificial notion, of interest only to those whose critical faculties have been weakened by prior exposure to the idea of stratification. However, there is a nontrivial result that makes essential use of this notion, and we will see it in section 6 where we show (theorem 4) that—for NF—duality for formulæ that are stratifiable-mod-2 is consistent relative to NF. Altho' we do not believe that this result is best possible it is nevertheless worth mentioning beco's it is a significant improvement on what has so far been known about duality. We still believe that duality for *all* formulæ is consistent relative to NF. If we achieve that, stratification-mod-n can perhaps go back to the shades whence it came. But perhaps by then it will have thrown useful light on other ideas: we shall see.

2 Stratification

Even readers who are familiar with the idea of stratification should probably read this section, since the treatment here is slightly more abstract than the usual one, and is tailored to the developments that follow.

Let $\mathcal{L} = \mathcal{L}(\in, =)$ be the language of set theory. We associate to every formula $\phi \in \mathcal{L}$ a digraph as follows. First we identify two variables 'v' and 'v'' if ϕ contains either of the atomic subformulæ '$v = v'$' or '$v' = v$', and so on, recursively. The vertices of the digraph are the equivalence classes of variables in ϕ, and we place a directed edge from one vertex v to another vertex v' if the atomic formula '$v \in v'$' is a subformula of ϕ.

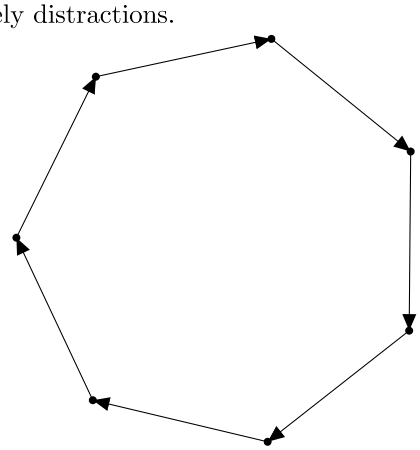
We call this graph the *derived graph* of ϕ, and write it G_ϕ.

Our digraphs are allowed to have loops at vertices, and may have multiple edges in the restricted sense that there could be a directed edge from v to v'

[1] Tho' recent work of Nathan Bowler seems to establish (modulo some very minor set-theoretic assumptions) that every stratifiable formula is equivalent to an acyclic formula. I do not yet understand his proof, and he hasn't published it. However I see no reason to doubt it.

as well as a directed edge from v' to v—but only one in each direction. In a digraph we can have a special notion of a path from v_1 to v_2 which allows us to "go the wrong way". The **length** of such a path is computed by adding 1 every time you follow an arrow the right way, and subtracting 1 every time you go the wrong way.

For $n \leq \aleph_0$ the n-**gon** G_n is the unique connected digraph with precisely n vertices where every vertex has indegree 1 and outdegree 1. It is a reduct of the integers mod n, in that it has successor-mod-n but does not have addition or multiplication. Despite this document bearing the title "stratification mod n" arithmetic mod n plays essentially no rôle in what follows: if we are to sensibly describe the circular stratification that is of interest to us here then it is the n-gon G_n that we need—rather than $\mathbb{Z}/n\mathbb{Z}$—because the additive and multiplicative structures of $\mathbb{Z}/n\mathbb{Z}$ do nothing for us when computing stratifications; they are merely distractions.



G_7

Unlike the integers-mod-n the n-gon G_n is not rigid: its automorphism group is the cyclic group[2] C_n. This matters because the set of stratifications-mod-n of a formula ϕ are "closed under rotation" so that if there is one there are n.

There is a slight problem when $n = 2$, since digraphs cannot normally have multiple edges, but we will tough this one out. And we still entertain hopes that the $\aleph_0$-gon will turn out to have a name already. For the moment let's call it the $\mathbb{Z}$-gon.

The theory of n-gons is Horn, so the class of n-gons is closed under products and homomorphisms. In particular there is a homomorphism $G_m \twoheadrightarrow G_n$ whenever n divides m, and we will exploit this fact, for example in the proof of remark 1.

DEFINITION 1

[2]Dana Scott points out that thinking of G_n as a polygon isn't *entirely* correct either, since polygons have reflections and reflections have no meaning in this context.

A **stratification graph** *is one where*

$$(\forall v_1)(\forall v_2)(\textit{all paths from } v_1 \textit{ to } v_2 \textit{ are the same length}).$$

A **stratification-mod-n** *graph is one with a homomorphism onto the n-gon.*

If we don't want to mention the 'n' we will say that a graph that is stratified-mod-n is **circularly stratified**.

A *'moiety'* is a set x such that $|x| = |V| = |V \setminus x|$.

A formula is **(Crabbé)-elementary** iff all its variables are related by the ancestral of the relation "v and v' occur in an atomic subformula together". We will tacitly assume in what follows that all our formulæ are Crabbé-elementary. Classically (though not constructively) every first-order formula is equivalent to a boolean combination of elementary formulæ (and every *closed* first-order formula is equivalent to a boolean combination of *closed* elementary formulæ) so there is little cost in making this simplifying assumption. Without it, some of the proofs below would become snarled up in annoying minor details, so we plead for the reader's indulgence.

Definition 2

A formula is **stratifiable** *iff its derived digraph is a stratification graph.*

A **stratification** *of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ to the $\mathbb{Z}$-gon;*

A **stratification-mod**-*n of a formula ϕ is a homomorphism from the derived graph G_ϕ of ϕ onto the n-gon.*

A formula is **stratifiable mod** *n iff its derived digraph is a stratification-mod-n graph.*

Again, if we do not want to mention the 'n' we will say of a formula that is stratifiable-mod-n that it is **circularly stratifiable**.

Equivalently a stratification graph is one where, for all vertices v, all paths from v to v are of length 0; a stratification-mod-n graph is one where, for all vertices v and v', all paths from v to v' are of the same length mod n, or—equivalently—for all vertices v, all paths from v to v are of length 0 mod n.

Remark 1

(i) A formula that can be stratified both mod-n and mod-m can be stratified mod-LCM(m, n), and conversely.

(ii) A formula that is stratifiable-mod-n for arbitrarily large n is just plain stratifiable, and a stratifiable formula is stratifiable-mod-n for all n.

Proof:

(i) Let ϕ be such a formula, and G_ϕ its derived graph. ϕ is both stratifiable-mod-n and stratifiable-mod-m which is to say that there are homomorphisms $f : G_\phi \twoheadrightarrow G_n$ and $g : G_\phi \twoheadrightarrow G_m$. Consider now the graph $G = \{\langle f(v), g(v)\rangle : v \in G_\phi\}$ with the obvious edge relation. We want to show that G is the LCM(m, n)-gon. It is a graph of size at most $n{\cdot}m$. There is a homomorphism $\lambda v.\langle f(v), g(v)\rangle$:

$G_\phi \twoheadrightarrow G$. Clearly every vertex in G has indegree 1 and outdegree 1, so it is either a gon (if it is connected) or a union of gons (o/w). It is also clear that if we apply the edge operation of the graph G n times to an ordered pair we reach an ordered pair with the same first component, and if we apply the edge operation m times to an ordered pair we reach an ordered pair with the same second component, so if we apply the edge operation LCM(m, n) times to an ordered pair we get back to that same ordered pair. And LCM(m, n) is the smallest number of times we can apply the edge operation of G to secure this effect. Therefore one of the connected components of G is the LCM(m, n)-gon, so G is the LCM(m, n)-gon as long as it is connected.

To establish that it is—indeed—connected, we show that, for all vertices v, v' in G, there is a path from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$. Recall that G_ϕ is a stratification graph, so there is a well-defined distance, d, from v to v'. We can now see that the distance from $\langle f(v), g(v)\rangle$ to $\langle f(v'), g(v')\rangle$ is precisely d, so G is connected.

For the converse, if ϕ is stratifiable-mod-LCM(m, n) then there is a homomorphism $f : G_\phi \twoheadrightarrow G_{LCM(m,n)}$. We compose f with the homomorphism from $G_{LCM(m,n)}$ onto G_n, thereby showing that ϕ is stratifiable-mod-n; similarly ϕ is also stratifiable-mod-m.

(ii) If $n >$ length(ϕ), then any stratification-mod-n of ϕ is (or, more correctly, can be easily modified into) a stratification. For the other direction, observe that, for every n, the $\mathbb{Z}$-gon maps onto the n-gon G_n. ■

So the picture is: we only have to worry about stratifiability-mod-p for p prime, and the various stratifiabilities-mod-p are the weakest conditions; stratifiability-mod-mn is stronger than stratifiability-mod-n, and all these are weaker than stratifiability *tout court*, which is the conjunction of them all. The various stratifiabilities-mod-p with p prime all seem to be equally weak, and they are all of minimal strength.

It may be worth noting that we cannot strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* both to a formula that is stratifiable-mod-n and to a formula that is stratifiable-mod-m, because of the axiom of counting. For every n, the axiom of counting is equivalent (modulo NF) to a formula that is stratifiable mod n (we will see a proof of this on p 12) so the analogue of remark 1 part (ii) would tell us that it is equivalent to a stratifiable formula. However, it is known that it is not equivalent (modulo NF) to any stratifiable formula. However, the axiom of counting is invariant, so it might be possible to strengthen remark 1 by modifying the asssumption on the formula to being merely *equivalent* (mod NF) both to a formula that is stratifiable-mod-n and to a formula that is stratifiable-mod-m, if the conclusion we want to infer is that the formula in question is merely invariant (modulo NF) rather than actually stratifiable.

Finally we wrap up some definitions which—altho' standard in an NF context—might not be well-known to other readers.

$B(x) = \{y : x \in y\}$.

ι is the singleton function: $\iota(x) = \{x\}$. If $\iota \upharpoonright x$ exists we say x is **strongly cantorian**.

2.1 A Feeble Attempt at Motivation

It would certainly help to motivate this study if there were natural examples of formulæ of the language of set theory that were stratifiable-mod-n for some n. There are not many, God knows, but there are some. "`I` has a Winning strategy in G_x" and "`II` has a Winning strategy in G_x" (where G_x is the $\in$-game in [10]) are both stratifiable-mod-2. The first is

$$(\forall y)(\mathcal{P}(\reflectbox{$\mathcal{P}$}(y)) \subseteq y \to x \in y)$$

and the second is

$$(\forall y)(\reflectbox{$\mathcal{P}$}(\mathcal{P}(y)) \subseteq y \to x \in y).$$

($\reflectbox{$\mathcal{P}$}(x)$ is $\{y : y \cap x \neq \emptyset\}$ and is thus dual to $\mathcal{P}$, which is why we write it with an upside-down '$\mathcal{P}$'.) Right way round?

The axiom of counting is unstratified and not equivalent modulo NF to any stratifiable formula but is, for each concrete n, equivalent modulo NF to a formula that is stratifiable-mod-n. It's also invariant. The same goes for AxCount$_\leq$ (with a bit more work) since—for any concrete k—AxCount$_\leq$ can be written as '$(\forall n \in \mathbb{N})(n \leq T^k n)$'. Wellfoundedness does not seem to be capturable by any formula that is stratifiable-mod-n, whatever n we choose. [this is worth proving if true] Recall from remark 1 that any formula that is stratifiable-mod-n for all n is stratifiable.

Maybe at later stages of this study we will find other examples.

3 Preservation Results for Stratification-mod-n

We start with a definition from [5].

Definition 3 $H(0, \tau) =: \mathbb{1}_V$; $\quad H(n+1, \tau) =: (j^n \tau) \cdot H(n, \tau)$.

This H notation will only ever be used with concrete naturals in first argument place.[3]

The effect of this notation is that, for any τ and any concrete n, $(\forall xy)(x \in \tau(y) \longleftrightarrow H(n,\tau)(x) \in H(n+1,\tau)(y))$. The intention behind the design of this family of permutations derived from a single τ is to prove that, when ϕ is stratifiable, ϕ^τ is equivalent to the result of replacing every occurrence of each free variable 'v' with '$H(n_v, \tau)(v)$' where n_v is the concrete natural number

[3] so we shouldn't use these purely concrete chaps as arguments; they should be hidden in the syntax? The trouble with this policy is that we don't want footnotesized things like '$LCM(n, m)$'.

associated to the variable 'v' in a fixed stratification of ϕ. In the treatment here, our stratifications are functions from $vbls(\phi)$ to the $\mathbb{Z}$-gon or the n-gon and do not take numbers as values. This can be remedied by composing a stratification with a decoration-by-numbers (satisfying the obvious adjacency condition) of the gon in question.

It might be worth minuting other facts about the family of permutations engendered in this way from a permutation σ. For example $H(n+m,\sigma) = j^m(H(n,\sigma)) \cdot H(m,\sigma)$. We don't think there is a nice formula for $H(n \cdot m, \sigma)$. This is another manifestation of the fact that there is no natural arithmétic structure on the set of type indices.

We have a theorem of Scott that stratifiable formulæ are preserved under the Rieger-Bernays permutation construction. This is an assertion of the form

$$(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi))) \tag{A}$$

or equivalently

$$(\forall\phi)(\phi \in \Gamma \to (\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)))$$

Assertions like (A) have converses of the form

$$(\forall\pi)[(\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)] \tag{B}$$

and of the form

$$(\forall\phi)[(\forall\pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma] \tag{C}$$

In this section we consider the project of proving assertions like these where Γ is the set of formulæ that are stratifiable-mod-n. This will involve us in identifying interesting properties of permutations to serve as the 'F' in the statement of the results

3.1 Instances of (A): $(\forall\pi)(F(\pi) \to (\forall\phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)))$

PROPOSITION 1 *If ϕ is stratifiable-mod-n then it is preserved under all Rieger-Bernays constructions using setlike permutations π s.t. $H(n,\pi) = \mathbb{1}$.*

Proof:

The proof is a straightforward adaptation of the proof given by Henson.

In Henson's treatment of the stratified case we fix a stratification s for ϕ. [In that treatment stratifications take values in $\mathbb{Z}$, not in the $\mathbb{Z}$-gon.] Then, whenever we look at a subformula '$x \in \sigma(y)$' in ϕ^σ we replace it by '$H(n,\sigma)(x) \in H(n+1,\sigma)(y)$' where n is the type given to 'x' by the stratification s. We then observe that, for every variable, all occurrences of that variable in the rewritten version of ϕ^σ are prefixed by a '$H(n,\sigma)$' where n is the type given to 'x' by the stratification s. Then we appeal to the fact that $H(n,\sigma)$ is a permutation, so we can reletter '$H(n,\sigma)(x)$' as 'x', and this manipulation turns ϕ^σ back into ϕ.

The difference here, in this case, is that our subscripts are no longer integers but are integers-mod-n, so that if $i \equiv j \pmod{n}$ we must have $H(i, \sigma) = H(j, \sigma)$. This is equivalent to requiring that $H(n, \sigma)$ be the identity. ∎

3.2 Instances of (C): $(\forall \phi)[(\forall \pi)(F(\pi) \to (\phi^\pi \longleftrightarrow \phi)) \to \phi \in \Gamma]$

There is a theorem, proved by Pétry and Forster ([7], [14], [15]) to the effect that: if a formula is preserved under all Rieger-Bernays constructions using setlike permutations then it is equivalent to a stratifiable formula.

Is there an analogous result to the effect that if a formula is preserved under all Rieger-Bernays constructions using setlike permutations $\sigma = H(n, \sigma)$ then it is equivalent to a formula that is stratifiable-mod-n? Something like that ought to be true, and it's probably worth proving.

3.3 Instances of (B): $(\forall \pi)[(\forall \phi)(\phi \in \Gamma \to (\phi^\pi \longleftrightarrow \phi)) \to F(\pi)]$

We start with a very easy example:

Remark 2 *If $f : V \to V$ (possibly a proper class) satisfying $\phi \longleftrightarrow \phi^f$ for all stratifiable expressions then f must be a setlike permutation.*

Proof: The axiom of extensionality is stratifiable, and any f that preserves it must be onto. If f preserves an $(n+1)$-stratifiable formula then $H(n, f)$ has to be defined, so f has to be n-setlike. ∎

One might expect that if π is a permutation that preserves all formulæ that are stratifiable-mod-n then $H(n, \pi) = \mathbb{1}$. Something with that sort of flavour should be true. The following is a straw in the wind.

Remark 3 *If $H(n, \sigma) = \mathbb{1}$ and $H(k, \sigma) = \mathbb{1}$ then $H(HCF(n, k), \sigma) = \mathbb{1}$.*

Proof: This is because, for every σ, the class of naturals n s.t. $H(n, \sigma) = \mathbb{1}$ is closed under subtraction[4] so we can, as it were, perform Euclid's algorithm. If $H(n, \sigma) = \mathbb{1}$ and $H(k, \sigma) = \mathbb{1}$, with $n > k$ then reflect that $H(n, \sigma)$ is $(j^k H(n - k, \sigma)) \cdot H(k, \sigma)$. So $j^k H(n - k, \sigma) = H(n, \sigma) \cdot H(k, \sigma)^{-1} = \mathbb{1} \cdot \mathbb{1} = \mathbb{1}$. But then $H(n - k, \sigma) = \mathbb{1}$ as well. ∎

This doesn't actually say that if σ both preserves formulæ that are stratifiable-mod-n and preserves preserves formulæ that are stratifiable-mod-k) then it preserves formulæ that are stratifiable-mod-$HCF(n, k)$, but it has that flavour.

One wants to say that a permutation that preserves *all* closed formulæ must be an $\in$-automorphism, but that doesn't seem to be strictly true. At any rate we don't know how to prove it! Perhaps we can prove it by reasoning about

[4] *prima facie* we cannot expect this thing to be a set, since it is defined by an unstratified expression.

Ehrenfeucht games. What we *do* know how to prove is that, if $V \simeq V^\sigma$, then σ is skew-conjugate to the identity. The only permutation that preserves *all* expressions (i.e., including open formulæ) is $\mathbb{1}$.

Must define 'skew-conjugate'

And, once we have identified predicates F that appear in theorems of flavour (B), one wants to find a structure for the set of all permutations on V such that, for each F, the class of permutations that are F is a sub*structure* not a mere sub*class*.

One thing one might have hoped to prove is that if ϕ is stratifiable-mod-n and is logically equivalent to a formula that is stratifiable-mod-m then it is logically equivalent to a formula that is stratifiable-mod-nm, but this possibility is denied us by the axiom of counting, as noted above (p 2).

Definitely work to be done in section 3!

4 Cylindrical Types

We should note that—in contrast to stratification *tout court*—stratification-mod-n is not a useful notion from the point of view of comprehension principles in a one-sorted language, since there are paradoxical objects that are the extension of formulæ that are stratifiable-mod-n; one thinks of the n-fold Russell class $\{x : x \not\in^n x\}$—being the extension of the formula '$x \not\in^n x$' (which is stratifiable-mod-n) which is a paradoxical object even in mere first-order logic. This is discussed in section 4 of [4] and also below). Of course there are no known paradoxical objects defined by stratifiable set abstracts.

So *that*'s a dead end, but there is an obvious link from formulæ that are stratifiable-mod-n to the theory T$\mathbb{Z}$T+ Amb^n. The usual Specker equiconsistency analysis leads one thence to type theories whose levels are indexed by the n-gon. One could perhaps call these theories "type theory mod n", and that is what i shall do here; the proper name will be "TC$_n$T" ("theory of n cylindrical types").

Let's be formal about it.

Definition 4 *The language $\mathcal{L}(TC_nT)$, where n is a concrete natural number, has two binary relation symbols: '$=$' and '$\in$'. Its variables each have a type index as an integral part, and those type indices are precisely the elements of the n-gon.*

The axioms of TC_nT are extensionality at each type, as with $T\mathbb{Z}T$, but there is a subtlety with the set comprehension axioms. One cannot allow $(\exists x)(\forall y)(y \in x \longleftrightarrow y \not\in^n y)$ to be an axiom (for obvious reasons) even tho' this formula is a wff of $\mathcal{L}(TC_nT)$ and has the syntactic form of a comprehension axiom, and '$y \not\in^n y$' is a wff of the language One allows set comprehension only for the old $T\mathbb{Z}T$ axioms. To be formal about it, a wff that looks like a comprehension axiom is adopted as an axiom only if it is possible to rejig the type indices in it so that the resulting formula is an axiom of $T\mathbb{Z}T$.

Thus the axioms of $\mathrm{TC}_n\mathrm{T}$ are "closed under rotation", or *ambiguous* in traditional parlance. The fact that the existence of $\{x : x \not\in^n x\}$ is not a comprehension axiom does not *ipso facto* mean that the sets $\{x^i : x^i \not\in^n x^i\}$ cannot exist at any of the n levels, though the impossibility of their existence can be shown. This fact is probably worth minuting.

REMARK 4 *$TC_nT \vdash R^i_n = \{x^i : x^i \not\in^n x^i\}$ does not exist for any $i \leq n$.*

Proof:
Reasoning in $\mathrm{TC}_n\mathrm{T}$ we pick on any level i and consider the possibility of the existence of $R^i_n = \{x^i : x^i \not\in^n x^i\}$. Consider $\iota^{n-1}(R^i_n)$; is it a member of R^i_n or not? If it is, then it belongs to an $\in$-loop of circumference n, so it is barred from membership of R^i_n. So it isn't a member of R^i_n. So there are $x_1 \ldots x_{n-1}$ with $\iota^{n-1}(R_n) \in x_1 \in x_2 \in \ldots \iota^{n-1}(R_n)$ an $\in$-loop of circumference n. But then $x_1 = R_n$ (peel off the brackets) showing that $R_n \in R_n$ after all. ■

This proof establishes that—according to $\mathrm{TC}_n\mathrm{T}$—$\{x : x \not\in^n x\}$ does not exist at *any* level. However, it does use some set-theoretic axioms. Readers might like to note the curiosity that inside first-order logic pure and simple (without using any set theory at all) we can show that $\{x^i : x^i \not\in^n x^i\}$ must fail to exist at least one level i. This, too, is probably worth minuting.

REMARK 5 *The fact that no model of TC_nT can contain $R^i_n = \{x^i : x^i \not\in^n x^i\}$ for all $i \leq n$ is a theorem of First Order Logic.*

Proof:
Suppose $\{x : x \not\in^n x\}$ exists at every level. Let us write 'R^i' for its manifestation at level i. Let i be an arbitrary concrete natural $\leq n$. Suppose $R^i \not\in R^{i+1}$. Then R^i belongs to an $\in$-loop of circumference n, and there must be x^{i-1} in R^i in this loop. But $x^{i-1} \in R^i$ implies that x^{i-1} cannot belong to any such loop. Thus we conclude $R^i \in R^{i+1}$. But i was arbitrary. So there is an $\in$-loop of circumference n consisting entirely of the R^i and this clearly cannot happen. ■

It doesn't seem to be possible to spice up this proof to show (in first-order logic) that *none* of the R^i exist.

Now that we know that NF is consistent (see Holmes [12], in preparation) we also know that $\mathrm{TC}_2\mathrm{T}$ is consistent: any model $\mathfrak{M}$ of NF straightforwardly gives rise to a model $\mathfrak{M}^{(n)}$ of $\mathrm{TC}_n\mathrm{T}$ (for any concrete n we please) and all such models are typically ambiguous. Altho' no model of $\mathrm{TC}_2\mathrm{T}$ can contain the double Russell class $\{x : (\forall y)(x \in y \to y \not\in x)\}$ at *both* levels, we don't know whether or not there can be a model of $\mathrm{TC}_2\mathrm{T}$ that contains this object at *one* of its two levels ... and there are of course more complicated analogues of this question for larger values of 2.

It's an old result (it was in Forster's Ph.D. thesis, with a much improved proof by Crabbé [2] subsequently) that $\mathrm{T}\mathbb{Z}\mathrm{T}+ \mathrm{Amb}^n$ refutes AC, and by essentially the same mechanism as does $\mathrm{T}\mathbb{Z}\mathrm{T}+$ Amb. The best guess is that all the theories $\mathrm{TC}_n\mathrm{T}$ are equiconsistent with NF. Section 4.1 introduces a hare which we can chase in that direction.

I noted above, in definition 4, that we have to make sure that our comprehension axioms are only those formulæ which become axioms of $\mathrm{T}\mathbb{Z}\mathrm{T}$ lest we get Russell-style paradoxes. It might be worth thinking a bit about how one might cautiously relax this restriction to admit some more comprehension axioms. There is an analogue of *strongly cantorian* and altho' one obviously cannot allow the class of analogue-stcan sets to be a set (for the usual reasons) there doesn't seem to be any objection to the collection of finite analogue-stcan sets being a set.

4.1 Analogues of TTT for Cylindrical Type Theory

The reader is assumed to be familiar with Holmes' elegant (bizarre but fruitful) device of *Tangled Type Theory* in [13]. We modify the conception slightly. [Before we present the theory we have to agree on a way of notating it; it'll have to be something like '$\mathrm{TTT}(n)$' or 'TTT_n' or 'TTT^n' or '$\mathrm{TTT}^{(n)}$' or '$\mathrm{TTT}^{[n]}$'.... Perhaps we can have a vote. For the moment we shall write '$\mathrm{TT}_n\mathrm{T}$'.]

The way in is as follows. Models of TTT consist of an ordered collection of levels with membership relations holding between lower levels and higher levels; in the version $\mathrm{TT}_n\mathrm{T}$ of TTT for Cylindrical Type Theory with n types we have an ordered family not of *levels* but of *models of* TST_n. In a model of $\mathrm{TT}_n\mathrm{T}$, whenever we have two such models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with $\mathfrak{M}_2$ higher than $\mathfrak{M}_1$, we supply a membership relation between the top level of $\mathfrak{M}_1$ and the bottom level of $\mathfrak{M}_2$. Clearly we can obtain a model of $\mathrm{TT}_n\mathrm{T}$ from a model of $\mathrm{TC}_n\mathrm{T}$ merely by making lots of copies; readers familiar with Holmes' methods in [13] will be able to see how to use them to go in the other direction and obtain a model of $\mathrm{TC}_n\mathrm{T}$ from a model of $\mathrm{TT}_n\mathrm{T}$. Later draughts of this document will treat this material in more detail. At this stage we do not know what notion of *tangled web of cardinals* (see [12]) corresponds to $\mathrm{TT}_n\mathrm{T}$—or indeed if there even *is* such a notion.

And this generalisation of the TTT machinery is just the start. There is no compelling reason to require that all the wee models of type theory that we conscript to make a model of the modified TTT should have the same number of levels. (What are we to call the resulting theory? '$\mathrm{TT}_{<\omega}\mathrm{T}$' ...?)

Be that as it may, the tho'rt was that the reduction of $\mathrm{Con}(\mathrm{TC}_n\mathrm{T})$ to $\mathrm{Con}(\mathrm{TT}_n\mathrm{T})$ might enable us to show that all the theories $\mathrm{TC}_n\mathrm{T}$ are equiconsistent with NF, but for the moment that goal seems as far off as ever.

5 Modulo-n analogues of *strongly cantorian*

5.1 Analogues in NF

In this section we consider the property "$\iota^n \upharpoonright x$ exists" which is stratifiable-mod-n. And we work in NF.

It's an analogue of *strongly cantorian*. Lots of things to be said about it. Is this generalisation of strong cantorian-ness a good notion of small set? In the categorial sense, that is?

I noticed years ago the fact that altho' the existence of $\iota \restriction x$ clearly implies the existence of $\iota^n \restriction x$, the converse does not seem to hold. If $\iota^2 \restriction x$ exists then certainly $x \sqcup \iota``x$ is cantorian but that (and its analogues for $n > 2$) seem to be as far as one can go. It would appear that, in principle, there might be sets x s.t. $\iota^n \restriction x$ exists for some n but which are nevertheless not strongly cantorian.

The property "$\iota^n \restriction x$ exists" is inherited by subsets in the same way that strong-cantorianness is, so it is an *analogue* of 'strong cantorian' rather than a mere *weakening* of it, unlike 'cantorian', which is merely a weakening.

The possible existence of such sets is worth noting in the present context, since for them one can prove an analogue of *subversion of stratification* for formulæ that are stratifiable-mod-n.

Subversion of stratification says that, if M is a strongly cantorian set, and ϕ an arbitrary formula, then $\{x \in M : \phi^M(x)\}$ exists. (ϕ^M is the result of restricting all quantifiers in ϕ to M.) The analogue here would say that, if $\iota^n \restriction M$ exists and ϕ is stratifiable-mod-n, then $\{x \in M : \phi^M(x)\}$ exists. Of course this will hold in $\mathrm{TC}_n\mathrm{T}$ ... which may be the correct setting for this observation: $\mathrm{TC}_n\mathrm{T}$ has subversion of stratification for x s.t $\iota^n \restriction x$ exists, in the sense that—at all levels—the following holds.

REMARK 6 *If $\iota^n \restriction x$ exists, and ϕ is stratifiable-mod-n then $\{y \in x : \phi(y)\}$ exists.*

Should really write out a proof

Just as subversion for strongly cantorian sets gives us interpretations into (extensions of) NF of fully unstratified set theories, subversion for sets x for which $\iota^n \restriction x$ exists will give us interpretations into (extensions of) NF of set theories satisfying syntactic contraints correspondingly less onerous than full stratification. Does this open up a vein of novel, more delicate, relative consistency proofs? Possibly, but not if we are adopting an axiom of infinity: the assumption that there is an (infinite) x s.t. $\iota^n \restriction x$ exists is as strong as the assumption that there is an infinite strongly cantorian set. This triviality is worth minuting because we will make use of it elsewhere (see p. 5).

REMARK 7
(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then x is strongly cantorian.
(ii) If there is an infinite x and a concrete n such that $\iota^n \restriction x$ exists then the axiom of counting holds.

Proof:

(i) If x is a wellorderable set s.t. $\iota^n \restriction x$ exists then the order type of any worder of x is certainly going to be less than all of Ω, Ω_1 ...[5], so we can assume without loss of generality that x is an initial segment X of the ordinals. This means that $\iota^n \restriction X$ exists, and that in turn means that $T^n \restriction X$ exists, and that

[5] Ω is the order type of the set of ordinals; $\Omega_1 = T\Omega$, and so on.

in turn means that we can prove by induction on the ordinals that $T^n \restriction X$ is the identity. So, for every $\alpha \in X$, $T^n\alpha = \alpha$. For every ordinal α (and so in particular for every $\alpha \in X$) we have $\alpha = T\alpha \vee \alpha < T\alpha \vee \alpha > T\alpha$. The second disjunct implies (apply T to both sides) $T\alpha <^2 \alpha$ giving $\alpha < T\alpha < \ldots T^n\alpha$ contradicting $T^n\alpha = \alpha$; the third disjunct is refuted similarly. So $T \restriction X$ exists beco's it is the identity, so $\iota \restriction X$ exists as well.

(ii) This property "$\iota^n \restriction x$ exists" is preserved by power set as well as by subset, so if there is even one infinite set which has it then $\mathbb{N}$ will have it as well. (Just as: $\mathbb{N}$ is strongly cantorian if there is even one infinite strongly cantorian set). But $\mathbb{N}$ is wellordered, so we can apply part (i). ■

The other direction (inferring "$\iota^n \restriction \mathbb{N}$ exists" for arbitrary concrete n from the axiom of counting) is easy. Thus, for every (concrete) n, the axiom of counting is equivalent modulo NF to a formula that is stratifiable-mod-n. This is presumably something to do with it being invariant. If ϕ is, for each n, equivalent (modulo NF) to something that is stratifiable-mod-n must it be (NF)-invariant?

However if the axiom of infinity is not assumed we do get some play.

Let Mac_n be Mac with separation restricted to formulæ that are Δ_0 and stratifiable-mod-n. Analogues of the result in [9] to the effect that Mac + TCl can be interpreted can be obtained, saying that Mac_n + TCl can be interpreted into KF, but these results are weaker than the result in [9]. However these refined constructions could turn out to be useful should there turn out to be theories of the form $\text{Mac}_n \cup \{A\}$ (where A is some formula not a theorem of Mac). However no such examples leap to mind. Not to the authors' mind anyway: $\exists$NO might have sounded like a starter but is is inconsistent with the existence of $\iota^n \restriction x$ for all x. (This last follows from remark 7 part (i).)

The upshot of this is that $\exists$NO is incompatible with Mac_n, the point being that $\iota^n \restriction$the representative set of wellorderings would exist and that the quotient would be strongly cantorian.

Lemma 1 *If $(\forall x)(\iota^n \restriction x$ exists$)$ then $(\forall x)(\iota^{n \cdot k} \restriction x$ exists$)$ for all concrete k.*

Proof: We know that RUSC(R) always exists, so $\text{RUSC}^k(R)$ exists for all R and all concrete k, so $\text{RUSC}^k(\iota^n \restriction x)$ exists and so $\iota^n \restriction x$ composed with $\text{RUSC}^n(\iota^n \restriction x)$ exists, and that is $\iota^{n \cdot 2} \restriction x$. And so on for all the other multiples of n. ■

5.2 Analogues in TC_nT

In TC_nT the assertion "$\iota^n \restriction x$ exists" is wellformed, so this expression gives us a good notion of strong cantorian-ness for TC_nT. Thus it comes to pass that in TC_nT we can state a version of the axiom of counting. Presumably it will be the case that if the axiom of counting holds at one level it holds at all levels. This kind of thing is making it look ever more likely that TC_nT and NF are equiconsistent.

5.3 Finitising the restriction of the scheme of Δ_0 separation to formulæ that are stratifiable-mod-n

We know how to finitely axiomatise stratifiable Δ_0 separation, and we can get full Δ_0 separation from that axiomatisation simply by adding the existence of $\iota \restriction x$ for all x. It seems fairly clear that the way to modify the collection of rudimentary functions to obtain separation for Δ_0 formulæ that are stratifiable-mod-n is to replace the function giving $\{\langle \iota(x), y\rangle : x \in y \in A\}$ by the function giving $\{\langle \iota^{n+1}(x), y\rangle : x \in y \in A\}$. It seems clear, but it might be an idea to write out the details; all it would involve is a simple modification of the proof in the second edition of the monograph [6].

5.4 Ambiguity in $\mathrm{TC}_n\mathrm{T}$

Take a simple example: $\mathrm{TC}_2\mathrm{T}$. Since every formula that is stratifiable-mod-4 is also stratifiable-mod-2 we can assert in the language of $\mathrm{TC}_2\mathrm{T}$ that the `yin` collection that would be the quartic Russell class $\{x : x \not\in^4 x\}$ exists. Ambiguity for formulæ that are stratifiable-mod-4 would then say that the corresponding `yang` set exists. See the discussion on page 10.

If we am right about all ambiguity^n schemes being of equal consistency strength then it should be easy to prove the consistency of $\mathrm{TC}_n\mathrm{T}$ + Ambiguity for formulæ that are stratifiable-mod-$m \cdot n$ relative to $\mathrm{TC}_n\mathrm{T}$. Yeah right.

5.5 CO models for $\mathrm{TC}_n\mathrm{T}$

It is simplicity itself to cook up a CO model of (the version of) $\mathrm{TC}_n\mathrm{T}$ that corresponds to AST. Let $\langle \mathbb{N}, E\rangle$ be the standard Oswald model. Define a new relation E' on $\mathbb{N}$ by

$$2n\, E'\, 2m+1 \text{ iff } n\, E\, m \text{ and}$$

$$2n+1\, E'\, 2m \text{ iff } n\, E\, m.$$

That way even numbers are `yin` and odd numbers are `yang`. I think the double Russell class will turn out to contain precisely the wellfounded sets... but this will need to be checked. It's clear how to do the same for $\mathrm{TC}_n\mathrm{T}$ for $n > 2$. You partition $\mathbb{N}$ into the n residue classes mod n and you say that i is a member of j in the new sense if $i + 1 \equiv j$ mod n and $(i \texttt{ DIV } n)\ E\ (j \texttt{ DIV } n)$.

Of course there is nothing special about E. We can do this for any Oswald model at all. The careless reader (like this careless writer, for one) might suspect that one can even produce a model of $\mathrm{TC}_2\mathrm{T}$ that has the double Russell class in one lobe but not in the other. But it's elementary to check that this is not possible. Think about the singleton $\{D\}$ of the double Russell class D. Do we have $\{D\} \in D$? If we do then we can't have $D \in \{D\}$, which is absurd. So $\{D\} \not\in D$. So $\{D\}$ must be a member of one of its members. But its only member is D, whence $\{D\} \in D$ after all. We saw this in remark 4.

What we might be able to do is get a model of the AST version of $\mathrm{TC}_2\mathrm{T}$ with a Boffa antiatom in one lobe but not in the other. It might be an instructive exercise to write this out in some detail.

We'll have two copies of $\mathbb{N}$: `yin` naturals and `yang` naturals. And we'll put a Boffa antiatom into level `yang` but not into level `yin`. In n is a `yin` natural and m a `yang` natural then we ordain than m is a member of n in the new sense iff $m\,E\,n$, where E is the membership relation of the Oswald model. Membership of `yang` naturals echoes the construction of CO models containing moieties. You look at `yang` naturals mod 4: that is to say, peel off the two least significant bits of a `yang` natural m and use them as a flag, which of course is $0, 1, 2$ or 3.

> If the flag is 0 then we say n belongs to m in the new sense iff the nth bit of the truncation is 1;
>
> If the flag is 1 then we say n belongs to m in the new sense iff the nth bit of the truncation is 1;
>
> If the flag is 2 then we say n belongs to m in the new sense iff (the nth bit of the truncation is 1 iff n belongs to the complement of the Boffa antiatom);
>
> If the flag is 3 then we say n belongs to m in the new sense iff (the nth bit of the truncation is 1 iff n belongs to the Boffa antiatom).

But questions of whether or not any given `yin` n belongs to the `yang` Boffa antiatom are answered by examining whether the Boffa antiatom is a member of n. And membership of `yang` sets in `yin` sets is unproblematic.

5.6 Generalise a Result of Specker?

Specker shows that in the situation where our language admits an automorphism $*$ of order 2, a conjunction of finitely many assertions of the form $\phi \longleftrightarrow \phi^*$ is another expression of that form. See Chad Brown's discussion of this question. Can we do anything similar here? Does it matter?

6 Applications to Duality

The special case of stratification-mod-n which will concern us here is $n = 2$. The context throughout this section is NF.

Definition 5 *The dual $\widehat{\phi}$ of a formula ϕ is the formula obtained from ϕ by replacing all occurrences of '$\in$' in ϕ by '$\not\in$'.*

It has been known for some time that $\phi \longleftrightarrow \widehat{\phi}$ is a theorem of NF whenever ϕ is a closed stratifiable formula. Permutation models can be found in which $\phi \longleftrightarrow \widehat{\phi}$ *fails* for some unstratifiable ϕ, but it remains an open question whether or not there are models in which $\phi \longleftrightarrow \widehat{\phi}$ *holds* for all ϕ. The natural conjecture is that there should be such models.

We do not prove the full conjecture here but we can prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ at least for all ϕ that are stratifiable-mod-2. This will be theorem 4 below, and it is the principal aim of this section to prove it.

However, in preparation for theorem 4 we need to do a lot of bush-clearing in regard to NF's theory of permutations (and specifically involutions) of V, and this necessitates a few subsections of prolegomena.

First we reflect that the duality scheme might in principle be witnessed by the existence of an antimorphism. An **antimorphism** is a permutation τ of V satisfying $(\forall x, y)(x \in y \longleftrightarrow \tau(x) \notin \tau(y))$. An antimorphism that is an involution is a **polarity**. Clearly if there is an antimorphism then duality follows.

6.1 Transversals, Partitions, Conjugacy and the Axiom of Choice for sets of pairs

We will need the concept of a *transversal* for a disjoint family; it is a set that meets every member of the family on a singleton.

Nathan has found an injection i from the set of pairs into the set of singletons. This enables us to infer (2) from (1):

1. Every set of disjoint pairs has a choice function;

2. Every set of pairs has a choice function.

Let P be a set of pairs. We desire a choice function for it, but we know only (1)—not (2). The set

$$\{p \times i(p) : p \in P\}$$

is a family of disjoint pairs and therefore, by (2), has a choice function, f. We can recover a choice function f^* for P by $f^*(p) =: \texttt{fst}(f(p \times i(p)))$. ■

We will also need the equivalence of (3) and (4):

3. Every partition of V into pairs has a choice function;

4. Every set of disjoint pairs has a choice function.

If we are given a set of pairs we can make disjoint copies of it by the trick we used earlier. In fact—by using an i whose range is a moiety of singletons—we can ensure that the sumset $\bigcup P$ of the disjoint family P of pairs we construct by this method has a complement that is the same size as V. The complement $V \setminus \bigcup P$ therefore has a partition P' into pairs. Then $P \cup P'$ is a partition of V into pairs. Any selection set for this partition will give us a choice function for the partition we started with.

Two more propositions:

5. Whenever we partition V into pairs we get the same number of pairs;

6. Whenever we partition V into pairs the two partitions are conjugate.

It turns out that (6) is equivalent to AC_2. (I mention 5 only as a foil, lest a reader think i'm talking about 5 when i am in fact talking about 6).

One direction—$AC_2 \to 6$—is easy.

Suppose P_1 and P_2 are two partitions of V into pairs. By AC_2 we have a selection set S for P_1 and P_1 is obviously a bijection between S and $V \setminus S$. So $|S| = |V|$ and $|P_1| = T|V|$. We argue for P_2 similarly of course. So there is a bijection π between P_1 and P_2. For each $p \in P_1$ there are precisely two bijections between p and $\pi(p)$ and we use AC_2 to pick one. The union of all such chosen bijections is a permutation conjugating P_1 and P_2.

For the other direction, assume 6. If P is a partition of V into pairs then by 6 it will be conjugate to the partition $\{\{x, V \setminus x\} : x \in V\}$. That is to say, there is a permutation π of V such that, for all $p \in P$, π"p is a pair $\{x, V \setminus x\}$. But clearly the partition $\{\{x, V \setminus x\} : x \in V\}$ has a choice function f ("pick the element that contains $\emptyset$") so the choice function for P that we want is $p \mapsto \pi^{-1}(f(\pi``p))$. ■

So we have established:

Remark 8 *The following are equivalent:*

Every set of pairs has a choice function (AC_2);
Every set of disjoint pairs has a choice function;
Any two partitions of V into pairs are conjugate;
Every partition of V into pairs has a choice function.

If we partition V into pairs how many do we get? No more than $T|V|$ (by this result of Bowler's) but can we get fewer? Try to connect this with the question of whether or not $|V|$ is decomposable.

Definition 6 *An involution with no fixed points and no transversal set is* **bad**.

I assume the reader can work out for themselves that if τ is a polarity then τ is a bad involution.

A conversation with Nathan

tf:

Suppose σ is a flexible permutation, and it lives on a moiety M. Then we can copy it over to a permutation living on $V \setminus M$, because there is an involution π mapping M onto $V \setminus M$. I now think of σ as a digraph. How do i move along an edge of σ? Well, i can move over into $V \setminus M$ by π (which is a good involution). Then i come back to M by means of the involution that swaps each x in ths support of the copied version of σ (that lives in $V \setminus M$) with $\sigma(\pi(x))$.

Nathan:

Call this second involution τ. Then for x in the support of σ, $\tau(\pi(x)) = \sigma(\pi(\pi(x))) = \sigma(x)$, which is a good sign. However, $\tau \cdot \pi$ also moves some other stuff. Let x be in the support of the copied version of σ. So $\pi(x)$ is in the support of σ. What does τ do to $\pi(x)$? Well, consider $y = \pi(\sigma^{-1}(\pi(x)))$. y is also in the support of the copied version of σ, and so τ swaps y with $\sigma(\pi(y)) = \pi(x)$. That is, $\tau(\pi(x)) = y$, so $\tau \cdot \pi$ does not equal σ, which fixes x.

Indeed, with sufficient lack of choice there cannot be a way to represent every permutation as a product of two involutions. Suppose that there is some permutation σ consisting of one cycle C_n of each finite odd size n, where there is no choice function on those cycles. Suppose further (for a contradiction) that $\sigma = \tau \cdot \pi$, where τ and π are involutions. Then $\tau \cdot \sigma \cdot \tau = \pi \cdot \tau = \sigma^{-1}$, so τ conjugates σ to σ^{-1}. In particular, τ takes fixed points of σ to fixed points of σ and elements of C_n to elements of C_n for each n. Identifying C_n with the integers modulo n, with the action of σ being addition of 1, we get $\pi(x+1) = \pi(x) - 1$, for any x, so that by induction $\pi(x) + x$ is constant on C_n. Say it takes the value k. Then $x = \pi(x)$ iff $x = k - x$ iff $2x = k$ iff $x = k/2$ modulo n. As n is odd, there is a unique solution of this equation modulo n. That is, π fixes precisely one element of C_n for each n. This gives a choice function on the C_n, which is the desired contradiction.

tf:

Ah, i think i see …The point is that $\tau \cdot \pi$ is not σ but the union of σ and its copy in $V \setminus M$.

Nathan:

Exactly so. But this is certainly progress. Suppose now that we have some permutation σ, supported on a moiety, that we want to represent as a product of involutions. By the argument you suggested, we can get the permutation σ' consisting of countably many copies of σ and countably many copies of σ^{-1}: σ' is a product of two involutions. Then composing σ' with the conjugate of σ' which cancels all the copies of σ^{-1} and all but one of the copies of σ, we get σ as a product of four involutions (this was my original argument, but not the argument in the paper).

…well, this isn't what I was thinking of, but it does work, and (with a little tweaking) shows that every flexible permutation is a product of at most 4 good involutions. Let's say we have some flexible permutation σ. Identify V with $V \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers, and let π be the permutation which moves each copy of the universe up one place: $\langle x, m\rangle \mapsto \langle x, m+1\rangle$. Let τ be the permutation which moves almost everything down one place: $\langle x, m\rangle \mapsto \langle x, m-1\rangle$ unless $m = 1$, and $\langle x, 1\rangle \mapsto \langle \sigma(x), 0\rangle$. Then τ and π are both products of $\mathbb{Z}$-cycles-with-distinguished-elements, so that each of τ and π is a product of two good involutions. σ is conjugate to $\tau \cdot \pi$, so is a product of 4 good involutions.

We consider the sequence of permutations: $\mathbb{1}$, c, $jc \cdot c$, $j^2c \cdot jc \cdot c \ldots$, where c is the complementation permutation. The superscripts are all small (they are all concrete numerals, in fact), so—rather than persist with the more general but slightly unwieldy $H(c, i)$ notation of [4] introduced above—we will revert to the simpler (original) notation of Henson, in which these permutations are written 'c_i', thus: $c_1 := c$; $c_{i+1} := j(c_i) \cdot c$.

We will need some lemmas.

We are interested in the behaviour of $j\tau \cdot c$ because τ is an antimorphism iff $\tau = j(\tau) \cdot c$.

Lemma 2 *AC_2 implies that, for all permutations τ, $j\tau \cdot c$ has fixed points iff τ has no odd cycles.*

Proof:

R $\rightarrow$ L

Suppose X is a fixed point for $j\tau \cdot c$. Then, for each τ-cycle C, we must have $\tau``(X \cap C) = C \setminus X$ and that means that $|C|$ must be even (or infinite). This direction does not need AC_2.

L $\rightarrow$ R

This direction needs AC_2. Suppose τ has no odd cycles. Each τ-cycle splits into precisely two τ^2 cycles. Use AC_2 to pick, for each τ-cycle, one of the two τ^2-cycles into which it splits. The union of the set of chosen τ^2-cycles is a fixed point for $j\tau \cdot c$.

■

The converse is true too. Suppose τ is a permutation with no odd cycles, and assume the consequent. Then $j\tau \cdot c$ has a fixed point. τ itself of course has no fixed point. The fixed point for $j\tau \cdot c$ is a transversal for τ!

Lemma 3
(i) All the c_i are involutions;
(ii) All the c_i commute with each other.

Proof:

We start by noting a key triviality: c commutes with $j\tau$ for all τ.

(i) We prove this by induction on i. Suppose c_i is an involution. $c_{i+1} = jc_i \cdot c$. So $(c_{i+1})^2 = (jc_i \cdot c)^2 = jc_i \cdot c \cdot jc_i \cdot c$. Now by the key triviality we can rearrange to $jc_i \cdot jc_i \cdot c \cdot c = \mathbb{1}$. In fact this even shows that all products of the c_i are involutions.

(ii) We prove by induction on i that, for all j, c_i commutes with c_j. Case $i = 0$. $c_0 = c$ and c commutes with $j(\pi)$ for all π. But every c_j is $j(\pi) \cdot c$ for some π, and (compose with c on the right) $j(\pi) \cdot c \cdot c = j(\pi)$ and if we compose with c on the left we get $c \cdot j(\pi) \cdot c$ which, too, is $j(\pi)$ because c commutes with $j(\pi)$.

Now for the induction.

$$c_{i+1} \cdot c_j = j(c_i) \cdot c \;\cdot\; j(c_{j-1}) \cdot c$$

and the RHS simplifies to

$$j(c_i) \cdot j(c_{j-1})$$

which is

$$j(c_i \cdot c_{j-1})$$

which by induction hypothesis is

$$j(c_{j-1} \cdot c_i)$$

which is

$$j(c_{j-1}) \cdot j(c_i).$$

We now sprinkle a couple of cs judiciously—by the triviality we know can insert them anywhere—obtaining

$$j(c_{j-1}) \cdot c \;\cdot\; j(c_i) \cdot c$$

which is of course

$$c_j \cdot c_{i+1}.$$

■

We will make much use of the fact that an involution without fixed points can be thought of as a partition of V into pairs.

We will make much use of the triviality that, if π is an involution without fixed points then the fixed points for $j\pi$ are precisely the transversals for π.

Lemma 4 *Any two involutions-without-fixed-points whose corresponding partitions-of-V-into-pairs have transversals are conjugate.*

Proof:

First we establish that if P is a transversal for a partition Π of V into pairs then its cardinality is $|V|$. Clearly $|\Pi| = T|P|$, since we can send each piece of Π to the unique singleton $\subset P$ that meets it. Observe that there is a bijection between $\iota``V$ and $\Pi \times \{0, 1\}$, as follows. For each x there is a unique $p_x \in \Pi$ with $x \in p_x$. If $x \in P$ we send $\{x\}$ to $\langle p_x, 0\rangle$; if $x \notin P$ we send $\{x\}$ to $\langle p_x, 1\rangle$.

Finally if π_1 and π_2 are two involutions-without-fixed-points equipped with transversals P_1 and P_2, then not only do we have $|P_1| = |P_2| = |V|$ but π_1 and π_2 are conjugate, as follows. P_1 and P_2 are in bijection, by a map π^*, say. Any such π^* can be extended to a permutation π of the universe by adding all the ordered pairs $\langle \pi_1(x), \pi_2(\pi^*(x))\rangle$ for $x \in P_1$.

■

Some minor points:

(i) The proof of lemma 4 given above tells us nothing about permutations that conjugate π_1 and π_2 beyond the fact that they exist. However the construction is effective and can be mined for more information. In lemmas **??** and 9 we consider a particular case in which we need more information and we go into more detail.

(ii) Notice that in lemma 4 the assumption on the two involutions is that the corresponding partitions have transversals. It is not the weaker assumption that the corresponding partitions are the same size. *Might* it be possible to prove in NF that any two partitions of V into pairs are the same size...? After all, Nathan Bowler [1] has shown us a proof in NF that there are as many pairs as singletons.

Sadly no, not unless NF $\vdash$ AC$_2$.

REMARK 9 *If whenever σ and τ are two involutions-without-fixpoints whose two partitions of V into pairs are of the same cardinality then σ and τ are conjugate, then AC_2 follows.*

Proof:

Let π be a set of pairs without a choice function. Without loss of generality the pairs are disjoint. Take the disjoint union of $\bigcup \pi$ with V. The result is the same size as V and can be canonically split into pairs using c (on the copy of V) and π (on the copy of $\bigcup \pi$). Copy this over into a partition of V into pairs. We have $T|V|$-many pairs, which is the same as the number of pairs in the partition corresponding to c. So—*if any two partitions of V into the same number of pairs are conjugate*—then this π must have a choice function. But π was arbitrary. ∎

REMARK 10
Let σ and τ be involutions of V.

> *(1) Let τ be an involution without fixpoints. Then T is a transversal for τ iff T is a fixpoint for $j\tau \cdot c$;*
>
> *(2) T is a fixpoint for σ iff $B(T)$ is a transversal for $j\sigma \cdot c$.*

Proof:

(1) Think of τ as a partition of V into pairs. Then, if T is a transversal, $V \setminus T$ (which is also a transversal) is precisely $\tau``T$.

(2) A piece of [the partition] $j\sigma \cdot c$ is a pair $\{x, V \setminus \sigma``x\}$—which of course might be a singleton. If $\sigma(T) = T$ then, for all $x$, precisely one of $x$ and $V \setminus \sigma``x$ will contain T. That is to say, $\{x, V \setminus \sigma``x\} \cap B(T)$ is a singleton, so $B(T)$ is a transversal.

For the other direction ... if $B(T)$ is a transversal for $j\sigma \cdot c$ then, for all x, precisely one of x and $V \setminus \sigma``x$ contains T, which is to say that $T \in x \longleftrightarrow \sigma(T) \in x$. But x could be $\{T\}$.

∎

Corollary 1
τ is bad iff $j\tau \cdot c$ is bad.

Proof:
$j\tau \cdot c$ bad implies τ bad:
Suppose $j\tau \cdot c$ is bad. Then it has no transversals. In particular for no T is $B(T)$ a transversal, so for no T is T a fixpoint for τ.
Suppose $j\tau \cdot c$ is bad. Then it has no fixpoints. So τ has no transversals.
Now for τ bad implies $j\tau \cdot c$ bad.
τ has no fixpoint so $j\tau \cdot c$ has no transversal.
τ has no transversal. Suppose, *per impossibile*, that $j\tau \cdot c$ has a fixpoint, x. Then $x = V \setminus \tau``x$ which says that x is a transversal .

■

One idea, new in this paper and due to the second author, is that of a *universal involution*.

Definition 7 *For permutations σ and τ of sets X and Y, a* **map of permutations** *from σ to τ is a function $\pi\colon X \to Y$ such that $\pi \cdot \sigma = \tau \cdot \pi$. If π is injective, we call it an* **embedding of permutations***.*

For the moment we need definition 7 only for involutions, and we will speak of *involution-embeddings* or *embeddings of involutions*. In all settings we will write "$\sigma \leq \tau$."

We will need the following analogue of Cantor-Bernstein for embeddings-of-involutions.

Lemma 5 *Let σ and τ be involutions of X and Y such that there are embeddings π of σ into τ and ρ of τ into σ. Then σ and τ are conjugate.*

Proof: Most proofs of the Cantor-Bernstein theorem extend to proofs of this fact. For the sake of brevity, we will use a proof based on the Knaster-Tarski theorem that any order-preserving function on a complete lattice has a fixed point. Applying this to the lattice of sets which are closed under the action of σ and the order-preserving function $S \mapsto X \setminus j(\rho)(Y \setminus j(\pi)(S))$ we obtain a fixed point P. Then the map defined by π on P and ρ^{-1} on $X \setminus P$ is an isomorphism from σ to τ. ■

Thus we can think of the permutation-embedding relation as giving rise to a partial order on conjugacy classes of permutations. It's a directed poset because of disjoint unions of copies of V. (Is it an upper semilattice? It certainly supports a + operation, but whether or not $[\sigma] + [\tau]$ is the sup of $[\sigma]$ and $[\tau]$ is another matter!) We observe without proof [at least for the moment] that if π is an embedding of permutations from σ to τ then $j(\pi)$ is an embedding of permutations from $j(\sigma)$ to j(τ). That is to say, conjugacy is a congruence relation for j, so we can think of j as acting on the congruence classes, and it will be an (injective) endomorphism of the poset (We will need this in the proof of the second part of lemma 6.)

Also worth minuting is the fact that

Remark 11 *Conjugacy is a congruence relation for $\pi \mapsto j\pi \cdot c$.*

Proof:

Suppose σ and τ are conjugate; so, for some π, $\pi \cdot \sigma \cdot \pi^{-1} = \tau$. Then
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) = j(\tau)$. Compose with c:
$j(\pi) \cdot j(\sigma) \cdot j(\pi^{-1}) \cdot c = j(\tau) \cdot c$. But c commutes with j of anything, giving:
$j(\pi) \cdot j(\sigma) \cdot c \cdot j(\pi^{-1}) = j(\tau) \cdot c$
which says that $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ are conjugate.

■

Notice that $j(\sigma) \cdot c$ and $j(\tau) \cdot c$ are conjugated by j of something, which is a stronger condition than simply being conjugate. There seems to be no obvious reason why the induced function $[\sigma] \mapsto [j\sigma \cdot c]$ on conjugacy classes should be injective. However it does seem to preserve that quasiorder we have just defined:

Remark 12 *$\sigma \mapsto j\sigma \cdot c$ is order-preserving.*

Proof:

Suppose $\sigma \leq \tau$. Then $\exists \pi$
$\pi \cdot \sigma = \tau \cdot \pi$. This gives
$j(\pi) \cdot j(\sigma) = j(\tau) \cdot j(\pi)$ and
$j(\pi) \cdot j(\sigma) \cdot c = j(\tau) \cdot j(\pi) \cdot c$ and
$j(\pi) \cdot j(\sigma) \cdot c = j(\tau) \cdot c \cdot j(\pi)$ which says that
$j(\sigma) \cdot c \leq j(\tau) \cdot c$.

■

It will turn out that the subposet consisting of conjugacy classes of involutions (which of course is also directed, for the same reason) also has a top element.

Definition 8
An involution is universal *if every involution can be embedded into it.*

We begin by giving some examples of universal involutions of V.

Lemma 6 *For all i, $j^i(c)$ is universal.*

Proof:

First we prove that $j(c)$ is universal.

Let θ be any bijection from V to $V \setminus \{\emptyset\}$. For any involution σ of any set X we define an embedding of involutions π from σ to $j(c)$ by $x \mapsto j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x))$. The function π is injective, with left inverse $y \mapsto j(\theta^{-1})(\{z \in y : \emptyset \notin z\})$. To see that π is a map of involutions from σ to $j(c)$ we calculate as follows:

$$\begin{aligned}
(j(c) \cdot \pi)(x) &= j(c)(j(\theta)(x) \cup j(c \cdot \theta)(\sigma(x))) \\
&= j(c \cdot \theta)(x) \cup j(c \cdot c \cdot \theta)(\sigma(x)) \\
&= j(\theta)(\sigma(x)) \cup j(c \cdot \theta)(\sigma(\sigma(x))) \\
&= (\pi \cdot \sigma)(x)
\end{aligned}$$

For the main result we argue as follows.

Clearly any involution into which a universal involution can be embedded is also universal, and any involution conjugate to a universal involution is again universal.

Since $j(c)$ is universal, there is an embedding of c into $j(c)$. This lifts to embeddings of $j^i(c)$ into $j^{i+1}(c)$, and composing these embeddings we get embeddings of $j(c)$ into $j^i(c)$ for any $i \geq 1$. Thus $j^i(c)$ is universal for any $i \geq 1$.

■

Thus $j(c)$ is a universal involution. In the medium term we are going to be interested in the possibility of permutations that are universal for other classes of permutations. For the moment let us just reflect that by considering the restriction of $j(c)$ to those sets that are not closed under complementation (i.e., the set of its *non*-fixpoints) we obtain an involution which is universal for involutions without fixpoints. For consider: if π is any such permutation, there is a permutation-embedding from π into $j(c)$ and that embedding must send π into that part of $j(c)$ that consists of pairs not singletons. It's easy to check that there are $|V|$-many sets that are not closed under complementation (*) so that part of $j(c)$ can be copied over to V.

[For (*) reflect that $\{x : V \in x \wedge \emptyset \not\in x\}$ is a subset of the collection of sets not closed under complementation, so it will suffice to show that it is of size $|V|$. But it's a moiety of a moiety, in the sense that $B(V)$ is a moiety and provably the same size as V, and its members fall into one two pieces depending on whether or not they contain $\emptyset$, and these two pieces are of course the same size as each other and the same size as V.]

Suppose AC_2 fails, so that there are ("bad") involutions with neither fixed points nor transversals. If τ is a bad involution then so is $j\tau \cdot c$ by remark 10. And if there are bad involutions then any involution that is maximal among involutions without fixed points will be bad. If we can show that there is a unique conjugacy class of bad involutions then we will have shown that if AC_2 fails then there is a permutation model containing a polarity. In your dreams innit.

What we might be able to do is this: If AC_2 fails then there is a bad involution, so any involution that is universal for involutions-without-fixpoints (uiwf) is bad. So all we need to show is that if τ is a bad involution that is uiwf then $j\tau \cdot c$ is uiwf. That doesn't sound obviously impossible: after all, we already know that one is bad iff the other one is. All we have to do now is show that one is uiwf iff the other is.

The key fact which we will need later is that any two universal involutions are isomorphic to each other. This follows from lemma 5, the "Cantor-Bernstein" theorem for involutions.

Lemma 7 *c_2 is conjugate to $j(c)$ and so is also universal.*

Proof:

Given a set of the form $x \triangle B(\emptyset)$ we can recover x since it is $(x \triangle B(\emptyset)) \triangle B(\emptyset)$. So $x \mapsto x \triangle B(\emptyset)$ is injective. But the same thought reassures us that it is surjective too, so it is genuinely a permutation of V and, actually, an involution. In fact we could write it $\prod_{x\in V}(x, x\triangle B(x))$ as a product of transpositions ... or π for short. To see that π conjugates c_2 to $j(c)$, we calculate as follows:

$$\begin{array}{rcl}
(j(c)\cdot\pi)(x) & = & j(c)(x \triangle B(\emptyset)) \\
 & = & j(c)(x) \triangle j(c)(B(\emptyset)) \\
 & = & j(c)(x) \triangle (V \setminus B(\emptyset)) \\
 & = & j(c)(x) \triangle (V \triangle B(\emptyset)) \\
 & = & (j(c)(x) \triangle V) \triangle B(\emptyset) \\
 & = & (V \setminus j(c)(x)) \triangle B(\emptyset) \\
 & = & (c\cdot j(c))(x) \triangle B(\emptyset) \\
 & = & c_2(x) \triangle B(\emptyset) \\
 & = & (\pi\cdot c_2)(x)
\end{array}$$

■

Corollary 2 *Any two universal involutions are conjugate.*

■

Corollary 3 *Every model of NF has a permutation model with an internal $\in$-automorphism.*

Proof: In particular, it follows from corollary 2 that $j(c)$ and $j^2(c)$ are conjugate, making $j(c)$ an example of a permutation which is conjugate to j of itself. It was shown in [6] that any model containing such a permutation π has a permutation model wherein π has become an (internal) $\in$-automorphism (one that is a set of the model). In [6] it is shown that there must be such a π, but on the assumption of AC_2, and of course we have here scrupulously eschewed AC_2. ■

Some questions

Must a polarity be a universal involution?

Under what operations is the class of universal involutions closed?

Can we prove that if τ is universal and AC_2 fails then $j\tau\cdot c$ is universal?

If π is universal among involutions without fixed points is $j\pi$ universal?

Are the universal involutions a normal generating subset of J_0?

Are there maximal permutations? We could start by asking for a maximal permutation of order 3.

A polarity is *bad* it has neither fixed points nor transversals. Must the conjugacy class of a polarity be maximal among the involutions without fixed points? Perhaps not, but if there is only one such conjugacy class we argue as follows. Suppose π is a bad involution. Then $j\pi\cdot c$ is also bad, and is therefore conjugate to π so there is a permutation model with a polarity.

6.2 Finding permutations that will prove duality[2]

For the main argument which follows later, it will suffice to find involutions σ and τ such that there is a permutation π conjugating σ to $j(\tau) \cdot c$ and τ to $j(\sigma) \cdot c$. Our aim in this section is to show that this is possible, taking σ to be c_1 and τ to be c_2. So we must find a permutation π conjugating c_1 to c_3 but commuting with c_2.

Let us itemize this fact so we can refer back to it later.

Lemma 8
There is an involution that conjugates c with c_3 and commutes with $j^2c \cdot jc$.

Proof:

We begin by choosing a fixed point a of c_2 and setting $b = c_1(a)$. Since a is a fixed point of c_2 we also have $b = c_1(c_2(a)) = j(c)(a)$. For any $s \subseteq \{a, b\}$ we define X_s to be $\{x : x \cap \{a, b\} = s\}$.

$X_\emptyset$ is closed under both $j(c)$ and $j^2(c)$; let $\sigma_\emptyset$ be the restriction of $j(c)$ to $X_\emptyset$ and $\tau_\emptyset$ the restriction of $j^2(c)$. Then there are embeddings of $j(c)$ into $\sigma_\emptyset$ and $j^2(c)$ into $\tau_\emptyset$, so by the results of the last section both $\sigma_\emptyset$ and $\tau_\emptyset$ are universal. Let $\pi_\emptyset$ be an isomorphism from $\sigma_\emptyset$ to $\tau_\emptyset$. Since $j(c) = c_1 \cdot c_2$ and $j^2(c) = c_3 \cdot c_2$ we have the equation $\pi_1 \cdot c_1 \cdot c_2 = c_3 \cdot c_2 \cdot \pi_1$, which we note for future use.

We now define $\pi \colon V \to V$ by

$$x \mapsto \begin{cases} \pi_\emptyset(x) & \text{if } x \cap \{a, b\} = \emptyset \\ x & \text{if } x \cap \{a, b\} = \{b\} \\ c_3(c_1(x)) & \text{if } x \cap \{a, b\} = \{a\} \\ c_3(\pi_\emptyset(c_1(x))) & \text{if } x \cap \{a, b\} = \{a, b\} \end{cases}$$

Then π is a union of bijections from X_s to X_s for each $s \subseteq \{a, b\}$, so it is a bijection.

It remains to check that for any x we have $\pi(c_1(x)) = c_3(\pi(x))$ and $\pi(c_2(x)) = c_2(\pi(x))$. For each equation there are 4 cases, depending on $x \cap \{a, b\}$. We now check these cases for the first equation.

- If $x \cap \{a, b\} = \emptyset$, then $c_1(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_1(x)) = c_3(\pi_\emptyset(c_1(c_1(x)))) = c_3(\pi_\emptyset(x)) = c_3(\pi(x)) .$$

- If $x \cap \{a, b\} = \{b\}$ then $c_1(x) \cap \{a, b\} = \{a\}$ and so

$$\pi(c_1(x)) = c_3(c_1(c_1(x))) = c_3(x) = c_3(\pi(x)) .$$

- If $x \cap \{a, b\} = \{a\}$ then $c_1(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_1(x)) = c_1(x) = c_3(c_3(c_1(x))) = c_3(\pi(x)) .$$

- If $x \cap \{a, b\} = \{a, b\}$ then $c_1(x) \cap \{a, b\} = \emptyset$ and so

$$\pi(c_1(x)) = \pi_\emptyset(c_1(x)) = c_3(c_3(\pi_\emptyset(c_1(x)))) = c_3(\pi(x)) .$$

The four cases for the other equation are similar.

- If $x \cap \{a, b\} = \emptyset$ then $c_2(x) \cap \{a, b\} = \{a, b\}$ and so

$$\pi(c_2(x)) = c_3(\pi_\emptyset(c_1(c_2(x)))) = c_3(c_3(c_2(\pi_\emptyset(x)))) = c_2(\pi_\emptyset(x)) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{b\}$ then $c_2(x) \cap \{a, b\} = \{b\}$ and so

$$\pi(c_2(x)) = c_2(x) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a\}$ then $c_2(x) \cap \{a, b\} = \{a\}$ and so

$$\pi(c_2(x)) = c_3(c_1(c_2(x))) = c_2(c_3(c_1(x))) = c_2(\pi(x))\,.$$

- If $x \cap \{a, b\} = \{a, b\}$ then $c_2(x) \cap \{a, b\} = \emptyset$ and so

$$\pi(c_2(x)) = \pi_\emptyset(c_2(x)) = \pi_\emptyset(c_2(c_1(c_1(x)))) = c_2(c_3(\pi_\emptyset(c_1(x)))) = c_2(\pi(x))\,.$$

■

Lemma 9 *Every model of NF has a permutation model containing two permutations σ and τ satisfying*

$$(\forall xy)(x \in y \longleftrightarrow \sigma(x) \notin \tau(y)) \quad \textit{and} \quad (\forall xy)(x \in y \longleftrightarrow \tau(x) \notin \sigma(y)).$$

which is to say: $\sigma = j\tau \cdot c$ and $\tau = j\sigma \cdot c$.

Proof:

Our construction will exhibit σ and τ that are commuting involutions.

Suppose we can find π such that $V^\pi \models (\exists\sigma)(\sigma = j^2\sigma \cdot jc \cdot c)$. We find that in this model, for all x, y and z

$$x \in y \ \wedge \ y \in z$$

iff

$$\sigma(x) \in j\sigma(y) \ \wedge \ j\sigma(y) \in j^2\sigma(z)$$

iff

$$\sigma(x) \notin c \cdot j\sigma(y) \ \wedge \ c \cdot j\sigma(y) \in jc \cdot j^2\sigma(z)$$

iff

$$\sigma(x) \notin c \cdot j\sigma(y) \ \wedge \ c \cdot j\sigma(y) \notin c \cdot jc \cdot j^2\sigma(z)$$

iff

$$\sigma(x) \notin c \cdot j\sigma(y) \ \wedge \ c \cdot j\sigma(y) \notin \sigma(z)$$

and, writing 'τ for '$c \cdot j\sigma$' this becomes

$$\sigma(x) \notin \tau(y) \ \wedge \ \tau(y) \notin \sigma(z)$$

giving us the two permutations we want.

But such a π was provided by lemma 8

■

Corollary 4 *Every model of NF has a permutation model that satisfies duality for formulæ that are stratifiable-mod-2.*

Proof: We use the permutation π from lemma 9). ■

It's worth bearing in mind that σ and τ retain in V^π all the stratified properties they had in their previous life in V, where they were c and c_2. Thus they commute, and $\sigma^2 = \tau^2 = \mathbb{1}$. Observe also that $j(\sigma\tau) = j\sigma \cdot j\tau = \tau \cdot c \cdot c \cdot \sigma = \tau\sigma = \sigma\tau$, so $\sigma\tau$ is actually an $\in$-automorphism of V^π. It is a nontrivial automorphism beco's σ and τ are not inverse to each other: τ has fixed points and σ does not. By the remark in the proof of part (i) of lemma 3 $\sigma\tau$ is an involution.

We should be able to express this as a fact inside the base model....

Can we use this technique to obtain models in which duality holds for formulæ that are stratifiable-mod-p for other primes? No. If we were to attempt to rejig the above development to obtain a proof for formulæ that are stratifiable-mod-3 then we would be looking for an i such that c_i and c_{i+3} are conjugate. However, as we saw in the discussion following corollary **??**, all the c_{2n} are conjugate to c_2 (which has fixed points) and the c_{2i+1} are conjugate to c (which has none); so c_i and c_{i+3} can never be conjugate.

We see this most starkly in the case of formulæ which are stratifiable-mod-1, which is to say *all* formulae. To find—by this method—a permutation model in which duality held for *all* formulæ we would want the model to contain an antimorphism: a permutation τ such that $\tau = j\tau \cdot c$. This would involve finding a permutation τ in our home model such that τ and $j\tau \cdot c$ were conjugate. Unfortunately—as lemma 2 tells us—AC_2 implies, for all permutations τ, that $j\tau \cdot c$ has fixed points iff τ has no odd cycles. So, in particular, τ and $j\tau \cdot c$ cannot be conjugate.

Very well, so we drop AC_2, in the hope that this might open up the possibility of an involution τ such that τ and $j\tau \cdot c$ have the same cycle type. Such a τ would not be definable. But then we would need AC_2, after all, to show that τ and $j\tau \cdot c$ are conjugate.

Clearly if we are to prove the relative consistency of the scheme $\phi \longleftrightarrow \widehat{\phi}$ for *all* ϕ we need a new idea.

I mentioned earlier that duality for sentences that are stratifiable-mod-2 is much weaker than the conjectured duality for all sentences. In one respect, however, the result we have just shown does more: the existence of the τ and σ combining as above would appear to be more than is needed to establish duality for sentences that are stratifiable-mod-2; The existence of the τ and σ stand to duality for sentences that are stratifiable-mod-2 in the same way that the existence of an antimorphism stands to full duality. In both cases the first party to the relation seems to be on the face of it much stronger than the second. The existence of an antimorphism certainly implies duality but the converse looks most unlikely, since the existence of an antimorphism strongly contradicts AC. It ought to be possible to obtain models of duality for sentences that are stratifiable-mod-2 without actually exhibiting functions that witness it.

6.3 Full Duality?

It may be that the set of things fixed by $\sigma\tau$ is a model of NF + full Duality. Something to check!

First we check that $\sigma\tau$ (which is the same as $\tau\sigma$) is an $\in$-automorphism. For all x and y we have $x \in y \longleftrightarrow \sigma(x) \notin \tau(y)$ so $\sigma(x) \notin \tau(y) \longleftrightarrow \tau\sigma(x) \in \sigma\tau(y) = \tau\sigma(y)$ so $\tau\sigma$ is an $\in$-automorphism as desired.

Next we check that if π is an $\in$-automorphism then the set of fixed points is a model of NF. The big gap here is extensionality. We would have to show that every nonempty fixed set has a fixed member.

Finally we check that the set of fixed points of $\sigma\tau$ is additionally a model of duality. Observe that, for all such fixed x we have $x = \sigma(\tau(x))$ whence $\sigma^{-1}(x) = \tau(x)$. But $\sigma^2 = \mathbb{1}$ so $\sigma(x) = \tau(x)$.

Now suppose x and y both fixed. Then $x \in y \longleftrightarrow \sigma(x) \notin \tau(y) = \sigma(y)$. So σ is an antimorphism of the fixed points.

But this relies on the set of fixed points being extensional. It may be that we can ensure this by a judicious choice of the permutation in lemma 9. We seek a π that conjugates c to $j^2c \cdot jc \cdot c$ and moreover has the extra feature that in V^π the set $\{x : \sigma(x) = \tau(x)\}$ is extensional. Must turn this into a condition on π.... We think

$$V^\pi \models (\forall x)(x \neq \emptyset \wedge \sigma\tau(x) = x \rightarrow (\exists y \in x)(\sigma\tau(y) = y))$$

is

$$(\forall x)(\pi(x) \neq \emptyset \wedge \sigma\tau(x) = x \rightarrow (\exists y \in \pi(x))(\sigma\tau(y) = y))$$

which becomes

$$(\forall x)(x \neq \emptyset \wedge j^2c \cdot jc(x) = x \rightarrow (\exists y \in \pi(x))(j^2c \cdot jc(y) = y))$$

where π conjugates c and $j^2c \cdot jc \cdot c$.

Let us write 'F' for $\{x : x = jc \cdot j^2c(x)\}$ to keep things readable. The π we seek has got to inject F into $\{y : y \cap F \neq \emptyset\}$—o/w known (see p. 6) as "$\text{\L}(F)$". Observe that $\text{\L}(x)$ is always a moiety, since it is $V \setminus (\mathcal{P}(V \setminus x))$, and the complement of a power set (of anything other than V) is always the same size as V. This is beco's every set (other than V itself) is included in the complement of a singleton, and the power set of a complement of a singleton is a principal prime ideal and therefore a moiety.

So there's no problem on *that* score.

It's not blindingly obvious to me that it cannot be done.

7 Work still to do

There remains of course the challenge of proving consistency of duality for all sentences, not merely those that are stratifiable-mod-2. But more to the point are the possibilities of extending to formulæ that are stratifiable-mod-n things known about the rather more restricted class of stratified formulæ—and these we haven't started thinking about. Here are some, in no particular order.

We should show in an NF context that, for each n, the assertion that "there are sets x s.t. $\iota^n \upharpoonright x$ exists" is invariant.

• Is there any interest in versions of Forti-Honsell Antifoundation along the lines "Every set picture that is a n-stratification graph is a picture of a set"?

• If ϕ is, for each n, equivalent (modulo NF) to something that is stratified-mod-n must it be (NF)-invariant?

• Randall has just (4/vi/2016) pointed out to me that $\mathrm{TC}_n\mathrm{T}$ is in some sense the same theory as NFU + $|V| = |\mathcal{P}^n(V)|$. It could be a good idea to spell this out.

• In a model of $\mathrm{TC}_k\mathrm{T}$ one can sensibly ask, for any m, whether or not Ambiguity holds for formulæ that are stratifiable-mod-$k \cdot m$.

• What is the correct notion of a permutation model of a model of $\mathrm{TC}_n\mathrm{T}$? The only permutations that are available to us are permutations that fix each level setwise. Perhaps one could start by thinking of a model of $\mathrm{TC}_2\mathrm{T}$ and trying to add a Boffa atom by a transposition $\tau = (x, B(x))$ where x is of type `yin`. Then $x_{\mathtt{yin}} \in_\tau y_{\mathtt{yang}}$ iff $x_{\mathtt{yin}} \in y_{\mathtt{yang}}$ and $x_{\mathtt{yang}} \in_\tau y_{\mathtt{yin}}$ iff $x_{\mathtt{yang}} \in \tau(y_{\mathtt{yin}})$.

It's easy to check that the result is a model of $\mathrm{TC}_2\mathrm{T}$. Then we have to check that genuinely stratifiable expressions are preserved while the truth-values of (at least some) stratifiable-mod-2 expressions (such as the existence of a Boffa atom) are altered.

• André Pétry suggests a generalisation of a result of his-and-mine alluded to earlier ([7], [14], and [15]) to the effect that if two structures are elementarily equivalent for formulæ that are stratifiable-mod-n then they have stratimorphic (as it were) ultrapowers.

• One could investigate whether the construction of [8] could be modified to encompass expressions that are stratifiable-mod-n. That looks messy.

• There are natural settings where one encounters embeddings that are elementary for stratifiable formulæ, and where one might hope to get embeddings that are elementary for some of these larger classes of formulæ. CO models is one setting: the embedding from the ground model into the hereditary low sets is elementary for stratifiable formulæ. (That particular example is probably not a good one, because if the inclusion embedding is elementary for formulæ that are stratifiable-mod-n for even one n then the hereditarily low sets cannot contain any Quine atoms). For another, let $\mathfrak{M}$ be a structure for $\mathcal{L}$. Consider the class of those $m \in M$ s.t. m is fixed by all permutations of M that, for all n, are j^n of something. It's an elementary substructure as long as it's extensional.

Now use instead those permutations π of M s.t. $j^m\pi = \mathbb{1}$. Now the class of fixed things is a substructure elementary for expressions that are stratifiable mod m (again, assuming extensionality).

• Str(ZF) is the theory axiomatised by the stratifiable axioms of ZF; by analogy str_n(ZF) will be the theory axiomatised by those axioms of ZF that are stratifiable-mod-n. ZF can be interpreted in str(ZF) + IO. (IO is the axiom "every set is the same size as a set of singletons"). Observe that IO is a theorem of str_n(ZF), since it proves that $\iota^n \upharpoonright x$ exists for all x, so every set is the same size as a set of singletons, so ZF can be interpreted in str_n(ZF). At this stage we cannot see how to prove that $\mathrm{str}_n(\mathrm{ZF}) = \mathrm{ZF}$. There are parallel questions about the fragments of Mac.

• Stratified parameter-free Δ_0 $\in$-induction seems to prove no more than the nonexistence of a universal set. How about stratifiable-mod-n parameter-free $\in$-induction... what does that do? One might hope that it would prove the nonexistence of $\in$-loops of circumference n but we can't see it offhand. But in any case we should start with the case $n = 2$ in order to not drown *immediately* in the deep end. We noted in section 2.1 that the collections `I` and `II` as in [10] are both the extensions of expressions that are stratifiable-mod-2. So stratifiable-mod-n parameter-free $\in$-induction will imply $\in$-determinacy. (tho' that induction is not Δ_0...) Needs looking into.

Stratifiable parameter-free $\in$-induction implies the nonexistence of the universal set. (If none of your members are the universal set, you can't be either). It's not known if the converse holds. However the strengthening of the converse one would consider in this context, namely "the non-existence of the universal set implies $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" clearly does not go through: $\in$-induction for parameter-free formulæ that are stratifiable-mod-n" implies $(\forall x)(x \notin^2 x)$, and that clearly doesn't follow from the nonexistence of V.

• Suppose we add to our favourite theory of wellfounded sets a scheme of $\in$-induction for formulæ that are stratifiable-mod-n, for some or all n. Is it the case that any such model is first-order indistinguishable from a wellfounded model? Can we prove anything with that flavour ...?

• Every weakly stratifiable theorem of first-order logic has a cut-free weakly stratifiable proof; every stratifiable theorem of first-order logic has a stratifiable proof (Crabbé, [3]); are there analogues for stratification-mod-n? Every theorem of first-order logic that is stratifiable-mod-n has a proof that is stratifiable-mod-n? Crabbé thinks so. Why should it *not* work, after all?

• There is an old question about whether the atoms of a model of NFU can be indiscernible. We know that they are indiscernible wrt stratifiable formulæ; now that we've started looking into stratification-mod-n it is natural to wonder whether one might be able to show that the atoms of a model of NFU must be indiscernible wrt expressions that are stratifiable-mod-2. At this stage it's not looking hopeful.

• We should investigate the consistency results relative to T$\mathbb{Z}$T obtained by omitting types, to see how many of them work for $\mathrm{TC}_n\mathrm{T}$. They make heavy use of Coret's lemma. Coret's lemma tells us how permutations preserve stratifiable formulæ. Any old permutation works. In the NF context we know that if we want to preserve all formulæ then we can't use any-old-permutation but only $\in$-automorphisms. Working in $\mathrm{TC}_n\mathrm{T}$ we want to preserve formulæ that are stratifiable-mod-n, and that means using permutations π s.t. $\pi = j^n(\pi)$, and such permutations are not just lying around. $\mathrm{TC}_n\mathrm{T}$ really does behave more like NF than like T$\mathbb{Z}$T.

• There is the old question of whether or not Amb^n is equiconsistent with NF. Suppose we work in KF, and consider $\mathrm{TC}_2\mathrm{T}$ to keep things simple initially. Suppose we have an x with $|x| = |\mathcal{P}^2 x|$. Is that going to give us a model of NF? Let α be the cardinal of such an x. Can we prove that $\alpha = 2^{T\alpha}$? We suspect not, because that would probably say something about theorems in $\mathrm{TC}_2\mathrm{T}$. A useful thought is the fact that α is $\beth_n$ of something for all concrete n. So we certainly have $\alpha = \alpha+1$, $\alpha = \alpha+\alpha$, $\alpha = \alpha\cdot\alpha$. The plan is to use these equations to show that $x \sqcup \mathcal{P}(x)$ gives us a model of NF. So we want $T(2^{\alpha+2^{T\alpha}}) = \alpha + 2^{T\alpha}$. Now $T(2^{\alpha+2^{T\alpha}}) = 2^{T\alpha} \cdot 2^{2^{T^2\alpha}}$. $T(2^{\alpha+2^{T\alpha}}) = 2^{T\alpha} \cdot \alpha$.

So we want $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$, and we hope to get it from the good behaviour of α. We have $\alpha = \alpha^2$ so we get $2^{T\alpha} = 2^{T\alpha^2} = (2^{T\alpha})^{T\alpha}$ which looks hopeful but isn't exactly what we want. The warning sign is that if this worked it would show that $2^{T\alpha}$ absorbs α and that sounds extremely implausible.

But even if $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$ it wouldn't help. We can exploit Bernstein's Lemma to show that we would have $\alpha = 2^{T\alpha}$ or—at the very least—that each $\leq^*$ the other, which is just as bad, as follows.

If we have $\alpha + 2^{T\alpha} = 2^{T\alpha} \cdot \alpha$ then Bernstein's Lemma gives $\alpha \leq 2^{T\alpha} \vee 2^{T\alpha} \leq^* \alpha$ and $\alpha \leq^* 2^{T\alpha} \vee 2^{T\alpha} \leq \alpha$ so a case analysis gives $\alpha = 2^{T\alpha} \vee \alpha \leq^* 2^{T\alpha} \leq^* \alpha$ which gives $2^\alpha = 2^{2^{T\alpha}} = T\alpha$, which is altogether too strong.

One has the impression that KF really does not want to prove that if there is x with $|x| = |\mathcal{P}^n(x)|$ then there is an x with $|x| = |\mathcal{P}(x)|$. The moral of this seems to be that $\mathrm{TC}_2\mathrm{T}$ is not as much like NF as it might be.

• Consider "$\Box$(Duality for sentences that are stratifiable-mod-2)"
Is this consistent? Does it imply AC_2?

• ZF + Foundation and ZF + antifoundation are alike extensions of ZF + Coret's axiom "every set is the same size as a wellfounded set" conservative for stratifiable sentences. (See [11]). Does this hold also for sentences that are stratifiable-mod-n?

Checking this last one should be simple!

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