Why the Sets of NF do not form a Cartesian-closed Category

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October 14, 2007

This is an old result, and was first published by McLarty [2] relatively recently. It was discovered independently at least three times—to my certain knowledge—and quite possibly at least five times since my guess would be that Dana Scott and Solomon Feferman discovered proofs in addition to the three proofs known to me; I know Randall Holmes did, and when I met Edmund Robinson in about 1980 the first question he asked me—on learning that I studied NF—was whether or not the category of sets in NF was cartesian-closed. My reply ("What is a cartesian closed category?") led to my first lesson in category theory and to Edmund and I answering the question together.

Of these independent discoveries McLarty's has a special significance. Noticing that a given mathematical insight is important is itself a—separate[1]—mathematical insight, and even if Colin was not the first to see that the sets of NF do not form a cartesian-closed category, he certainly was the first to see that this failure mattered. In any case my purpose here is not to score points about priority, but rather to exhibit a proof which is quite different from Colin's. I have just cobbled it together as a result of a question asked me by Peter Lumsdaine, and thanks go to Peter for the stimulus. My guess is that it is the same proof Edmund and I discovered, tho' i might be wrong. There are evidently at least three proofs—since Holmes assures me that the proof he discovered is different from McLarty's and mine—and if there are three there may be more.

I start from the two uncontroversial assumptions

1. For the category of sets of a theory to be cartesian closed it is necessary for the theory to believe that the graph of the function

   $$\texttt{curry}\colon\ (A \times B) \to C. \quad \to . \quad A \to (B \to C)$$

   is a set (at least locally, in the sense that its restriction to any set is a set)[2];

[1] This is a slightly racier version of Dedekind's argument for the axiom of infinity according to whicvh each thing is different from the concept of that thing.

[2] Thanks to Peter Johnstone for confirming that this is the case: I know no Topos theory!

2. In any sensible pairing function for NF, the expression '$x = \langle y, z \rangle$', when written out in primitive notation, must be stratified with 'y' and 'z' having the same type, and 'x' having a type which is not lower then the type of 'y' and 'z'.[3]

In NF the qualification at the end of item (1) makes no difference, since there is a universal set and the graph of `curry` local to it will be the graph of `curry` itself.

If the graph of `curry` is a set then in particular so is the graph (call it f_1) of the function that for each x sends $(\{\emptyset\} \times \{\emptyset\}) \to x$ to $\{\emptyset\} \to (\{\emptyset\} \to x)$. $\{\emptyset\} \to x$ is one type higher than x so—by NF comprehension—the graph (call it f_2) of the function sending $\{x\}$ to $(\{\emptyset\} \times \{\emptyset\}) \to x$ is a set. By the same token $\{\emptyset\} \to (\{\emptyset\} \to x)$ is two types higher than x, and—by NF comprehension again—the graph (call it f_3) of the function sending $\{\emptyset\} \to (\{\emptyset\} \to x)$ to $\{\{x\}\}$ is also a set.

Then the composition $f_3 \cdot f_1 \cdot f_2$ sends $\{x\}$ to $\{\{x\}\}$. This immediately gives us the graph of the singleton function as a set and this is known to be impossible in NF. ■

(Another way of characterising cartesian-closed categories is by the presence of an evaluation function $ev : (A \to B) \times A \to B$. Naturally a similar exercise will show that the graph of this function cannot be a set.)

How serious is this breakdown? It may be much less serious than one thinks. NFistes have known for a long time that whenever a desired result P—familiar from ZF—fails in NF then the proof method that gave rise to P can be tweaked to prove a variant P' which (i) is equivalent to P in ZF; and (ii) differs from P in having a T function inserted judiciously, or in having some occurences of a variable 'X' replaced by 'ι"X', or some similar modification. P' discharges many of the functions of P and life goes on roughly as normal. One thinks of the theorem $|\iota``x| < |\mathcal{P}(x)|$ which does duty for Cantor's theorem in NF. Perhaps the failure of cartesian-closedness of NF will turn out to be similarly un-concerning.

References

[1] Forster, T.E. [2007] Implementing Mathematical Objects in Set Theory Logique et Analyse *to appear*. Also available from `www.dpmms.cam.ac.uk/~tf/pairs.pdf`

[2] McLarty, C. [1992] Failure of cartesian closedness in NF. *Journal of Symbolic Logic* **57** pp. 555–6.

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[3]There is a fairly detailed discussion of this point in [1].

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