Ramsey and Paris-Harrington Again

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I return to the material in [1] "Paris-Harrington in an NF context". Various people[1] had commented that the concept of a *relatively large set of natural numbers* is unstratified, and in that essay I mused about whether or not the extra strength of PH over finite Ramsey was to do with this failure of stratification. In the present—self-contained—note I shall show that—somewhat to my annoyance—it is not: Paris-Harrington has a stratified formulation.

I was able to exhibit in [1] formulations of PH and Finite Ramsey which differed only in their quantifier prefix, and this suggested that the difference in strength is located in the difference between the quantifier prefixes, as below, quoted from [1]:

Finite Ramsey:

> For all n, m, j in $\mathbb{N}$
> There is k in $\mathbb{N}$ so large that
> For every set X of size k and
> For every m-colouring χ of $[X]^j$
> there is an enumeration e of X and
> there is $X' \subseteq X$ with $|X'| = n$ and X' monochromatic wrt χ and relatively large wrt e.

Paris-Harrington:

> For all n, m, j in $\mathbb{N}$
> There is k in $\mathbb{N}$ so large that
> For every set X of size k and
> For every m-colouring χ of $[X]^j$ and
> For every enumeration e of X
> there is $X' \subseteq X$ with $|X'| = n$ and X' monochromatic wrt χ and relatively large wrt e.

The only difference is in the quantifiers in the fifth line. Locating the difference in the quantifier prefix suggests that stratification is not the key to understanding the

[1] Subject: [FOM] PA Incompleteness; Sun, 14 Oct 2007 10:02:58 -0400): Harvey Friedman wrote

> "In "relatively large", an integer is used both as an element of a finite set and as a cardinality (of that same set). This is sufficiently unlike standard mathematics, that an effort began, at least implicitly, to find PA incompleteness that did not employ this feature, or this kind of feature."

situation. However, both these formulations make use of the concept of *relatively large subset*, and so are not stratified. I can now exhibit a pair of formulations of finite Ramsey and PH, both stratified, which differ only in the quantifier prefix. This shows that, rather to my disappointment and surprise, stratification plays no role in the extra strength of PH.

First, some notation: $\mathcal{P}_n(x)$ is $\{y \subseteq x : |y| = n\}$, and (this is new) $\scalebox{1}[-1]{$\mathcal{P}$}(x)$ is $\{y : y \cap x \neq \emptyset\}$. (It's an upside-down '$\mathcal{P}$' reflecting[2] the fact that the operation is dual to power set.) By abuse of notation we will write '$\scalebox{1}[-1]{$\mathcal{P}$}(x)$' when $x \subseteq X$ (X clear from context) to mean $\{y \subseteq X : y \cap x \neq \emptyset\}$.

The new thought is that we should think of PH as saying not so much that there is a monochromatic set with special properties, but rather that (in contrast to Ramsey, which only promises *one* monochromatic set) the set of χ-monochromatic subsets of X is *large* in the sense that, for every total order $<$ of X, it meets $\scalebox{1}[-1]{$\mathcal{P}$}$ of the initial segment containing the first n elements of $\langle X, < \rangle$. That sounds like a quantifier.

Finite Ramsey says

For all $n, m, j \in \mathbb{N}$
There is $k \in \mathbb{N}$ so large that
For every set X of size k
For every m-colouring χ of $[X]^j$
There is $X' \subset X$ with X' monochromatic wrt χ.

We can rephrase the last line to get

For all $n, m, j \in \mathbb{N}$
There is $k \in \mathbb{N}$ so large that
For every set X of size k and
For every m-colouring χ of $[X]^j$
The set M_n^χ of n-sized subsets of X monochromatic for χ is nonempty.

Now for the new formulation of PH:

For all $n, m, j \in \mathbb{N}$
There is $k \in \mathbb{N}$ so large that
For every set X of size k and
For every m-colouring χ of $[X]^j$
The set M_n^χ of n-sized subsets of X monochromatic for χ meets everything in $\scalebox{1}[-1]{$\mathcal{P}$}``\mathcal{P}_n(X)$.

...and the difference between these two [purely in the fifth line] is that one says that the set of monochromatic subsets of size n is nonempty, whereas the other says that it meets every member of a fairly large set.

It's probably worth saying a few words about why this version of PH is equivalent to the usual version that asserts the relative largeness of the monochromatic set. A subset of $\mathbb{N}$ of size n is relatively large simply if it has a member smaller than n. But

[2]"reflecting" (joke!—geddit??)

this is simply to say that an n-sized subset $X' \subseteq X$ is relatively large wrt an ordering $<$ iff one of its members is among the first n members of X according to $<$. For the other direction, if Y meets some element $X' \in \mathcal{P}_n(X)$ then it is relatively large wrt any enumeration of X that counts X' using only the naturals $\leq n$.

This formulation prompts some natural questions. Does this version have a slick compactness proof? Does it give a slick proof of Con(PA)? Does it suggest formulations of analogues of PH for uncountable cardinals?

The new formulation of PH asserts that M^n_χ meets every set in the family $b``\mathcal{P}_n(X)$. Now we know the size of this family (it's $\binom{k}{n}$) and we know the size of all the members of that family (and all these values of b are of size $(2^n - 1) \cdot 2^{k-n}$) so we have a lower bound on the size of M^n_χ purely in terms of k and n. Specifically we can find k/n pairwise disjoint subsets of X of size n, and no monochromatic set of size n can meet more than n of them, so $|M^n_\chi|$ must be at least k/n^2.

References

[1] Thomas Forster "Paris-Harrington in an NF context", in OHYAST (100 Years of Axiomatic Set Theory) *Cahiers du Centre de Logique* 2010 **17** pp 97–109. Also available from `www.dpmms.cam.ac.uk/~tf/parisharringtontalk.pdf`