The Axiom of Choice and Inference to the Best Explanation

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ABSTRACT

An argument often given for adopting the Axiom of Choice as an axiom is that it has a lot of obviously true consequences. This looks like a legitimate application of the practice of Inference to the Best Explanation. However, the standard examples of obvious-truths-following-from-AC all turn out, on closer inspection, to involve a fallacy of equivocation.

The Argument

It is common practice in the teaching of mathematics at university level to gloss over applications of the axiom of choice, and proclaim such standard propositions as—for example—"A countable union of countable sets is countable" with some sketchy argument which does not render explicit the use of the axiom, and indeed might not even mention it by name at all. The students in consequence do not form a mental image of the axiom, and tend subsequently not to recognise when it is being used. Typically when confronted with it later on in their education they either deny that it is being used, or acknowledge that it is being used, but say there is no problem, since the axiom is obviously true.

Not surprisingly, they end up defending the axiom in roughly the following terms:

> There are various obviously true assertions, such as that the union of countably many countable sets is countable, or that a perfect binary tree has an infinite path, or that a finitely branching infinite tree must have an infinite branch, which—it transpires—cannot be proved without exploiting the axiom of choice. Given that they are obviously true, any sensible system of axioms for set theory will have to prove them, so we'd better include the axiom of choice in our set of axioms.

Although this argument sounds a bit like a fallacy of affirming the consequent, it's actually nothing of the kind. Arguments like this are so common and so natural and so *legitimate* that it is hardly surprising that this method has been identified by philosophers as a sensible way of proceeding and that there is a nomenclature for it and a literature to boot. It is probable that this is (at least part of) what Peirce had in mind when he coined the word **abduction**; nowadays it is captured by the expression *Inference to the best explanation* "IBE"; see [1] for an excellent treatment.

The people who use this argument for AC are working in a context where the other axioms of ZF—which are the means by which we can infer these desirable consequences of AC—are not up for debate. It is also the case that these desirable consequences do not repay the compliment by entailing AC in turn: so this is not a case of inference to the *only* explanation.

Let us consider some of these apparently obvious truths.

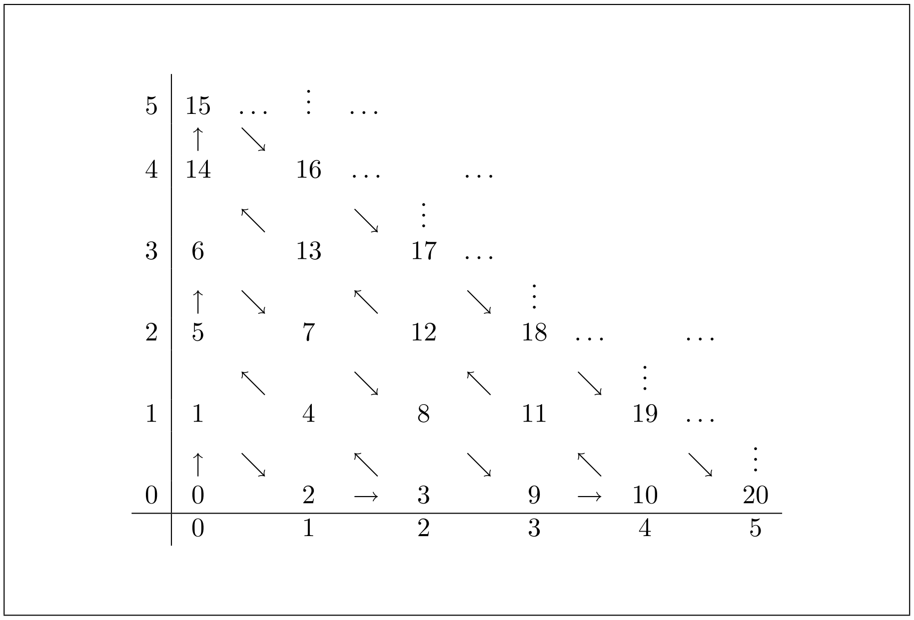
Socks

In [2] (p 126) we find the *sutra* of the millionaire whose wardrobe contains a countable infinity of pairs of shoes and a countable infinity of pairs of socks. It is usually felt to be obvious that there are countably many shoes and countably many socks in his attic. Most people will claim that it is obvious that there is a bijection between the shoes and the natural numbers, and obvious too that there is a bijection between the socks and the natural numbers. Obviousness is all very well but it is not reliably a path to understanding. Fortunately with a bit of prodding most students can be persuaded to say that the left shoe from the nth pair can be sent to $2n$ and the right shoe from the nth pair can be sent to $2n+1$. This indeed shows that there are countably many shoes. "And the socks?" one then asks. With any luck the student will reply that the same technique will work, at which point the victim can be ribbed for being a sad mathmo with odd socks. Old jokes are the best. In fact this joke is so good that it even survives being explained.

If we have succeeded in finding a bijection between the set of socks and the natural numbers then we have given each sock a number. This means, at the very least, that we now have a uniform way of choosing one sock from each pair, namely the one with smaller number. Conversely, if we have a uniform way of choosing one sock from each pair, then we can send the chosen sock from the nth pair to $2n$, and the rejected sock to $2n+1$. So the set of socks is countable iff there is a choice function on the pairs of socks. Analogously the set of shoes is countable iff there is a choice function on the pairs of shoes. Clearly there is a way of choosing one shoe from each pair, because we can uniformly distinguish left shoes from right. But socks? Clearly there is no *uniform* way of telling socks apart. So we need the axiom of choice to tell us that the set of socks is countable nevertheless.

A union of countably many countable sets is countable

Let $\{X_i : i \in \mathbb{N}\}$ be a family of countable sets. Is its sumset, $\bigcup\{X_i : i \in \mathbb{N}\}$, countable? The usual answer is: yes. Draw the X_i out in a doubly infinite array, and then count them by zigzagging. What could be more obvious? The picture below illustrates how we do it. Let $x_{i,j}$ be the jth member of X_i. Put the members of X_i in order in row i, so that $x_{i,j}$ is the jth thing in the ith row. Then we can count them by zigzagging.



A Counted Union of Counted Sets is Counted

What could be simpler?! If—as they tell us—we need the axiom of choice to prove this, then we'd better assume it.

Every perfect binary tree has an infinite path

A perfect binary tree is a tree with one root, wherein every node has precisely two children. If one has a blackboard to hand when telling this story, one is tempted to start off drawing a perfect binary tree:

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    /   \
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...which makes it obvious that this perfect binary tree at least (the one you have started drawing) has an infinite path. Always choose the leftmost branch. What could be easier!? Again we are told that this needs the axiom of choice. Very well, let us therefore adopt the axiom of choice.

The Fallacy of Equivocation

However, in all three cases we have a fallacy of equivocation. Let us take the binary tree case first, since it is fresh in our minds.

Perfect binary trees

The fact that is obvious is not the fact that

> A perfect binary tree has an infinite path; . (i)

but the fact that

> A perfect binary tree *with an injection into the plane* has an infinite path; (ii)

since we cannot follow the rule "take the leftmost child in each case" unless we can tell what the leftmost child is, and this information is provided for us not by the tree itself but by its injection into the plane.

Are not (i) and (ii) the same? They certainly will be if any two perfect binary trees are isomorphic. And aren't any two perfect binary trees isomorphic? Isn't that obvious?

No, what is obvious is *not*

> Any two perfect binary trees are isomorphic; (iii)

but

> Any two perfect binary trees *equipped with injections into the plane* are isomorphic; (iv)

and (iii) and (iv) are not the same. We are back where we were before. We should not fall into a *petitio principii*.

The countable union of countable sets

The most illuminating discussion of this known to me is one I learned from Conway (oral tradition). Conway distinguishes between a **counted set**, which is a structure $\langle X, f \rangle$ consisting of a set X with a bijection f onto $\mathbb{N}$, and a **countable set**, which is a naked set that just happens to be the same size as $\mathbb{N}$, but which does not come equipped with any particular bijection. As Conway says, elliptically but memorably: a counted union of counted sets is counted; a countable union of counted sets is countable, but a counted union of countable sets, and *a fortiori* a countable union of countable sets could–on the face of it–be anything under the sun. The fact that is obvious is not

A union of countably many countable sets is countable; (v)

but

A union of countably many counted sets is countable; (vi)

and

A counted union of counted sets is counted; (vii)

(vii) is of course a shorthand for the claim that a counted set of counted sets has a union with an obvious counting: indeed a counting that can be recovered—in the way displayed by the diagram above—by combining the counting of the set and the countings of its members. Although (vi) and (vii) are provable without choice—and the diagram gives a visual proof of (vii)—(v) is not. The fallacy of equivocation is to mistake (v) for one of (vi) and (vii).

Socks

There are at least two lines of thought that might lead one to think that there are countably many socks.

All countable sets of pairs look the same

"Surely to goodness" one might think "All sets that are unions of countably many pairs must be the same size! Just replace the members of the pairs one by one ...". In particular there must be the same number of socks in the attic as there are shoes. We know we can count the shoes (left shoes go to evens, right shoes go to odds) so we must be able to count the socks too.

But why should we believe that all sets that are the union of countably many pairs are the same size?

If I want to show that the union of countably many pairs of blue socks is the same size as the union of countably many pairs of pink socks I want to pair off the blue socks in the nth pair of blue socks with the pink socks in the nth pair of pink socks, and I can do this in two ways. However we need AC-for-pairs to pick one of the two ways, and persuading ourselves of the countability of the set of socks by this means exploits the axiom of choice and we would be arguing in a circle if we attempted to use IBE to get from socks to the axiom of choice.

Intuitions of space

Another line of thought that leads us to believe that the set of socks is countable proceeds as follows. The very physical nature of the setting of the parable has smuggled in a lot of useful information. It cues us to set up mental pictures of infinitely many shoes (and socks) scattered through space. The shoes and socks are—all of them—extended regions of space and so they all have nonempty interior. Every nonempty open set contains a rational, and the rationals are wellordered. This degree of asymmetry is enough to enable us to choose one sock from each pair. The physical intuitions underlying this last argument make it very clear to us that we can pick one sock from each pair—as indeed we can. Space is just sufficiently asymmetrical for us to be able to explicitly enumerate the socks in countably many pairs scattered through it.

So we have another example of a fallacy of equivocation, this time between:

$$\text{Every countable set of pairs has a choice function} \tag{viii}$$

and

$$\text{Every countable set of pairs of open sets of reals has a choice function} \tag{ix}$$

As before, it is the first member of the pair that needs the axiom of choice, but it is the second member of the pair that is obvious.

From a pedagogical point of view it may be worth making the following observation, even though it may initially seem to have no direct bearing on the above argument. When we reflect how straightforward is the construction that matches left shoes with evens and right shoes with odds, one is struck by how difficult it is to induce students to come out with it—or with anything like it. This suggests that the reason why students think the set of shoes is countable is because they think they can count it directly, in exactly the way they think they can count the socks directly—namely by illegitimately exploiting the physical intuitions cued by the background information in the parable.

Perhaps this does have direct bearing on the argument of this article after all, for—by drawing attention to the importance of physical intuitions in lending plausibility to mathematical claims—it underlies how easy it is to commit the fallacy of equivocation that got us into this mess.

Conclusion

I have argued that all the standard examples of consequences-of-the-axiom-of-choice-that-seem-obvious partake of fallacies of equivocation. This does not amount to an argument that the axiom of choice is *false*. However it should serve some of the rhetorical purpose of such an argument, since—by depriving the axiom of choice of its obviousness—it prepares the way for an opposing case.

I would like to thank Charles Pigden and my students (in particular Jason Grossman) for helpful discussions.

References

[1] Peter Lipton. Inference to the best explanation (second edition) International library of Philosophy, Routledge 2004

[2] Russell, B. A. W., *Introduction to Mathematical Philosophy* Routledge, 1919.

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