Erdös-Rado without choice

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ABSTRACT

A version of the Erdös-Rado theorem on partitions of the unordered n-tuples from uncountable sets is proved, without using the axiom of choice. The case with exponent 1 is just the Sierpinski-Hartogs' result that $\aleph(\alpha) \leq 2^{2^{2^{\alpha}}}$.

The liberal use made by Erdös and Rado in [2] of the cardinal arithmetic versions of the axiom of choice enables them to give their result a particularly simple expression. So simple, indeed, that the elegant construction underlying it is not brought to the fore. As it happens there is a nontrivial combinatorial theorem about uncountable monochromatic sets which this construction serves up, and it deserves to be isolated.

The referee for this paper made some very helpful comments and these are recorded in an appendix to the online version of this paper at
`www.dpmms.cam.ac.uk/~tf/erdosrado.pdf`.

Arithmetic notations

An aleph is a cardinal of a wellordered set. $\aleph_\alpha$ is the αth aleph, and in this usage α is of course an *ordinal*. $\aleph(\alpha)$ is the least aleph $\not\leq \alpha$, and in this usage α is a *cardinal* not an ordinal, and the Hebrew letter is being used to denote Hartogs' aleph function. When α is itself an aleph we often write 'α^+' for '$\aleph(\alpha)$'. By abuse of notation we will often use a notation denoting an aleph, such as '$\aleph(\alpha)$' or '$\aleph_\kappa$' to denote also the corresponding initial ordinal. Finally $\beth_0(\alpha) =: \alpha$; $\beth_{n+1}(\alpha) =: 2^{\beth_n(\alpha)}$.

Combinatorial notations

$[X]^n$ is the set of unordered n-tuples from X. "$\alpha \to (\beta)^\gamma_\delta$" means: take a set A of size α, partition the unordered γ-tuples of it into δ bits. Then there is a subset $B \subseteq A$ of size β such that all the unordered γ-tuples from it are in the same piece of the partition. Here γ will always be in $\mathbb{N}$, and α, β and δ will be infinite cardinals.

Points of a tree are **nodes**, and they have **children**; the set of children of a node is a **litter**.

The result

The proof will follow closely a standard procedure for proving Ramsey's theorem that $(\forall nm)(\omega \to (\omega)^n_m)$. We start by considering the case $n = 2$ and proceed to larger n by induction. We start by proving without AC the special "binary" case, first shown in [1].

REMARK 1. $\aleph(2^{2^{2^\kappa}}) \to (\aleph(\kappa))^2_2$.

Proof:

Let $\langle K, <_K \rangle$ be a wellordering of length $\aleph(2^{2^{2^\kappa}})$, and Δ a two-colouring of $[K]^2$ making every unordered pair red or blue. We will find a set monochromatic for Δ.

The idea is to remove ordered pairs from $<$ to obtain a wellfounded tree with

> **The Nice Path Property**: for all a, if b and c lie on the same branch as a and beyond a, then $\{a, b\}$ and $\{a, c\}$ are the same colour.

We will delete ordered pairs by a recursion on $<_K$.

At each stage we have in hand an element a and a wellfounded strict partial ordering $<_a$—which is initialised to $<_K$. We use a to discard some ordered pairs.

We reach stage a equipped with the strict partial order $\bigcap\limits_{b<a} <_b$, which we abbreviate to $<<_a$. We will weed out some ordered pairs from $<<_a$ to obtain $<_a$. We consider the members of $\{x : x >>_a a\}$. To ensure that the nice path property holds of the new—stricter—order we end up with we must ensure that whenever b and c in $\{x : x >>_a a\}$ are joined to a by edges of different colours then the new order believes they are incomparable. We will tease apart the points $>>_a a$ into two *rays*, the first containing of those elements joined to a by a red edge and the second containing those joined to a by a blue edge; points in different rays will have no common upper bound according to $<_a$.

Accordingly, if $a <<_a b <<_a c$, and b and c are joined to a by edges of different colours, then we delete the ordered pair $\langle b, c \rangle$ from $<<_a$. The result is $<_a$.

After κ steps we have performed this for every $a \in K$ and the set of those ordered pairs that remain—let us call it $<'$—is a wellfounded partial order with the nice path property. We had better check this.

1. $<'$ is wellfounded because any subset of a graph of a wellfounded relation is itself a graph of a wellfounded relation.

2. $<'$ has the nice path property. The two clauses that need checking are transitivity and the property that incomparable elements have no common upper bound.

   (i) transitivity. It will fail to be transitive if there is $b <' c$ and $c <' d$ but $b \not<' d$. In this case we know $c <_K d$. But since $b <' c$, we know that for any $a <_K b$, $\{a, b\}$ and $\{a, c\}$ are the same colour. Similarly since $c <' d$, we know that for any $a <_K c$, $\{a, c\}$ and $\{a, d\}$ are the same colour. But then, for any $a <_K b$, $\{a, b\}$ and $\{a, d\}$ are the same colour.

   (ii) the incomparable element condition. If we "separate" b and c as above then we cannot keep d above both b and c because $\{a, d\}$ must be the same colour as both $\{a, c\}$ and $\{a, b\}$ which is impossible.

So $\langle K, <' \rangle$ is a tree, and a *binary* tree at that, because there are only two colours.

We want this tree to have a branch in it of length at least (size) $\aleph(\kappa)$. If all branches die within $\aleph(\kappa)$ steps then there are at most $2^{\aleph(\kappa)}$ points in K, so $\aleph(2^{2^{2^\kappa}}) \leq 2^{\aleph(\kappa)}$. This step relies on the fact that a perfect binary tree of height α has precisely 2^α points. This is true as long as there is a choice function on the set of litters, but this is a nontrivial assumption. Fortunately it is true here: the litters are uniformly ordered because one child in the litter is joined to the parent by a blue edge and one by a red edge.

So if all branches die within $\aleph(\kappa)$ steps then $\aleph(\kappa) \leq^* 2^{2^\kappa}$ whence we would have $\aleph(2^{2^{2^\kappa}}) \leq 2^{\aleph(\kappa)} \leq 2^{2^{2^\kappa}}$, contradicting the definition of $\aleph(2^{2^{2^\kappa}})$.

So there is a branch of length $\aleph(\kappa)$. Every element in this branch can be thought of as a *red* point (if it is joined to all later points in that branch by a red edge) or as a *blue* point (if it is joined to all later points in that branch by a blue edge). So there are either $\aleph(\kappa)$ red points or $\aleph(\kappa)$ blue points, so one way or another we get a monochromatic set of size $\aleph(\kappa)$.

We proved slightly more than we will need or exploit. What we proved was that the tree has a branch of length *greater* than $\aleph(\kappa)$ whereas we do not actually exploit anything beyond the fact it has a branch of length *at least* $\aleph(\kappa)$.

Increasing the exponent

Next we tackle the "higher exponent" version, as seen in [2] (item (95) p 471).

We will prove the following by induction on n

Theorem 1. $\aleph(\beth_{3n}(\kappa)) \to (\aleph(\beth_{n-1}(\kappa)))^{n+1}_2$

We have just proved the case $n = 1$, so let us attack the inductive step.

Let $\langle K, <_K \rangle$ be as before, and Δ a 2-colouring of $[K]^{n+1}$. We want to discard ordered pairs to be left with a tree ordering $<$ such that whenever $a_1 < a_2 < \ldots < a_n < b_1 < b_2$ then $\{a_1 \ldots a_n, b_1\}$ and $\{a_1 \ldots a_n, b_2\}$ are the same colour.

So, given an n-tuple $a_1 <_K a_2 <_K \ldots <_K a_n$ and $b_1 <_K b_2$ beyond a_n we can "separate" b_1 from b_2—as in the binary case—whenever $\{a_1 \ldots a_n, b_1\}$ and

$\{a_1 \ldots a_n, b_2\}$ are coloured differently. We do this by deleting the ordered pair $\langle b_1, b_2 \rangle$ from $<_K$, and we can do this simultaneously for all $b_1 < b_2$ beyond a_n.

So the algorithm performs this procedure iteratively, once for each n-tuple $a_1 < a_2 < \ldots < a_n$, and considers these n-tuples in lexicographic order.

Now we must check as before that the set of ordered pairs that remain is a wellfounded tree with the nice path property.

The chief complication is with the branching number. To be able to run the cardinality argument as we did above we need to be able not merely to bound the size of the litters but to order all the litters uniformly. We need to be able to inject all litters simultaneously into some fixed set of size $2^{\aleph(\alpha)}$.

Each litter is indexed by colours, in the sense that for no node ν can two children have the same colour. 'Colour' here does not refer directly to Δ: two children c and c' of ν have different colours as long as there is an n-tuple $\langle a_1 \ldots a_{n-1}, \nu \rangle$ such that $\langle a_1 \ldots a_{n-1}, \nu, c \rangle$ and $\langle a_1 \ldots a_{n-1}, \nu, c' \rangle$ are coloured differently by Δ. How many colours does that make? Clearly, 2-to-the-power-of the number of such n-tuples. But this tells us how to label the colours uniformly. Fix a wellordering $\langle W, \leq_w \rangle$ of length $\aleph(\kappa)$. Think of the colour of c as the set of n-tuples $\vec{i}$ from W such that Δ colours $\langle a_{i_1} \ldots a_{i_n}, c \rangle$ blue. This works uniformly for all litters, and means that we can argue that the tree is one where every node has at most $2^{\aleph(\kappa)}$ children, and is of height $\aleph(\kappa)$ and has therefore at most $(2^{\aleph(\kappa)})^{\aleph(\kappa)} = 2^{\aleph(\kappa)}$ points, which is impossible as before.

Any path through this tree has the property that for every n-tuple $\langle a_1 < \ldots < a_n \rangle$ on it, all tuples $\langle a_1 \ldots a_n, c \rangle$ with $c > a_n$ are coloured the same by Δ. Thus we have a two-colouring of n-tuples from a set K of size $\aleph(\kappa)$, and we have obtained this from a two-colouring of the $n+1$-tuples from an $\aleph(2^{2^{2^\kappa}})$-sized superset of K

Bibliography

1. P. Erdös. Some set-theoretical properties of graphs. Revista Universidad Nacional de Tucuman, Série A vol 3 (1942) pp 363-367.

2. P. Erdös and R. Rado. A partition calculus in Set theory. Bull. Am. Maths. Soc 1956 pp 427-498.

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1 Referee's comments on "Erdös-Rado without Choice

Before reading this paper I was not familiar with the original proof of the Erdos-Rado Theorem. In graduate school I learned the proof using the downward Lowenheim-Skolem argument as is given, for example, in Chang and Keisler, Model Theory or in Jech, Set Theory. I think this proof is due to Steve Simpson. (Abstract in the Notices of AMS 1970).

I did look up the proof given in:

[EHMR] Erdos, Hajnal, Maté, Rado, Combinatorial set theory: partition relations for cardinals. Studies in Logic and the Foundations of Mathematics, 106. North-Holland Publishing Co., Amsterdam, 1984.

which I am guessing is equivalent to the original proof. It seems to me that their construction of a partition tree is essentially equivalent to the proof in this paper. They also mention the tree relation on κ see §18 p.100 but I didn't read that section carefully.

Let $\aleph(X)$ be the Hartog's cardinal for X, i.e., the least ordinal κ for which there is no embedding of κ into X. I sketched their proof below to make sure it isn't using the axiom of choice and for my own amusement.

Prop. [EHMR] page 88

$$\aleph(2^{<\omega_1}) \to (\omega_1)^2_2$$

proof:

Let $\kappa = \aleph(2^{<\omega_1})$ and suppose $c : [\kappa]^2 \to 2$ is any coloring. For each $f \in 2^{<\omega_1}$ inductively define $S(f) \subseteq \kappa$ as follows:

1. $S(\emptyset) = \kappa$.
2. If $dom(f) = \lambda$ is a limit, then $S(f) = \cap_{\alpha<\lambda} S(f|\alpha)$.

   Otherwise $dom(f) = \alpha + 1$:
3. If $S(f|\alpha)$ is empty, put $S(f) = \emptyset$ and $s(f) = 0$.
4. If $S(f|\alpha)$ is nonempty, choose $s(f|\alpha)$ to be the minimal element of $S(f|\alpha)$ and define

$$S(f) = \{\beta \in S(f|\alpha) \;\; : \;\; \beta > s(f|\alpha) \text{ and } c(s(f|\alpha), \beta) = f(\alpha)\}$$

For nonempty $S(h)$,

$$S(h) = \{s(h)\} \cup S(h0) \cup S(h1)$$

and this union is disjoint. Since it is impossible to embed κ into $2^{<\omega_1}$ there must be some $\alpha \in \kappa$ which is not equal to $s(h)$ for any $h \in 2^{<\omega_1}$. But for such an α there exists a unique $F \in 2^{\omega_1}$ with $\alpha \in S(F|\beta)$ for every $\beta < \omega_1$. By construction $\{s(F|\beta) : \beta < \omega_1\}$ is an end-homogeneous set. We pull from it a homogeneous set of order type ω_1.

QED

The proof seems to generalize to show that

$$\aleph(2^{<\gamma}) \to (\gamma)^2_2$$

for any indecomposable[1] ordinal γ.

It seems to me that $2^{<\omega_1}$ embeds[2] into 2^{2^ω} and hence $\aleph(2^{<\omega_1}) \leq \aleph(2^{2^\omega})$ and thus

$$\aleph(2^{2^\omega}) \to (\omega_1)^2_2$$

But I am using here that $\omega^2 = \omega$ which wouldn't be true in general.

In the paper (see Theorem 4.1)

[K] Kruse, A. H. Some results on partitions and Cartesian products in the absence of the axiom of choice. Z. Math. Logik Grundlagen Math. 20 (1974), 149–172.

it is shown that

Prop. [K]. $\aleph(2^{<\omega_1}) \leq (\aleph(2^\omega))^+$

proof:

Suppose $h : \delta \to 2^{<\omega_1}$ is one to one. For each $\beta < \omega_1$ let

$$S_\beta = \{\gamma < \delta \ : \ h(\gamma) \in 2^\beta\}$$

Let γ_β be the order type of S_β. Since there is a bijection between 2^ω and 2^β it is clear that $\gamma_\beta < \aleph(2^\omega)$. Hence using the canonical order type embedding maps we get that δ is embeddable into $\omega_1 \times \aleph(2^\omega) = \aleph(2^\omega)$. QED

So as a Corollary we get

[1] if $A \cup B = \gamma$ then either A or B has order type γ

[2] Map $\sigma \in 2^\alpha$; (where $\omega \leq \alpha < \omega_1$) to the set

$$\{(R, x) \in P(\omega \times \omega) \times 2^\omega : \exists\theta \ \ (\alpha, <) \simeq^\theta (\omega, R), \ \ \forall\beta < \alpha \ \ \sigma(\beta) = x(\theta(\beta))\}$$

where the notation above means that $\theta : \alpha \to \omega$ is an isomorphism.

$$(\aleph(2^\omega))^+ \to (\omega_1)^2_2$$

I think if you use Kruse's 4.1 you get that in general

$$\aleph(2^{<\aleph(X)}) \leq \aleph(2^X)^+$$

and so we have

$$\aleph(2^X)^+ \to (\aleph(X))^2_2.$$

I am not sure if this is always better than Forster's:

$$\aleph(2^{2^{2^X}}) \to (\aleph(X))^2_2.$$

In models of the axiom of choice,

$$\aleph(2^X)^+ < \aleph(2^{2^{2^X}})$$

and the axiom of determinacy gives $\aleph(2^\omega)^+ = \omega_2$ while $\aleph(2^{2^\omega})$ is at least the order type of the Wadge ordering which (I believe) is much greater than the first weakly inaccessible.

Q. Is it consistent with ZF that there is a set X with

$$\aleph(2^{2^{2^X}}) < \aleph(2^X)^+?$$

In regard to the Sierpinski-Hartog Theorem mentioned in the abstract,

Hickman, John L. Λ-minimal lattices. Z. Math. Logik Grundlag. Math. 26 (1980), no. 2, 181–191.

shows (Thm 10) that it is relatively consistent with ZF to have a set X such that

$$\aleph(X) = \aleph(2^{2^X}).$$

I believe that in Hickman's model $\aleph(X) = \aleph_\omega$ but[3]

$$\aleph_\omega \not\to (\aleph_\omega)^2_2$$

hence in Hickman's model:

$$\aleph(2^{2^X}) \not\to (\aleph(X))^2_2.$$

[3]consider the coloring $c(\alpha, \beta) = 0$ iff $\exists n \;\; \aleph_n < \alpha, \beta < \aleph_{n+1}$.

Let ker be the least κ such that $\kappa \to (\omega_1)^2_2$. Then by using the Sierpinski partition[4]

$$\aleph(2^\omega) \leq \ker$$

Hence by Kruse

$$\aleph(2^\omega) \leq \ker \leq (\aleph(2^\omega))^+$$

I wondered if anything more could be said about ker.

Example 1. In a model of the axiom of determinacy,

$$\omega_1 = \aleph(2^\omega) = \ker < \omega_2$$

Probably the consistence strength of this is much less. To get the consistency of $\omega_1 \to (\omega_1)^2_2$, I think should only need the consistency of a weakly compact cardinal. Jech showed the consistency of ω_1 measurable using the consistency of a measurable. Probably someone has done the same for weakly compact.

Example 2. Take the Levy collapse G of the smallest inaccessible κ in L to ω_1 and look at $L[\mathbb{R}]$ in $L[G]$, then we get

$$\omega_1 = \aleph(2^\omega) < \ker = \omega_2$$

To see that $\omega_1 \not\to (\omega_1)^2_2$ note that there is a κ-Aronszajn tree $T \subseteq 2^{<\kappa}$ in L. In $L[G]$ the tree T is an ω_1-Aronszajn tree[5]. In L enumerate $T = \{x_\alpha : \alpha < \omega_1\}$ (where ω_1 means $\omega_1^{L[G]} = \kappa$) and take the Sierpinski partition. But a homogeneous $H \in [\omega_1]^{\omega_1}$ gives a branch $f \in 2^{\omega_1}$ thru T: i.e. put $\sigma \subseteq f$ iff $\sigma \subseteq x_\alpha$ for all but countably many $\alpha \in H$. (Probably there is a simpler argument for this case.)

Q. Can we get the consistency of $\aleph(2^\omega) < \aleph(2^{<\omega_1})$ without using the consistency of an inaccessible cardinal?

Q. Does $\aleph(\omega^{<\omega_1}) \to (\omega_1)^2_\omega$?

The EHMR proof seems to break down at the last step, extracting a homogeneous set from an end-homogeneous set. The problem is that ω_1 might be the countable union of countable sets.

[4]given $x_\alpha \in 2^\omega$ distinct for $\alpha < \kappa$ put

$$c(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha < \beta \text{ and } x_\alpha <_{lex} x_\beta \\ 0 & \text{if } \alpha < \beta \text{ and } x_\alpha >_{lex} x_\beta \end{cases}$$

Even without using the axiom of choice, the order type of ω_1 cannot be embedded into the lexicographical order on 2^ω (or for that matter the usual order on $\mathbb{R}$).

[5]To see that T has no branch, note that in L given any sequence $(p_\alpha : \alpha < \kappa)$ of elements of the collapsing poset we can find $\Gamma \in [\kappa]^\kappa$ such p_α and p_β are compatible for any $\alpha, \beta \in \Gamma$. Hence if $f \in (2^\kappa \cap L[G])$ is a branch of T, then in L we could find p_α and $\sigma_\alpha \in T$ such that

$$p_\alpha \Vdash f|\alpha = \sigma_\alpha$$

But then $(\sigma_\alpha : \alpha \in \Gamma)$ would be a branch thru T in L.

Ali Enayat writes:

I ran across your paper on Erdos-Rado without choice on your website. I have downloaded the paper to look at it more carefully (but I did look at the referees comments). I just wanted to bring your attention to one place in the literature where a similar result is established: there is an old paper of Keisler and Morley on elementary end extensions of models of set theory in Israel J. Math, 5 (1968), pp. 49-65), where in the *appendix* to section 4, almost at the end of the paper, a version of Erdos-Rado without choice is proved. This choice-free version is needed in their work in order to prove that many of their results in the paper that seemingly only hold for models of ZFC, indeed hold for models of ZF as well.