An Introduction to Ordinals

Thomas Forster

March 8, 2012

Contents

These notes are written up from a talk I gave at a Trinity Mathematics Society symposium on 4/iii/2012. I am grateful to Mary Fortune and Jonathon Lee, officers of that society, for inviting me to talk and thereby provoking me to set my thoughts in an order suitable for a nonspecialist audience. (And for treating me to dinner afterwards!)

I am making these notes available even before i have managed to do the graphics properly. If i await that i'll never put this stuff up at all.

It's not always entirely clear when a mathematical object is a number. Natural, rationals, reals and complexes are numbers. Are p-adics numbers? They form a field... but then rational functions form a field and they're definitely not numbers. Integers mod p? Quaternions? *Octonions* ...?? Conway numbers are numbers (even tho' Conway games are not). My take on this is that ordinals are *definitely* numbers. This is partly because most flavours of numbers measure things that are out there in the real world, in the sense of not being mathematical objects. Naturals count the number of coins in my pocket, integers count my bank balance, rationals the efficacity with which blood dilutes my alcohol stream. We think of ordinals as numbers becuase —as we are about to see—they have something to do with length of processes.

1 Cantor's Discovery of ordinals

For reasons which need not detain us here[1] Cantor was interested in the operation of *derivative* on closed sets of reals. The derivative X' of X is X shorn of its isolated points.

If the output is the same type as the input we can iterate, and Cantor was led to consider the result of applying the derivative operation over and over again. Now the derivative is an operation that is monotone on the poset of sets-of-reals-with-$\subseteq$, (that is to say, $X \subseteq Y \rightarrow X' \subseteq Y'$) and the effect of this is that for any closed $X \subseteq \mathbb{R}$ there is a well-defined set which is the result of taking the derivative n times for all $n \in \mathbb{N}$.

This monotonicity is important. There is the story in the Philosophical literature of *Thompson's Lamp* (see [7]). It is off at time $t = 0$, it is switched on at time $t = 1/2$, off again at time $t = 3/4$, on again at time $t = 7/8$ and so on. Is it on at time $t = 1$? Or off? In the philosophical literature the problem is taken to be that its state at time $t = 1$ is *overdetermined* by the events at time $t < 1$ whereas of course the truth of the matter is that its state is *underdetermined*. And underdetermined because the Thompson's lamp process is not in any useful sense *monotone*.

It is this fact—that monotone operations can be iterated infinitely often—that opens up to us the new world of transfinite ordinals.

1.1 Ordinals as rank functions for parallel computations

[Draw four trees: two wellfounded, one loop and one ω^*-chain not wellfounded. Lisa].

Distinguish processes that *cannot be finished in finite time* from processes that *cannot even be started.*

every compound process whose assembly instructions are nice in the sense that this one is nice and these aren't has an ordinal rank.

[1] Tho' the reader should certainly make sure at some other time that they find the leisure to be detained by them. See e.g. Dauben, [2], Kanamori, [5].

If you are worried about time consumption bear in mind that you can use the same trick as with Thompson's lamp. Every edge can be decorated with a rational number in such a way that the sum of the decorations along any ascending path is finite.

We will later make precise what this sense of niceness is, and also why there are enough ordinals.

1.2 (Infinite!) ordinals for finite hierarchies of access

Consider a computer system for storing sensitive information like people's credit information, or criminal records, and suchlike. It is clearly of interest to the subjects of these files to know who is retrieving this information (and when and why), and there do exist systems in which each file on an individual has a pointer to another file which contains a list of the the userids of people accessing the head file, and dates of those accesses. Is there a spike of reads of this file whenever agent X is in the office. . . ? One can even imagine people wishing to know who has accessed *this* information, and maybe even a few steps further. A well-designed system would be able to allocate space for new and later members of this sequence of files as new reads by users made this necessary. These files naturally invite numerical subscripts. The system controllers might wish to know how many files had been generated by these reads, and know how rapidly new files were being generated, or what statistical relations existed between the number of reads at each level, and suchlike. This information would have to be stored in a file too, and the obvious subscript to give this file is ω. (It wouldn't be sensible to label it 'n', for n finite (even if large) because there is always in principle the possibility that we might generate n levels of data files.) Then we start all over again, with a file of userids and dates of people who have accessed the ωth file. Thus we can imagine a system where *even though there are only finitely many files* some of those files naturally have transfinite ordinals as subscripts.

2 Ordinals as a Recursive Datatype

We add to the constructors for $\mathbb{N}$ (which are of course 0 and `suc`, the successor function) the extra constructor `sup` which creates an ordinal from a set of ordinals. Annoyingly the datatype we obtain is not free in the way $\mathbb{N}$ is. Altho' each natural number can be constructed in only one way. In contrast ω can be constructed by `sup` in uncountably many ways. There are $2^{\aleph_0}$ unbounded sets of finite ordinals and every one yields us ω when we whack it with `sup`.

This obstructs the proof that $<_{On}$, the engendering relation on On, is a total order. (It's easy to show that it's wellfounded—engendering relations on recursive datatypes always are). However, it can be shown that it is a total (and therefore a *well*-) order nevertheless. In fact the usual proof of the Bourbaki-Witt theorem[2] is actually a proof that $<_{On}$ is a total order.

[2] which I learnt from PTJ, can be found in [3] p.54. I can't find it in [4] tho' it is certainly

If we have a set X where everything depends on something else in X then nothing in X can get constructed. You can never *break into* X. A rectype can have no subsets like this. We say of a relation whose domain has no bad subsets that it is **wellfounded**. Engendering relations of rectypes are always wellfounded. Indeed wellfoundedness is the characteristic property of engendering relations; all natural examples of wellfounded relations are either literally engendering relations or simple-minded modifications of them.

So we have a principle of induction, and definition by recursion. For details, see ch 2 of [3].

Now that we have the concept of wellfoundedness we can define a *wellordering* as a wellfounded total order. $<_{\mathbb{N}}$ but not $<_{\mathbb{R}}$. The class of wellorderings has the very nice feature that for any two of its members there is an canonical (indeed *unique*) bijection between one and an initial segment of the other. For the moment the proof of this fact is left to the reader.

A lexicographic product of two wellorderings is a wellordering.

3 Ordinals as isomorphism classes of wellorderings

(or perhaps virtual objects arising from is'm classes)

Ordinals wellordered by the obvious order relation.

Draw the picture

3.1 Sylver coinage shows how infinite ordinals can be useful in finite maths

We are now in a position to appreciate the next illustration of how infinite ordinals can crop up in the description of purely finite problems.

The game of Sylver Coinage was invented by Conway (see [**?**] p ??) and gives rise to a perfect exercise that introduces ideas from the second half of the course by clothing them in ideas from the first.

It is played by two players, `I` and `II`, who move alternately, with `I` starting. They choose natural numbers greater than 1 and at each stage the player whose turn it is to play must play a number that is not a sum of multiples of any of the numbers chosen so far. The last player loses.

Notice that by 'sum of multiples' we mean 'sum of *positive* multiples': this is really a game about the invention of new denominations of coins, and nobody gives change. What the players are doing is trying at each stage to invent a new denomination of coin, one that is of a value that cannot be duplicated by handfuls of coins already in circulation. (Conway calls the game 'Sylver Coinage' with a 'y' because of a theorem of Sylvester that says that every sufficiently large

in the lectures on which that book is based.

multiple of (m_1, m_2) can be expressed as $am_1 + bm_2$ with a and b both natural numbers. In fact the same goes for sufficiently large multiples of the highest common factor of m_1, m_2, m_3, $\ldots$.)

The aim is to show that every play of this game is finite. How do we do that?

Observe that—because of the theorem of Sylvester—at every stage there are only finitely many multiples of H (H is the HCF of all the numbers played so far) which are available to the player whose turn it is. There may be infinitely many other numbers available to that player of course, but if (s)he plays any of them then the quantity H decreases. The quantity to keep your eye on is the ordered pair $\langle H, N\rangle$ where N is the number of multiples of H that are available (i.e., are below Sylvester's bound). Every move by either player decrements the value of this quantity—in the sense of the lexicographic order on $\mathbb{N} \times \mathbb{N}$, and the ordinal of this wellordering is ω^2.

4 Rosser's Counting Principle

The fact that $\langle X, <_{On}\rangle$ is a wellordering (proved above p 4) means that every set of ordinals is naturally wellordered and has a length, which is of course an ordinal. In particular every initial segment of the ordinals has a length which is another ordinal. Which ordinal? Rosser's axiom of counting from [6] says that the length of any initial segment X of $\mathbb{N}$ is the least number not in X.

Lists are polymorphic: if `wombat` and `dingbat` are different types then `wombat-list` and `dingbat-list` are different types. However the naturals that count the lengths of `wombat-list`s and the naturals that count the lengths of `dingbat-list` are all of the one single type: natural numbers are *mono*morphic. This is clearly sensible: there are no obvious benefits to be derived from distinguishing these two kinds of naturals from each other. It is only beco's we take finite ordinals to be monomorphic that we are even able to *state* Rosser's counting principle, but once we *do* take them to be monomorphic then Rosser's counting principle is completely uncontroversial. (Prove it by induction if you have any doubts).

> One thing I have never been able to ascertain is whether or not there is anywhere on this planet a human culture that *doesn't* have monomorphic natural numbers, that is to say a culture that counts various things, but doesn't identify the numbers it uses for counting (say) humans with the number it uses for counting (say) cattle. It is true that—to this day—farmers in some parts of the Pennines use *p*-celtic number words (basically Welsh) for counting sheep, even tho' for everything else they use english number words… but that isn't quite what I wanted.

Usual mathematical practice goes beyond Rosser's axiom of counting when ordinals (which as we have seen, are a kind of transfinite generalisation of $\mathbb{N}$) are concerned. The *Extended Counting Principle* states that every ordinal is the order type of the set of its predecessors in their natural order (and this order, let us not forget, is a wellorder and therefore has an ordinal). In other words, the length of any initial segment $\langle X, <_{On} \rangle$ of the ordinals is the least ordinal not in X.

Sensible tho' it sounds, the extended counting principle has the potential to cause trouble if we make certain additional assumptions. What about the ordinal that is the length of the (improper) initial segment that is $\langle X, <_{On} \rangle$ itself? It must be the first ordinal not in On, which is absurd. This absurdity is the Burali-Forti paradox. The usual response is to say that this is a proof that On is not a set.

H I A T U S

Once one realises that polymorphism is an option (made available to us for Health and Safety reasons) and that perhaps ordinals might be *prima facie* as naturally polymorphic as are the lists that it is their purpose to count, one starts to wonder if polymorphism might not be a way of dealing with Burali-Forti, a better way (perhaps) than stressing about whether or not the collection of ordinals is a set. Seeing every mathematical (or at least every logical) foundational problem in terms of whether or not something in the picture is a set is *sooo* last-century and so steeped in the error of set-theoretic foundationalism. The monomorphism/polymorphism distinction is a new idea from CS and it might be helpful.

So we might try for size the idea that, altho' *small* ordinals are uncontroversially monomorphic, there might come a point at which the ordinals start to fray, and the decision to think of them as monomorphic is no longer safe. The interesting question then comes: when do we reach this point?

Uncountable Ordinals and Hartogs' theorem

Hartog's thm says that for every set x there is a set y with a wellordering such that $|y| \not\leq |x|$. Its real meaning is that if a definition by transfinite recursion fails then it's not beco's we have run out of ordinals. This is beco's the ordinals are a terminal object in the category of rectypes of infinite character.[3]

Where do we fit in: wellfoundedness guarantees termination and every wellfounded relation has a rank?

Where do we fit in the canonical bijection between wellorders?

References

[1] Conway, Berlekamp and Guy, Winning ways Academic Press

[2] Dauben, J.W. Georg Cantor: his Mathematics and Philosophy of the Infinite

[3] Forster, T.E. Logic, Induction and sets

[3] Well, *something* like that!

[4] Johnstone, P.T, Notes on set theory and Logic

[5] Kanamori, A. [title] Handbook of set theory

[6] Rosser, J.B. Logic for mathematicians

[7] J.M. Thompson. Tasks and super-tasks. *Analysis* **15** (1954) pp 1–13.