A new Datatype of `scansets` and some Applications: Interpreting Mac in KF

Thomas Forster

April 21, 2014

Contents

It's generally understood that KF and Mac are equiconsistent. I have stumbled upon a rather cute proof of this fact. Altho' it's simple it eluded me for years, and it's surprisingly easy to get it *wrong*, so—on both counts—i tho'rt i'd write it up. One direction is easy—since Mac is a superset of KF—so what we need is an interpretation of Mac into KF, and that is what we supply here. In fact we can interpret Mac + TCl into KF.

The obvious way to interpret an unstratified set theory (such as Mac) into a stratified set theory (such as KF) is by recourse to the device of the inner model of hereditarily strongly cantorian sets, for then one can exploit the fact that strongly cantorian sets *subvert stratification*. (The phrase is Holmes'). It is obvious, indeed, but there is an obstacle—slightly less obvious admittedly—but one that nevertheless obtrudes itself once one tries writing out the details. The problem arises with the axiom of sumset. It might seem obvious that KF proves that a strongly cantorian union of strongly cantorian sets is strongly cantorian, but close inspection reveals a lacuna. The desired map $\iota\restriction\bigcup X$ obviously wants to be obtained as the union of all the $\iota\restriction x$ for $x \in X$. The problem is that there is no reason to suppose that the set of those restrictions of ι is a set—even on the assumption that X is strongly cantorian. There was a time when i worried about this a lot. Is a [strongly] cantorian union of [strongly] cantorian sets [strongly] cantorian? And so on. I even wrote about it in [2]. (I hope my readers have now forgiven me). The problem is the existence of certificates... and although that is still a problem it is onei now know how to work round.

My initial thought was to invent a new datatype: objects that are (hereditarily) strongly cantorian sets equipped with a certificate that they are strongly cantorian. (A bit like count*ed* sets contrasted with count*able* sets.) But actually there is a solution along the same lines that is better and even simpler: instead of studying strongly cantorian sets equipped with restrictions of ι we take our objects of study to be those restrictions of ι themselves. The appropriate "membership" relation between these new objects is the membership relation between the sets to which they are the restrictions of ι. Then finally we of course restrict attention to the "hereditary" objects of this kind. Thus a `scanset` will be a restriction of ι to a set of scansets. This recursive datatype actually kills two birds with one stone, for it not only solves the sumset problem but also gives rise to a membership relation with a flexible definition that enables us to subvert stratification, and thereby interpret Mac (*sans* infinity) into KF[1]. (If we add infinity then all bets are off: clearly KFI cannot prove that there is an infinite strongly cantorian set.)

1 Definitions

We start with some background.

When x is an ordered pair we write '$\texttt{fst}(x)$' and '$\texttt{snd}(x)$' for its first and

[1]I hope the reader does not need to be warned that the form of words 'T *sans* infinity' should not be overinterpreted to denote T + ¬Infinity.

second components. The domain of a relation R is $\texttt{fst}``R$ and the range is $\texttt{snd}``R$; I shall sometimes write 'dom(R)' instead of '$\texttt{fst}``R$' and 'range($R$)' instead of '$\texttt{snd}``R$'.

Since our background theory has neither the axiom of infinity nor the axiom scheme of replacement, our ordered pairs have to be Wiener-Kuratowski. It is standard that the Wiener-Kuratowski pairing/unpairing apparatus is stratified and Δ_0, and that Zermelo set theory supports Wiener-Kuratowski pairing-and-unpairing. It is folklore among NFistes that the usual proofs in Zermelo Set Theory (that—*inter alia*—$\texttt{fst}``x$ and $\texttt{snd}``x$ exist for all x) work equally well in KF. We record all these facts—since we are going to make use of them—but we are not going to prove them.

We assume the reader is familiar with the notion of stratification in set theory, and with the axiomatisation of Quine's set theory NF.

Mac is Zermelo set theory with separation restricted to Δ_0 formulæ. KF is Mac with separation further restricted to *stratified* Δ_0 formulæ. KF is usually (as here) assumed not to include either foundation or infinity, and we will take Mac to be similarly limited. str(ZF) is the theory axiomatised by the stratified axioms of ZF. We always take the axiom of infinity in the stratified form "There is a Dedekind-infinite set".

TCo is the assertion "every set has a transitive superset";
TCl is the assertion "every set has a transitive closure".

'ι' denotes the singleton function. A set x is *strongly cantorian* if the graph of $\iota\restriction x$ is a set. Mac = KF + "Every set is strongly cantorian".

2 Scansets

We now define the new datatype. A datatype of things that are hereditarily something-or-other can be either the least fixed point (henceforth *lfp*) (so we get only the wellfounded objects) or the greatest fixed point (henceforth *gfp*). The gfp gives us more generality—which is what we want: after all we do not automatically assume foundation in the KF/Mac setting, and we want the construction to show that Mac can be interpreted into KF, and not merely that Mac + Foundation can be interpreteted in KF + Foundation. That second assertion is a refinement which we will reach in section

Definition 1
x is a **scanset** *iff* $(\exists Y)(x \in Y \wedge (\forall y \in Y)(\exists Y' \subseteq Y)(y = \iota\restriction Y'))$;
A set Y satisfying $(\forall y \in Y)(\exists Y' \subseteq Y)(y = \iota\restriction Y')$ *will be said to be a* **witness***;*
If $x \in Y$, with Y a witness, we say that Y **is a witness for** *x.*

Remark 1 *Every witness is strongly cantorian.*

Proof: Every member of a witness is a restriction of ι, and every set of restrictions of ι is strongly cantorian, as follows. If W is a set of restrictions of ι then we can do the following, where 'R' ranges over members of W.

$$\{R\} \;\mapsto\; \{\iota^2\text{“}\mathrm{dom}(R)\} \;\mapsto\; \iota\text{“}(\iota^2\text{“}\mathrm{dom}(R)) =^{(*)} \iota^2\text{“}\mathrm{range}(R) \;\mapsto\; R.$$

The equation $(*)$ holds beco's R is a restriction of ι. The intrusive 'ι^2“' is there because our pairs are Wiener-Kuratowski, so the domain and range of a relation are two types lower than the relation itself, rather than the *same* type, which they would have been had we been using Quine pairs ... in which case the displayed formula would have been:

$$\{R\} \;\mapsto\; \{\mathrm{dom}(R)\} \;\mapsto\; \iota\text{“}(\mathrm{dom}(R)) \;=\mathrm{range}(R) \;\mapsto\; R.$$

■

This will come in useful when we verify TCl in the scansets.

Next we define a membership relation between scansets.

Definition 2 *When x and y are scansets we say $x \,\mathcal{E}\, y$ iff $x \in dom(y)$.*

One way to get a feel for the datatype of scansets is to observe that there is a natural isomorphism π between the class of scansets and the class of hereditarily strongly cantorian sets defined by $\pi(s) = \pi\text{“}(\mathrm{dom}(s))$. This π is visible only from outside of course.

The existence of π makes the point that a scanset y is secretly the set $\mathrm{dom}(y)$ ... [well, sort-of]. Now observe that, if y is a scanset, then $\mathrm{dom}(y)$ is not only $\mathtt{fst}\text{“}y$ but is also equal to $\bigcup(y\text{“}V)$, which is one type *lower* than $\mathtt{fst}\text{“}y$. Thus $x \,\mathcal{E}\, y$ can be written either as "$x \in \mathrm{dom}(y)$" (in which case 'x' is *three* types[2] lower than 'y') or as "$x \in \bigcup(y\text{“}V)$" (in which case '$x$' is *four* types lower than 'y'). It would have been nice had $\mathcal{E}$ turned out to be homogeneous, but this is nearly as good, beco's of the following factoid which reassures us that we can subvert stratification just as effectively as if it has been homogeneous.

Lemma 1 *(The Subversion Lemma)*

Every formula ϕ in the language of set theory is equivalent (modulo the theory of extensionality) to a formula ϕ' in $\mathcal{L}(\in, =)$, the language of set theory, that admits a function from its variables to $\mathbb{N}$ with the property that if '$x \in y$' is a subformula of ϕ' then the number assigned to 'y' is greater—by 3 or by 4—than the number assigned to 'x'.

Proof:

See [1][3]. The proof is essentially the same as the proof of the folklore fact that if one liberalises stratification of '$x \in y$' so that 'y' may be given any type strictly greater than the type of 'x' then one can derive a paradox. The key observation in both cases is that '$x = y$' is equivalent to '$(\forall z)(z \in x \longleftrightarrow z \in y)$', and this last can be stratified in the new way with 'y' having a type one greater than that of 'x'. Thus '$x \in y$' can be replaced by '$(\exists z)(x \in z \wedge (\forall w)(w \in z \longleftrightarrow w \in y))$' and this possibility gives us all the freedom of manœuvre that one needs.

■

[2] Remember that our pairs are Wiener-Kuratowski.

[3] Do we need this reference?

It's worth checking that the subversion that enables us to prove lemma

We will need the following facts

Lemma 2

1. *If $X' \subseteq X$ and X is a scanset with Y a witness for X then $Y \cup \{X'\}$ is a witness for X'.*
2. *If X' and X are scansets then $(X' \subseteq X)^{\mathcal{E}} \longleftrightarrow X' \subseteq X$.*
3. *If Y is a witness for x_1 and $x_2 \,\mathcal{E}\, x_1$ then Y is a witness for x_2 as well.*

Proof:
We prove (3) only
We have: $x_2 \in \texttt{fst}``x_1$ and $x_1 \in Y \wedge (\forall y \in Y)(\exists Y' \subseteq Y)(y = \iota{\restriction}Y')$. Y is a witness for x_1 so there is $X_1 \subseteq Y$ with $x_1 = \iota{\restriction}X_1$.
Now
$x_2 \in \texttt{fst}``x_1 = X_1 \subseteq Y$ whence $x_2 \in Y$ as desired.

■

3 Verifying the Axioms of Mac + TCl in the Scansets of KF

Theorem 1
We interpret Mac + TCl in KF by restricting our variables to scansets, and sending '$\in$' to '$\mathcal{E}$'.

Proof: The proof is fiddlier than one might expect, so it's worth writing it out in some detail.

- Extensionality. Suppose x and x' are scansets with $(\forall \text{ scansets } y)(y \,\mathcal{E}\, x \longleftrightarrow y \,\mathcal{E}\, x')$. $x = \iota{\restriction}X$ and $x' = \iota{\restriction}X'$ for two sets X and X' that are sets of scansets. The fact that x and x' have-the-same-members$^{\mathcal{E}}$ means that X and X' have the same scansets as members and therefore—since all their members are scansets—they are coextensive, and therefore identical by extensionality. But then $x = \iota{\restriction}X = \iota{\restriction}X' = x'$.
- Pairing. If x_1 and x_2 are two scansets then $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$ is their (unordered pair)$^{\mathcal{E}}$, and it exists by pairing. Suppose Y_1 is a witness for x_1 and Y_2 is a witness for x_2. The set $Y_3 = Y_1 \cup Y_2 \cup \{\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}\}$ will be the witness we need for $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$.

  Y_3 is a witness because every member of it is either

(i) a member of Y_1 and is therefore the restriction of ι to a subset of Y_1, since Y_1 is a witness, or
(ii) a member of Y_2 [similarly] or
(iii) is $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$ which is $\iota\restriction\{x_1, x_2\}$ ... and $\{x_1, x_2\}$ is a subset of $Y_1 \cup Y_2$.

Finally $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$ is a member of Y_3 so Y_3 is a witness for $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$ as desired.

- Sumset. Let X be a scanset; we want to find a scanset that is to be the (sumset of $X)^{\mathcal{E}}$. Suppose $y\,\mathcal{E}\,X$ with Y a witness for X. $y\,\mathcal{E}\,X$ is $y \in \mathrm{dom}(X)$, and $\mathrm{dom}(X) \subseteq y$ so $y \in Y$ and Y is additionally a witness for y. By the same token, all y s.t. $y\,\mathcal{E}^2\,X$ will be in Y. Consider now the set B of all y s.t. $y\,\mathcal{E}^2\,X$; it will be a subset of Y. So $Y \cup \{\iota\restriction B\}$ is a witness for $\iota\restriction B$. This object is thus a scanset, and is clearly the (sumset of $X)^{\mathcal{E}}$; finally it exists beco's it is $\bigcup \mathrm{dom}(X)$.

- Power set. The (power set$)^{\mathcal{E}}$ of a scanset X is $\{\langle \mathtt{fst}``y, \{\bigcup \mathtt{snd}``y\}\rangle : y \subseteq X\}$. This object is a set beco's the defining formula is weakly stratified; it remains to find a witness for it. To this end observe that whenever Y is a witness so is $Y \cup \{w : (\exists x \in Y)(w \subseteq x)\}$. I think that if Y is a witness for X then

$$Y \cup \{w : (\exists x \in Y)(w \subseteq x)\} \cup \{\{\langle \mathtt{fst}``y, \{\bigcup \mathtt{snd}``y\}\rangle : y \subseteq X\}\}$$

is a witness for $\{\langle \mathtt{fst}``y, \{\bigcup \mathtt{snd}``y\}\rangle : y \subseteq X\}$.

- Δ_0 separation. Suppose X is a scanset, and we want to have the scanset $\{y \in X : \phi(y, \vec{z})\}$. Here we exploit subversion so that, when we rewrite ϕ with '$\in$' replaced by '$\mathcal{E}$' and restrict our our variables to live inside x and the $\vec{z}$, then the result is stratified—with the result that $\{y \in X : \phi(y, \vec{z})\}^{\mathcal{E}}$ is a set. Call it 'A' for short. Then $\iota\restriction A$ is also a set, since it is the same as $X\restriction A$. It is a scanset by part (i) of lemma

- Transitive closure. Suppose x is a scanset and W a witness for x. W is strongly cantorian so $\iota\restriction W$ is a set, and so too is $W \cup \{\iota\restriction W\}$, which is a witness to the fact that $\iota\restriction W$ is a scanset. But then this last object is a transitive$^{\mathcal{E}}$ scanset of which x is a member$^{\mathcal{E}}$. That gives transitive *containment*; we then obtain transitive *closure* by appeal to Δ_0 separation.

■

Some of the foregoing would have been easier had we been able to appeal to the fact that an arbitrary union $\bigcup_{i \in I} S_i$ of scansets is a a scanset. It would seem obvious that this should be the case, beco's an arbitrary union of restrictions of ι is another restriction of ι and all the members of its domain are scansets. The problem is that there is no obvious way of producing a witness. An arbitrary union of witnesses is a witness, so that, if X is a set of scansets, the obvious witness for $\bigcup X$ is $\bigcup\{y : y \text{ is a witness for a member of } X\}$... but there is no reason why this object should be a set. Even if we were assuming AC it wouldn't help: we cannot just pick witnesses for the S_i and take a union of them because the W_i (W_i is the collection of witnesses for S_i) that we want to pick from have an unstratified defining condition and might not be sets. And, even if they are, the family $\{W_i : i \in I\}$ might not be set.

As we will see in section

4 Verifying the Axioms of Mac + Foundation + TCl in the Scansets of KF

For this project we are of course going to need wellfounded$^{\mathcal{E}}$ scansets.

On the face of it there are lots of ways of defining "wellfounded$^{\mathcal{E}}$ scanset". One could start with gfp scansets and define wellfounded in the $\mathcal{E}$ language. One could also define lfp scansets. Lfp scansets have two definitions which are inequivalent *sans* infinity. We had better straighten out these various ideas before we attempt to use them.

The obvious way to define lfp-scanset is:

x is a wellfounded scanset if

$$(\forall Y)((\forall z)(\text{stcan}(z) \wedge z \subseteq Y. \to .\iota{\restriction}z \in Y) \to x \in Y) \qquad \text{(lpf1)}$$

However there is an 'upside-down' definition of lfp-scanset which one can obtain by analogy with a definition of natural number I learned from Quine [3][4]. Let P be the usual predecessor function on cardinals defined by: $P(0) := 0$, and when $y \in x$, $P(|x|) := |x \setminus \{y\}|$. Then we can define $\mathbb{N}$ to be

$$\{m : (\forall Y)((m \in Y \wedge (P``Y \subseteq Y)) \to 0 \in Y)\}.$$

Analogously one obtains

x is a wellfounded scanset if

$$(\forall Y)((x \in Y \wedge (\forall zwu)[\iota{\restriction}u = w \in \text{dom}(z) \wedge z \in Y. \to w \in Y]) \to \emptyset \in Y) \quad \text{(lpf2)}$$

The significance of the availability of these two definitions is that (lfp1) is almost certainly vacuous unless there are infinite sets—since any set closed under the relevant operation is virtually guaranteed to be infinite. In contrast (lfp2) is sensible even *sans* infinity.

[4]I don't think the idea originates with Quine. Must check...

4.1 Interpreting Mac + Foundation (*sans* Infinity) into KF (*sans* Infinity)

We will need definition (lfp2) for this.

Theorem 2
We interpret Mac + Foundation + TCl sans Infinity into KF sans Infinity by restricting our variables to wellfounded scansets, and sending '$\in$' to '$\mathcal{E}$'.

- Extensionality. The proof is the same *verbatim* as the proof of extensionality in theorem
- Pairing. If x_1 and x_2 are two scansets then $\{\langle x_1, \{x_1\}\rangle, \langle x_2, \{x_2\}\rangle\}$ is their (unordered pair)$^{\mathcal{E}}$, and it exists by pairing.
- Sumset.
- Power set.
- Δ_0 separation.
- Transitive closure.

■

5 Permutations

Must check various things like:

> *Every model of NFC has permutation models in which there are infinite scansets*;

or

> *Every model of NF has permutation models in which every strongly cantorian natural contains a scanset.*

If $\text{AxCount}_{\leq}$ fails then there should be a permutation model with a finite upper bound on the size of wellfounded scansets.

Let's do some hand-calculations:

x is an infinite scanset iff

$$(\exists Y)(x \in Y \wedge (\forall y \in Y)(\exists Y' \subseteq Y)(y = \iota\restriction Y')) \wedge |x| \not< \aleph_0$$

The hard part is to calculate '$(y = \iota\restriction Y')^{\pi}$'. Now $y = \iota\restriction Y'$ is

$$(\forall w)(w \in y \longleftrightarrow (\exists z \in Y')(\mathtt{fst}(w) = z \wedge \mathtt{snd}(w) = \{z\}))$$

and

$$(\forall w)(w \in y \longleftrightarrow (\exists z \in Y')(\mathtt{fst}(w) = z \wedge \mathtt{snd}(w) = \{z\}))^{\pi}$$

is

$$(\forall w)(w \in \pi(y) \longleftrightarrow (\exists z \in \pi(Y'))(\mathtt{fst}(w) = \pi(z) \wedge \mathtt{snd}(w) = \{z\}))$$

which is

$$y = (\pi \circ \iota) \restriction \pi(Y').$$

So suppose that V^{π} contains an infinite scanset.

$$((\exists Y)(x \in Y \wedge (\forall y \in Y)(\exists Y' \subseteq Y)(y = \iota \restriction Y')) \wedge |x| \not< \aleph_0)^{\pi}$$

$$(\exists Y)(x \in \pi(Y) \wedge (\forall y \in \pi(Y))((\exists Y' \subseteq \pi(Y))(y = \iota \restriction Y'))^{\pi}) \wedge |x| \not< \aleph_0$$

So we want

$$(\exists Y)((\forall y \in Y)((\exists Y' \subseteq Y)(y = \pi \circ \iota \restriction Y'))) \wedge |Y| \not< \aleph_0$$

Work still to do here

6 Remaining bullet points

- Check that an arbitary union of wellfounded scansets is another scanset;
- We need to check that these definitions (lpf1) and (lpf2) are equivalent if one has the Axiom of Infinity. It is fairly clear that definition (lpf1) can be vacuous if the Axiom of Infinity fails, since there might be no sets closed under the relevant operation;
- One can then ask if they imply the gfp definition;
- Interpret Mac + Foundation + Infinity in KF + Counting; a discusson of why this doesn't give us a relative consistency proof of AC. The first and most onbious point is that extensionality fails so one has to take a quotient. One then presumably needs AC to deal with the quotients. It would be an instructive exercise.

References

[1] Forster, T. E. and Esser, O. E. "Relaxing Stratification" Bull. Belg. Math. Soc. Simon Stevin, **14**, (2007), pp. 247–258. Available from `www.dpmms.cam.ac.uk/~tf/relaxing.pdf`.

[2] Forster, T. E., "Set Theory with a Universal Set" Oxford Logic Guides.

[3] Quine, W. v O. "Set Theory and its Logic". Belknap Press, Harvard, 1967