Games played on an illfounded membership relation

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April 9, 2012

The paper was originally presented at one of the LMS set theory meetings orgaanised by Charles Morgan and Mirna Dzamonja some time in the 1990's and appeared in the Boffa festschrift in 2001. There are slight differences between this version and the published version, but only slight. Some typos have been corrected and the formatting subtly changed. The only significant change is the reference in the bibliography to subsequent work on this topic done by Denis Savelieff in ignorance of the work you are reading—which may be of interest to some readers.

Abstract

The $\in$-game as an attempt to salvage something from failure of foundation. Solitaire version gives rise to concept of wellfounded set and rank, binaire version gives us pseudofoundation and pseudorank. These both give rise to games defining a binary relation of relative rank (resp. pseudorank). Greatest and least fixed points. Natural constructions of models of $\in$-determinacy. A paradox of inductively defined sets.

Long before it was fashionable, Maurice Boffa was working on set theory without the axiom of foundation, and his prescience in spotting its foundational importance caused him to become the focus of—and an inspiration and example to—a circle of younger scholars and students to whose number it was my privilege to belong. It is now a pleasure for me to be able to contribute to his festschrift an article on this topic that informed his early work, helped build his reputation and brought me to sit at his feet. My enthusiastic fallacious proofs always elicited from Maurice the response "Mais Thomas, il faut l'écrire!". So here is some illfounded set theory in writing for his eagle eye.

Definitions

All games here are two-player games of perfect information, where the players move alternately. These are sometimes called **combinatorial** games. A completed sequence of moves is a **play.** A game (in which the players move alternately) is **open** iff the set of plays in the game that are wins for the first player (player I) form an open subset of the set of all plays in the product topology. (Remember that the set of plays is a product of countably many discrete spaces.) That is to say, in an open game, if player I is going to win this fact has become apparent after finitely many moves. In a closed game player I wins by not having lost at any finite stage, and if player II wins this has become apparent after finitely many steps.

In all the games that follow I shall use the convention that 'Wins' with a capital 'W' means 'has a winning strategy for'. We will also generally have the **normal play convention**. Any player finding himself or herself unable to move thereby loses.

A bit of terminology used here is possibly not standard: **binaire** vs **solitaire**. Binaire games are the usual two player games with two distinct players. A solitaire version of a two-player game is the degenerate version where one player takes both rôles. Although solitaire games are often much easier than the corresponding binaire games (The sequence 1: P-K4 P-K4; 2: B-QB4 P-Q3; 3 Q-KB3 P-KKt3; 4 Q $\times$ KBP mate; is a win in solitaire chess) including them enables one to give a smoother general treatment than would otherwise be possible..

Introduction

Some say that set theory without the axiom of foundation is just the theory of an extensional relation, and is not part of set theory proper. One could start a quarrel by saying 'In that case, what is wellfounded set theory but the theory of one wellfounded extensional relation?' but without taking sides in this debate one can still recognise interesting problems in (illfounded) set theory, even if one prefers to describe them as problems in something other than set theory.

The point will of course be made that whatever can be said about games played on an illfounded $\in$-relation can be said also about games played on any illfounded binary relation, and that this will reveal illfounded set theory to be—as it was charged with—merely the theory of one illfounded binary relation. However this same generalisability point can be made against the theory of wellfounded sets. What is distinctive about the theory of wellfounded sets is $\in$-induction, and this too can be—and indeed very properly is—generalised to a principle of induction over any wellfounded relation.

The people who believe that the axiom of foundation is true (I call them 'Fundi's) believe that there can be found a range of entities described by the theory of one wellfounded binary relation with equality in a manner in which one cannot find entities described by any theory of illfounded binary relations with equality.

In what follows I shall be developing ideas for new axioms for illfounded set theory, and since I want to keep them as open-ended as possible the discussion will perforce have to be conducted in naïve set theory. Readers should not take this as an invitation to carp or panic, but rather as a warning to hold hard onto their seats.

1 Simultaneous displays

There are several constructors that create new games out of old games that can be naturally thought of as simultaneous displays.

If G and G' are two combinatorial games, with G played between `I` and `II`, and G' between `I`$'$ and `II`$'$, then we can imagine a simultaneously display put on by two artistes called **Arthur** and **Bertha**, with Arthur playing `I`$'$ in G' and `II` in G, and Bertha playing `II`$'$ in G' and `I` in G.

Prima facie the two players move simultaneously, and so this would be a game of imperfect information. We can turn it into a game of perfect information either by ruling that at each stage Arthur makes his move and then Bertha does, or by ruling that Bertha plays first and then Arthur.

Even this description is incomplete. For one thing, the player who plays second (which is to say the player who makes even-numbered moves) seems to have the choice of which of G and G' to play in. (Notice that the "odd" player has no such liberty). It may be sensible to define the game as restricting the "even" player to move in the game *other* than the game that the odd player has just moved in. That gives us eight ways of combining G and G' in a simultaneous play even before we remember that each game can be taken as open or closed.

We can dispose briefly of the games in which the even player is free to chose which component game to move in. Whenever the even player has a winning strategy in either of the component games (s)he can chose to play entirely in that game, and can thereby win the simultaneous display. No new structure arises. Accordingly we consider only simultaneous displays where the even player is constrained to respond in the component game that the odd player has *not* just moved in. It is probably worth noting at this point that Ehrenfeucht-Fraïssé games can be seen as arising from Hintikka games in precisely this way.

Finally there is a kind of simultaneous display that makes sense only when the two games being displayed are *solitaire*. In this construction Arthur plays first by making a move in one of the games—he choses which. Bertha must reply with a move in the *other* game. Arthur's freedom of manœuvre means that he can swap between rôles `I` and `II` in both games so the distinction between `I` and `II` is lost, and the games may as well be thought of as *solitaire*. We have the normal play convention as usual.

Evidently Bertha has a winning strategy if the two games are the same game: she simply copies Arthur's move. We will think of this game as a game played to test whether or not the two component games are the same.

2 The $\in$ game

The foregoing has slightly more generality than we will need here. All the combinatorial games that we will be stitching together with these constructors will be instances of the $\in$ game, to which we now turn.

The $\in$ game has two players: `I` and `II`. We can define G_x by: `I` picks a member x' of x (he loses if he can't), and then they play $G_{x'}$, with `II` starting.

With a bit of overloading we can also say that $\mathtt{I}$ is $\{x : \mathtt{I} \text{ Wins } G_x\}$ and $\mathtt{II}$ is $\{x : \mathtt{II} \text{ Wins } G_x\}$.

A bit of notation: $b(x)$ is the set of things which meet x. Thus $b(x) = -\mathcal{P}(-x)$. The 'b' is an upside-down '$\mathcal{P}$' to remind us that these operations are dual.

If $\mathtt{I}$ Wins G_y for all $y \in x$ then $\mathtt{II}$ Wins G_x. Dually if $\mathtt{II}$ Wins G_y for even one $y \in x$ then $\mathtt{I}$ Wins G_x. This tells us that $\mathtt{I}= b(\mathtt{II})$ and $\mathtt{II}= \mathcal{P}(\mathtt{I})$, and that $\mathtt{I} = b(\mathcal{P}(\mathtt{I}))$ and $\mathtt{II} = \mathcal{P}(b(\mathtt{II}))$. Obviously we want $\mathtt{I}$ and $\mathtt{II}$ be the *least* fixed points for these two operations, and since $\mathcal{P}$ and b are both monotone functions on the complete poset $\langle V, \subseteq\rangle$ there will be such least fixed points. We can define

$$\mathtt{I} \ = \bigcap\{y : b(\mathcal{P}(y)) \subseteq y\} \ \text{ and } \ \mathtt{II} \ = \bigcap\{y : \mathcal{P}(b(y)) \subseteq y\}.$$

Least fixed points always allow a definition "from below" by iteration over the ordinals.

$$\mathtt{II}_1 =: \{\emptyset\}; \qquad \mathtt{I}_1 =: \{V\};$$

$$\mathtt{II}_\alpha = \mathcal{P}(\bigcup_{\beta<\alpha} \mathtt{I}_\beta); \qquad \mathtt{I}_{\beta+1} =: b(\mathtt{II}_\beta); \qquad \mathtt{I}_\lambda =: \bigcup_{\beta<\lambda} \mathtt{I}_\beta \ \text{ for } \lambda \text{ limit.}$$

Then the *rank* ($\rho(x)$) of a set in $\mathtt{I}$ or $\mathtt{II}$ is the least α such that it belongs to $\mathtt{I}_\alpha$ or to $\mathtt{II}_\alpha$. Notice that things in $\mathtt{II}$ (unlike things in $\mathtt{I}$) can have limit rank. When I need to distinguish this concept of rank from the rank function of wellfounded sets I shall call it 'pseudorank'.

Then

$$\mathtt{I} \ = \bigcup_{\beta \in On} \mathtt{I}_\beta; \qquad \mathtt{II} \ = \bigcup_{\beta \in On} \mathtt{II}_\beta.$$

If x is in $\mathtt{I}$ (or $\mathtt{II}$ respectively) then the appropriate player has a (nondeterministic) winning strategy, namely "if i am player $\mathtt{I}$, pick a member of $\mathtt{II}$", or "If i am player $\mathtt{II}$, pick a member of $\mathtt{I}$" and the rank of x is also simply the rank of the tree of all plays played according to this strategy.

2.1 The solitaire game and wellfoundedness

In solitaire G_x player $\mathtt{I}$ builds an descending $\in$-chain, and loses if he reaches an empty set. So x is wellfounded iff every strategy for $\mathtt{I}$ in solitaire G_x is losing.

The rank of a wellfounded set is an indication of how wellfounded it is: $\rho(x) < \rho(y)$ says that x is more wellfounded than y. We can arrive at the same comparison of rank of wellfounded sets by considering the apparently unrelated phenomenon of simultaneous solitaire games of G_x and G_y. The simultaneous display of solitaire G_x and G_y is put on by two artistes called 'Arthur' and 'Bertha'. Arthur plays solitaire G_y and Bertha plays solitaire G_x, and as usual the first player who is unable to move loses. (Normal play convention) The intention is that Arthur has a winning strategy if y is less wellfounded than x.

Consider first the version where Arthur shows his hand first.

Definition 1 *The rules for $G_{x<y}$ are as follows.*

1. *Arthur picks y' in y (loses if he can't); and then*
2. *Bertha picks x' in x (loses if she can't);*

then they play $G_{x'<y'}$. (So that then

1. *Arthur picks y'' in y' (loses if he can't); and then*
2. *Bertha picks x'' in x' (loses if she can't);*

and so on)

This isn't really a definition of a single game, because it says nothing about who wins infinite plays. To complete it—as a definition of a game—we would have to supply a function from { set of infinite plays } $\rightarrow$ {`I`, `II`}. However, for the moment it is probably simplest to think of this as a game that allows draws—every infinite play is drawn—so that even without supplying information about who wins infinite plays we can at least say that if x is wellfounded but y isn't, then Arthur has a winning strategy, which is simply his winning strategy for solitaire G_y: he ignores Bertha completely. If both x and y are illfounded then both players have strategies to avoid defeat. If they are both wellfounded then Arthur has a winning strategy as long as $\rho(x) < \rho(y)$. He simply picks a member of his last element whose rank is greater than the rank of the last element played by Bertha. The converse is also true: if x and y are both wellfounded and Arthur has a winning strategy in $G_{x<y}$ then $\rho(x) < \rho(y)$. This can be proved by induction on the rank of y.

Definition 2 *The rules for $G_{x\leq y}$ are as follows.*

1. *Bertha picks x' in x (loses if she can't); and then*
2. *Arthur picks y' in y (loses if he can't);*

then they play $G_{x'\leq y'}$.

In this version (where Arthur moves second) he has a winning strategy iff $\rho(x) \leq \rho(y)$. He simply picks a member of his last element whose rank is at least the rank of the last element played by Bertha.

These two games are set up so that Arthur has a winning strategy in the game iff $\langle x, y\rangle$ is in the appropriate relation. Thus

- the games (be they open or closed) in which Arthur moves first correspond to fixed points (be they greatest or least) for the operation that sends R to $\{\langle X, Y\rangle : (\exists y \in Y)(\forall x \in X)(R(x, y))\}$.
- the games (be they open or closed) in which Bertha moves first correspond to fixed points (be they greatest or least) for the operation that sends R to $\{\langle X, Y\rangle : (\forall x \in X)(\exists y \in Y)(R(x, y))\}$.

Thus $\leq$ corresponds to $\forall\exists$ and $<$ corresponds to $\exists\forall$, which is why the subscript in the game where Arthur moves first includes a '$<$'.

There is a notation in use for the second operation, due I think to Roland Hinnion, who writes "R^+", but I know of no standard notation for the first. Both these operations take quasiorders to quasiorders. The set of all quasiorders is a complete lattice under $\subseteq$ and both operations have lots of fixed points.

Naturally we will be interested in the greatest and least fixed points for these two lifts. The greatest fixed points correspond to the versions of the games where Arthur wins all infinite plays, and the least to those where Bertha wins all infinite plays.

2.2 The *binaire* game and pseudofoundation

A simultaneous display of the *binaire* versions of G_x and G_y has a connection with pseudorank analogous to the connection simultaneous *solitaire* G_x and G_y has to (ordinary set-theoretic) rank.

Arthur will play `I` in G_y and `II` in G_x, Bertha the other way around.

Definition 3 *The rules for $G_{x<y}$ are as follows.*

- *Arthur picks y' in y (loses if he can't)*
- *Bertha picks x' in x (loses if she can't)*

then they play $G_{y'<x'}$.

For the moment we consider only the game where Arthur moves first: hence the '$<$' in the subscript.

Naturally we will be interested mainly in the open game (all infinite plays won by Bertha) and the closed game (all infinite plays won by Arthur) rather than the others.

This gives rise to two relations $x <_o y$ (if Arthur Wins the open game $G_{x<y}$) and $x <_c y$ (if Arthur Wins the closed game $G_{x<y}$). Naturally $<_o$ is the least fixed point and $<_c$ the greatest fixed point.

We can also define a transfinite sequence of relations $x <_\alpha y$ recursively as $(\exists y' \in y)(\forall x' \in x)(\exists \beta < \alpha)(y' <_\beta x')$. The effect of this is that $<_\alpha$ is the αth iterate of $=$ under $+$, taking unions at limit stages.

Now we can give a nice game-theoretic demonstration that this least fixed point has the properties it should. Suppose Arthur has a strategy σ in $G_{x<y}$ and a strategy τ in $G_{y<z}$. The following picture shows how he can use these to construct a strategy for $G_{x<z}$. This (entirely standard) manœuvre is known to game-theorists as *strategy stealing*.

$$
\begin{array}{ccccc}
x & & y & & z \\
\text{Bertha plays} & \leftarrow & \text{Arthur fakes using } \sigma & \leftarrow & \text{Arthur plays} \\
\downarrow & & & & \\
\text{Arthur plays} & \rightarrow & \text{Arthur fakes using } \tau & \rightarrow & \text{Bertha plays} \\
& & & & \downarrow \\
\text{Bertha plays} & \leftarrow & \text{Arthur fakes using } \sigma & \leftarrow & \text{Arthur plays} \\
\downarrow & & & & \\
\text{Arthur plays} & \rightarrow & \ldots & &
\end{array}
$$

When Arthur plays in the column under 'z' he is using strategy τ on the pretence that all his moves in the middle column (under 'y') noted as being made using σ were made by Bertha, and moves he made in the middle column noted as being made using τ were made by him.

When Arthur plays in the column under 'x' he is using strategy σ on the pretence that all his moves in the middle column (under 'y') noted as being made using τ were made by Bertha, and moves he made in the middle column noted as being made using σ were made by him.

Finally when Arthur fakes in the column under 'y' he is alternately (on even moves) using strategy σ on the pretence that all his moves in middle column (under 'y') noted as being made using τ were made by Bertha (and moves made by him in the middle column using σ were made by him), or (on odd moves) using τ on the pretence that all his moves in middle column (under 'y') noted as being made using σ were made by Bertha (and moves made by him in the middle column using τ were made by him).

That way the left-hand and middle columns together look like a play of $G_{x<y}$ in which Arthur is playing according to σ and the middle and right-hand columns together look like a play of $G_{y<z}$ in which Arthur is playing according to τ.

This picture (and the corresponding picture for composing Bertha's strategies) should be enough to prove that

THEOREM 4 *$<_o$ and $<_c$ are transitive.*

This gives rise to the following observation.

REMARK 5 *$x \in \mathtt{II} \wedge y \in \mathtt{I} \rightarrow x <_o y$.*

Proof:

Suppose $\mathtt{I}$ has a winning strategy τ in G_y and $\mathtt{II}$ has a winning strategy σ in G_x. Arthur then has a winning strategy in the open game $G_{x<y}$ as follows.

$$
\begin{array}{ccc}
x & & y \\
\text{Bertha replies with } x_1 \in x & \leftarrow & \text{Arthur picks } y_1 \in y \text{ using } \tau \\
\downarrow & & \\
\text{Arthur picks } x_2 \in x_1 \text{ using } \sigma & \rightarrow & \text{Bertha replies with } y_2 \in y_1 \\
& & \downarrow \\
\ldots & \leftarrow & \text{Arthur picks } y_3 \in y_2 \text{ using } \tau
\end{array}
$$

Since τ and σ are winning Arthur is never at a loss for a move. Since τ is winning in G_y this play outlined will come to an end. The only way it can end is if $\mathtt{II}$ is unable to move in G_x. But that means that Arthur has won that play of $G_{x<y}$. We have made no particular assumptions about what Bertha does, so this will happen whatever she does. So this strategy is winning for Arthur. ■

This is susceptible of progressive refinement.

REMARK 6 $(\forall \alpha \in On)$

1. $(\forall xy)((y \in \mathtt{I} \wedge \rho(y) = \alpha \wedge (x \in \mathtt{I} \rightarrow \rho(x) > \rho(y))) \rightarrow x <_\alpha y)$
2. $(\forall xy)((y \in \mathtt{II} \wedge \rho(y) = \alpha \wedge (x \in \mathtt{II} \rightarrow \rho(x) < \rho(y))) \rightarrow x <_\alpha y)$

Proof:
By induction on α. The base cases, where $\alpha = 0$ are easy to verify. For the induction we prove the two clauses in order.

(i) Suppose that this is true for all $\beta < \alpha$, and that $y \in \mathtt{I}$, $\rho(y) = \alpha$ and $(x \in \mathtt{I} \rightarrow \rho(x) > \rho(y))$. We need to find a $x' \in y$ such that for all $y' \in x$, $x' <_\beta y'$ for some $\beta < \alpha$. Now nothing in $\mathtt{I}$ has limit rank, so α is successor, and every $x' \in y$ that is in $\mathtt{II}$ is of rank precisely $\alpha - 1$. There may be other things in y that are not in $\mathtt{II}$ at all, but there must be at least one thing in $\mathtt{II}$ of rank precisely $\alpha - 1$. Arthur should pick one of those to be x'.
What can Bertha pick from x? We know only that $(x \in \mathtt{I} \rightarrow \rho(x) > \rho(y))$. If $x \in \mathtt{I}$ then the only things in $\mathtt{II}$ Bertha can pick are of rank greater than $\alpha - 1$ and we can use clause (ii) of the induction hypothesis. If $x \notin \mathtt{I}$ then nothing in x is in $\mathtt{II}$ and again we can use clause (ii) of the induction hypothesis.

(ii) Suppose that this is true for all $\beta < \alpha$, and that $y \in \mathtt{II}$, $\rho(y) = \alpha$ and $(x \in \mathtt{II} \rightarrow \rho(x) < \rho(y))$. We need to find a $x' \in y$ such that for all $y' \in x$, $x' <_\beta y'$ for some $\beta < \alpha$. Every member of y is in $\mathtt{I}$ and is of lower rank. Arthur wants to pick something that is of higher rank than anything Bertha can pick from x. It will be sufficient to pick something of rank at least $\rho(x)$.
What can Bertha pick from x? We know only that $(x \in \mathtt{II} \rightarrow \rho(x) < \rho(y))$. If $x \in \mathtt{II}$ then all its members are in $\mathtt{I}$ and of lower rank, so Arthur has succeeded. If $x \notin \mathtt{II}$ then Bertha can—if she wishes—pick something not in $\mathtt{I}$, but that would be of no use to her.

■

Similarly we can prove

THEOREM 7

1. $x <_o y \wedge x \in \mathtt{I} \rightarrow y \in \mathtt{I}$.
2. $x <_o y \wedge y \in \mathtt{II} \rightarrow x \in \mathtt{II}$.

Proof: of (1)

Suppose I has a winning strategy τ in G_x and Arthur has a winning strategy σ in the open game $G_{x<y}$. Player I in G_y (let's notate him '$\mathtt{I}^y$') can now play according to the following diagram:

$$
\begin{array}{ccc}
x & & y \\
\mathtt{I}^y \text{ plays } x_1 \in x \text{ using } \tau & \leftarrow & \mathtt{I}^y \text{ picks } y_1 \in y \text{ using } \sigma \\
\downarrow & & \\
\mathtt{I}^y \text{ plays } x_2 \in x_1 \text{ using } \sigma & \rightarrow & \mathtt{II} \text{ replies with } y_2 \in y_1 \\
 & & \downarrow \\
\ldots & \leftarrow & \mathtt{I}^y \text{ picks } y_3 \in y_2 \text{ using } \sigma
\end{array}
$$

As before, since σ and τ are winning, $\mathtt{I}^y$ is never at a loss for a move. Since I is playing according to τ in G_x, the game will end, and the only way it can is by II being unable to move in G_y, so $\mathtt{I}^y$ Wins.

The proof of (2) is dual and is omitted. ■

Corollary 8

I *is an upward-closed subset of* $\langle V, <_o \rangle$.
II *is a downward-closed subset of* $\langle V, <_o \rangle$.

It was obvious from the outset that I is an upward-closed subclass of $\langle V, \subseteq \rangle$ and that II is a downward-closed subclass of $\langle V, \subseteq \rangle$, so this corollary is telling us that we should think of this corollary as telling us that $<_o$ is a generalisation of $\subseteq$. It is obviously a refinement of $\subseteq$.

We can close this section by sketching part of the top and bottom of $\langle V, <_o \rangle$. Notice that $x <_o y$ iff $\{y\} <_o \{x\}$ so this poset contains an upside-down copy of itself.

$$\emptyset < \{V\} < \{V \setminus \{V\}\} < \{V \setminus \{V \setminus \{V\}\}\} \ldots < \ldots V \setminus \{V \setminus \{V\}\} < V \setminus \{V\} < V$$

3 Connections with the equality game

If we combine G_x and G_y in the manner of the last construction of section 1 we have a game that tests whether or not $x = y$. Let us call it '$G_{x=y}$'. Let the first player be $\neq$, and second player $=$. As usual, there are two versions, an open version and a closed. It is easy to check that the relation '$=$ has a winning strategy in the open game $G_{x=y}$' is an equivalence relation. Inconveniently, as Isaac Malitz noticed, it is not equality. He points out that $=$ will win the open game $G_{V=-\{V\}}$. For consider: what can $\neq$ do? He cannot pick something in $-\{V\}$ that isn't in V so his only hope is to pick something in V that isn't in $-\{V\}$, namely V. But even if he does pick V, $=$ need only pick $-\{V\}$ and they are back where they started. Anything else allows $=$ to copy his moves blindfold and, if not actually win in finitely many moves, at least never lose in finitely many moves, which is enough to ensure that she can Win the open game.

A moment's reflection will reveal that this reasoning depends only on very general properties of V and $-\{V\}$, and that what Malitz has shown is that if $x \in x$ and $(x \setminus \{x\}) \in x$ then the open game $G_{x=(x \setminus \{x\})}$ cannot distinguish x and $x \setminus \{x\}$.

This relation is a congruence relation for the quasiorders defined by simultaneous play of the $\in$ game, as follows.

Obviously $=$'s winning strategy in $G_{x=y}$ can be used by Bertha to Win $G_{x<y}$.

Clearly if Bertha has strategies to win the open games $G_{x \leq y}$ and $G_{y \leq x}$ then $=$ can use them to Win $G_{x=y}$.

Also if Arthur has a winning strategy in $G_{x<y}$ (resp. $G_{x \leq y}$) and $=$ has a winning strategy in $G_{y=z}$ then he also has a winning strategy in $G_{x<z}$ (resp. $G_{x \leq z}$).

Analogously if Arthur has a winning strategy in $G_{x<y}$ (resp. $G_{x \leq y}$) and $=$ has a winning strategy in $G_{x=z}$ then he also has a winning strategy in $G_{z<y}$ (resp. $G_{z \leq y}$).

4 Pseudofoundation

People who believe in the axiom of foundation regard illfounded set theory—if they accord it any legitimacy at all—as a theory of equality and a single extensional relation, and tend not to think of it as part of Set Theory. One can argue against this view—and correctly—that it distorts history but I think that defence misses the point. The neatness and naturalness of the interpretation of ZF + Forti-Honsell AFA into ZF by means of isomorphism types of extensional relations shows that ZF + AFA really is, indeed, a theory of one extensional relation. One should not be attempting to defend ZF + AFA as a theory of sets.

Of the axioms incompatible with foundation the one that most obviously can arise only from an endogenous concept of sets is the axiom of complementation: the charge of only being a theory of an extensional relation cannot be levelled against theories with an axiom of complementation. If one adopts this axiom it is natural to seek to add as well other axioms that preserve as many of the consequences of foundation as one can while still assuming complementation. Such axioms one might call *pseudofoundation* axioms. An obvious candidate is $\in$-determinacy: $V = \mathtt{I} \cup \mathtt{II}$. It is implied by foundation; as we have seen it implies some of the consequences of foundation (nonexistence of Quine atoms, for example); it remains only to exhibit some models of complementation + $\in$-determinacy. Two illustrations follow.

4.1 The theory of negative types

As well as recalling from Forster [2] the definition of the theory of negative types we will need two facts, both proved there. The theory of negative types is the simple theory of types, but with types T_i indexed by the (positive and negative) integers, not $\mathbb{N}$. For each type T_i the symmetric group on T_i acts in an obvious way on T_{i+j}. A set in T_{i+j} that is fixed by the action of the symmetric group on T_i is said to be **j-symmetric**. Thus the empty set at each type is 1-symmetric, the unordered pair of the empty set and the universe is 2-symmetric and so on. In fact:

(i) Every (set denoted by a) set abstract is symmetric, and (ii) every n-symmetric set is of rank $n+2$ at most.

We can find models of TNT in which every set is of finite rank by omitting the type that says, for each concrete n, that both `I` and `II` have strategies to avoid defeat for n moves.

Suppose $\phi(x)$ is a predicate that realises this type. Think about $\{x : \phi(x)\}$. It is a definable set, so is of finite rank. But then either every member of it is in `II` and is of finite (indeed bounded) rank, or it has a member in `I`, also of finite rank. But neither of these is possible, so $\{x : \phi(x)\}$ must be empty.

Of course if every set is of finite rank then $\in$-determinacy holds.

4.2 The models of Oswald and Church

Oswald's [4] model $\langle \mathbb{N}, E\rangle$ of a set theory with an axiom of complementation is defined as follows.

> $n\ E\ m$ iff either
>
> 1. m is even and the nth bit of the binary expansion of $m/2$ is 1; *or*
> 2. m is odd and the nth bit of the binary expansion of $(m-1)/2$ is 0.

This obviously derives from the old trick (due to Ackermann) of defining $n\ E\ m$ $(n, m \in \mathbb{N})$ iff the nth bit of the binary expansion of m is 1.

The Oswald model has the rather nice property that $\in$ restricted to finite sets (not even hereditarily finite sets!) is wellfounded: the map from finite sets to $\mathbb{N}$ is a homomorphism sending $\in$ to $<_{\mathbb{N}}$. Thus E is really wellfounded, seen from outside.

We can now give an easy proof that

Remark 9 *(Forster [3]) $\in$-determinacy holds in the Oswald model.*

Proof: Notice first that the Oswald model contains all of V_ω, so it has infinitely many elements in `I` and infinitely many in `II`.

So any cofinite set contains an element of `II` and so is in `I`.

Any indeterminate set must therefore be finite and nonempty. Indeed it must have indeterminate members which in turn are finite and nonempty. But then the class of indeterminate sets is a class of finite sets with no $\in$-least member, contradicting the wellfoundedness of $\in$ restricted to finite sets. ■

There is a generalisation of this construction due to Church [1], in which one starts with an arbitrary wellfounded model of ZF, rather than specifically V_ω. Models arising in this way have a concept of "small" set, where a set is small iff it is the same size as a wellfounded set. They tend to obey the pseudofoundation axiom "$\in$ restricted to small sets is wellfounded" and we can use that analogously to show that $\in$-determinacy holds in them.

5 A Paradox

In naïve set theory both $\mathtt{I}$ and $\mathtt{II}$ are sets. All members of $\mathtt{I}$ are wins for player $\mathtt{I}$ so $G_{\mathtt{I}}$ itself must be a win for player $\mathtt{II}$. So $\{\mathtt{I}\}$ must be a win for player $\mathtt{I}$. But this being the case, $\mathtt{I}$'s first move in $G_{\mathtt{I}}$ can be $\{\mathtt{I}\}$, to which $\mathtt{II}$ can reply only with $\mathtt{I}$, leaving them back in the same position as they start. So $G_{\mathtt{I}}$ isn't a win for player $\mathtt{II}$ after all! So $\mathtt{I}$ and $\mathtt{II}$ cannot be sets. More generally, but less idiomatically, we can write:

There is no $\subseteq$-least set x such that $\mathcal{P}(b(x)) \subseteq x$.

Proof: Suppose $\mathcal{P}(b(x)) \subseteq x$. This implies that $b(x) \in x$, so $x \setminus \{b(x)\}$ is a proper subset of x. We will show that $\mathcal{P}(b(x \setminus \{b(x)\})) \subseteq (x \setminus \{b(x)\})$ so x is not $\subseteq$-minimal.

$\mathcal{P}$ and b are both monotone (*) so $\mathcal{P}(b(x \setminus \{b(x)\})) \subseteq \mathcal{P}(b(x))$. Now $\mathcal{P}(b(x)) \subseteq x$ by assumption whence $\mathcal{P}(b(x \setminus \{b(x)\})) \subseteq x$. We want '$x \setminus \{b(x)\}$' not '$x$' to the right-hand-side of the '$\subseteq$'. So we want $w \subseteq b(x \setminus \{b(x)\}) \to w \neq b(x)$. This is just a long way of saying $b(x) \not\subseteq b(x \setminus \{b(x)\})$. We established that $b(x) \in x$ so $x \setminus \{b(x)\}$ is a proper subset of x whence $b(x) \not\subseteq b(x \setminus \{b(x)\})$ by injectivity of b. (*)

So x is not minimal. ■

This establishes

Remark 10 $\mathtt{I}$ *and* $\mathtt{II}$ *cannot be sets.*

In fact this generalises to a sort of omnibus paradox of inductively defined sets. At the points in the argument marked with an asterisk we have used the assumption that b is monotone and injective, and indeed that is all we need. The fact that $\mathcal{P}$ and b are dual seems to play no rôle.

Remark 11
Suppose f is monotone and injective: $(\forall xy)(x \subseteq y \longleftrightarrow f(x) \subseteq f(y))$.
Let $A := \bigcap\{x : \mathcal{P}(f(x)) \subseteq x\}$.
Then A is not a set.

Proof:
We will show that $f(A)$ both is and is not a member of A.

For any x we have

$$\mathcal{P}(f(-\{f(x)\})) \subseteq -\{f(x)\}$$

iff

$$f(x) \notin \mathcal{P}(f(-\{f(x)\}))$$

iff

$$f(x) \not\subseteq f(-\{f(x)\})$$

iff

$$x \not\subseteq -\{f(x)\}$$

iff

$$f(x) \in x$$

Now $\mathcal{P}(f(-\{f(x)\})) \subseteq -\{f(x)\}$ certainly implies that $A \subseteq -\{f(x)\}$, and $A \subseteq -\{f(x)\}$ is just a long-winded way of saying that $f(x) \notin A$. So we have

$$(\forall x)(f(x) \in x \rightarrow f(x) \notin A)$$

In particular $f(A) \notin A$.

On the other hand, let x be an arbitrary set such that $\mathcal{P}(f(x)) \subseteq x$. $A \subseteq x$ so $f(A) \subseteq f(x)$ so $f(A) \in \mathcal{P}(f(x)) \subseteq x$ so $f(A) \in x$. But A is the intersection of all such x, so $f(A) \in A$. ■

If we take f to be the identity we get Mirimanoff's paradox. If we take f to be b we get the paradox on `I` and `II` with which we started.

There is clearly a lot more to be said about these paradoxes than can be said here and now, but the following observations may be in order. In general, as we all know, if the operation one is trying to close under is finitary or first-order there is no paradox: the set of natural numbers is not a paradoxical object. However finitary-ness is a sufficient condition not a necessary one: V_ω can be defined as the intersection of all sets that contain all their finite subsets, and the collection of hereditarily countable sets can be defined similarly and so on for larger cardinals *ad libitum*. Although even at this stage there are complications (The proof that V_ω—as defined above—is not finite has no normal form) we do not seem to get outright paradox until we consider collections inductively defined by second-order operations for which there is no boundedness theorem. Roughly: the sethood of collections inductively defined as the closures under operations of bounded character seems not to be paradoxical, even if some of the facts about these collections (not being members of themselves for example) seem only to have pathologioical proofs.

Closure under constructors of infinite character has long been known to result in paradox: the hereditarily wellordered sets and the hereditarily transitive sets are both paradoxical collections, and it is significant that there are theorems telling us that there are unboundedly large transitive or wellordered sets.

The task before us now is to see if these two paradoxes (and possibly others) can be related precisely to the paradox of this paper to give a accurate understanding of when a second-order inductive definition is legitimate.

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