Propositional Compactness Implies Uniqueness of Dimension for Vector Spaces

(Edited by Thomas Forster from a message from Andreas Blass)

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Suppose X and Y are two bases for the one vector space. We want $|X| = |Y|$. Because we can use Schröder-Bernstein it will suffice to find an injective map $f : X \to Y$.

Every $x \in X$ is a linear combination of ("depends on") *finitely many* $y \in Y$. Let us call this finite set '$Y(x)$'. We want an injection f from X into Y; if we require further that $f(x) \in Y(x)$ for all $x \in X$ then our task actually becomes *easier*, because there are now only finitely many options for each of the values $f(x)$. This means that the problem of producing such an f can be rewritten as the problem of satisfying a certain set of propositional formulæ.

The propositional variables $p_{x,y}$ that we use to build up the formulæ in the set will arise from the elements of $X \times Y$, with $p_{x,y}$ saying that $f(x) = y$. The sentences to be satisfied are:

- $\neg(p_{x,y} \wedge p_{x,y'})$ whenever $y \neq y'$;
- $\neg(p_{x,y} \wedge p_{x',y})$ whenever $x \neq x'$;
- for each $x \in X$, the disjunction $\displaystyle\bigvee_{y \in Y(x)} p_{x,y}$.

By propositional compactness, these sentences will be simultaneously satisfiable as long as each finite subset is satisfiable. But—for a finite subset—we have only finitely many x's to worry about. So it suffices to show that, for each finite subset $X_0 \subseteq X$, there is an injection from X_0 to Y that sends each $x \in X_0$ to some $y \in Y(x)$. Now we need Hall's Marriage Theorem.[1]

By Hall's matching theorem it suffices to show that, for each finite subset $X_1 \subseteq X_0$, $|Y(X_1)| \geq |X_1|$. Now a finite subset of a finite subset of X is just a finite subset of X, so it will suffice to show that $X' \subseteq X$, $|Y(X')| \geq |X'|$.

But this is clear, by ordinary finite-dimensional linear algebra: we can't write the $|X'|$ linearly independent vectors in X_1 as linear combinations of strictly fewer than $|X'|$ vectors from Y.

[1]This states that given a set B of blokes and a set C of chicks, and a compatibility relation R between blokes and chicks, every bloke can be allocated a chick as long as, for each set B' of blokes, the set $\{c \in C : (\exists b \in B')(R(b, c)\}$ of chicks is of size at least $|B'|$.