An introduction to WQO and BQO theory

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Contents

Loose ends

Two things that we shouldn't completely lose sight of. (i) Can we use the characterisation of multisets in terms of surjections to prove BQOness of various multiset orderings (and in general characterisations of certain datatypes in terms of domain equations to prove lots of theorems about BQOs) (ii) The connections between $\text{AxCount}_{\leq}$ and WQOs in NF.

Proofs of wellfoundedness of RPO and LPO. The topological story. Minimal bad array theorem. Proof of the BQO tree theorem and Laver's theorem. Sets of children or lists of children??? Inductive vs coinductive dfn of infinite trees. Minimal bad array theorem. and therest of the topological stuff. Excluded substructure characterisation using $RADO(n)$. Commutation of lifts.

Can we use the fact that WQOness of X is the same as wfness of pow X to extend the notion of the wf part of a relation to a wqo part of a relation?

Minimal bad array lemma. I haven't got anywhere near understanding it. Sse $\langle X, \leq_X \rangle$ is a quasiorder that is not BQO. Then $\langle \mathcal{P}_{\aleph_1}(X), \leq_\infty \rangle$ is not wellfounded. Look at the wellfounded part. Is this WQO? If it is, then it's the whole thing, so it isn't. So there is an MBS. Does this MBS sift into a bad array that in minimal in the right sense?

Let Θ be the structure whose carrier set is $H_{\aleph_1}(Q)$ where Q is a countable set of Quine atoms, quasiordered by the obvioius lift of the identity quasiorder. Presumably every countable wellfounded quasiorder embeds in it. Something like that, anyway...

If r is a ray of a block then `tl`"r is another block

Should say a lot about domain equations and greatest and least fixed points

The RADO QO cannot be an intersection of two wellorderings. One of them would have to be the worder of $<_{\mathbb{N}}$ to length ω^2. Any total order whose intersection with that gives RADO must be illfounded beco of $\langle 1, 2\rangle > \langle 2, 3\rangle > \langle 3, 4\rangle > \ldots$.

index entries for

list stretching coinduction, excluded substructures FFF sifting block RADO

Notation

Does one write sequences in functional notation or subscript notation? That is, does one write "$x_1, x_2, \ldots$" or "$f(1), f(2), \ldots$"? I am coming round to the idea that the second is infinitely superior. With the first, the letter 'x' does nothing. The second notation also makes it much easier to deal with subsequences. I think i'll try to remove all subscripted-sequence-talk as i process this document.

Preface

This book has grown out of lectures for a graduate ("Part III") course entitled 'Logic and Combinatorics' given at the University of Cambridge. Since motivations for interest in WQO theory are various I should perhaps explain that that course arose from my desire to show to my students a beautiful result of Harvey Friedman's which goes some way to explaining why there should be courses with this title in the first place. That result is "FFF": Friedman's Finite Form—of Kruskal's theorem on the wellquasiorderings of trees. Logicians have known ever since the days of Gödel's Incompleteness theorem that for any axiomatic system of arithmetic there are logically simple assertions of arithmetic not provable in that system, but until the advent of FFF no examples were known that were mathematically natural. FFF arguably still remains the most natural and pleasing example of such a formula. (The closest competition, the Paris-Harrington formula, was also in that course, but did not make it into this book because it doesn't involve WQOs or BQOs). I still remember the talk where i first heard it, given by my Doktorvater, Adrian Mathias, in the early 1980's.

Others will have different reasons for interest. Theoretical computer scientists are interested in WQO and BQO theory because it underpins their craft of proving termination of algorithms. (Indeed my sole original contribution to BQO theory appeared in the Journal of Theoretical Computer Science.) Finally, people interested in descriptive set theory will have had their attention drawn to BQO theory by the important and influential work of Steve Simpson, showing that descriptive set theory provides a smooth and illuminating treatment of BQOs.

This book came to be written in a way that I suspect many textbooks are written. It is the book that I wish I had had when i embarked on my attempt to understand BQO theory. I do not claim to be an expert on BQO theory, and this book is not the definitive pronouncement of a master, but rather the log of the labours of a journeyman with the false starts and fruitless errors removed, and offered to the public as such, in the hope that it may be useful in the way that expert texts are not.

Being by nature a lazy reader, I have worked most of this out for myself, with the clues I could find in the literature. I have cited everything I have read, and a great deal that I haven't. Most of the proofs I supply are proofs I found myself, and altho' I have given credit to other authors where I know it to be due, I make no claims of priority for unattributed results. worked exercises burble.

It is a pleasure to be able to thank my long-suffering correspondents Steve Simpson, Richard Laver and Alberto Marcone, who patiently and courteously answered the questions—many of them no doubt quite daft—with which I plied them during my attempts to teach myself this material. One of my reasons for writing this book is to ensure, by setting down in writing (some of) what I have learned from them, that they are in future slightly safer from the prospect of importunate correspondents like me.

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Background

More chat along the following lines....

Computer scientists do not need to be reminded of the importance of wellfounded relations in their subject: their utility in proofs of termination is enough by itself to compel attention. The typical way for a wellfounded relation to arise is from declarations of recursive datatypes, but some seem to have different roots, and an important class of relations that can be wellfounded (or have natural wellfounded parts) is the class of wellquasiorders, and a special subclass of that family is the class of better quasiorders.

miniexercises

The original audiences for the lectures on which this book is based were fourth year cohorts that had in their third year been subjected to the lectures on which [16] were based, so it is hardly surprising that that book provides the requisite logical background

1.1 Definitions and Notation

The larger the number of mathematical *groupuscules* that have reason for being interested in a topic, the greater is the need for an introductory text on that topic. By the same token, sadly, the greater will be the divergence of notations that students bring with them to the endeavour of reading such a book.

Combinatorists, proof theorists, set theorists and theoretical computer scientists all have reasons for being interested in BQOs, and they have different notations. I have had to make choices about notation but I have tried to remain impartial in other ways. One way of preserving an air of impartiality is to ensure that applications and illustrations come from all areas equally. And one way of doing this—namely to provide no applications at all—recommends itself in other ways too. There is always a case, in introductory texts, for concentrating on ideas. This is particularly so in the case of WQO and BQO theory where the basic ideas are rebarbative to the point of prickliness. I have exploited illustrations only where they illuminate the underlying ideas, which—God knows—are quite hard enough for readers to grasp as it is, without their [anaphora] being hectored by their guide on how useful they are. They know that already.

Delicate balancing act: CS people are likely to prefer constructive arguments where these are available, and this means not only eschewing the axiom of choice but also excluded middle. In contrast combinatorists generally blithely assume the axiom of choice, even the uncountable axiom of choice. Descriptive set theorists are in the middle somewhere, assuming dependent choice always, but full AC (Zorn's lemma etc) only at times. The author is from none of these tribes, but is sympathetic to all three viewpoints. Although the axiom of choice generally looms large (even if unawoved, and in the background) in the proofs of the classical results from WQO and BQO theory, it is a deep and significant fact that most of the interesting mathematics to be had there takes place in countable structures, and that therefore only countable choice will be needed. At the time of writing I have no plans to make any use of uncountable choice at all. I shall follow my usual classroom practice of preferring AC-free proofs where they are available and flagging all uses of AC when I am forced to use it.

Definitional equality $=:$

A list is either the empty list (here written `nil`) or the result of `cons`-ing a head onto another list, and `cons`-ing is usually written in infix notation as $h{::}t$. Thus il $l = h{::}t$, h is the **head** of l (written $\mathtt{hd}(l)$) and t is the **tail** (written $\mathtt{tl}(l)$). Thus if $l = h{::}t$ is a list of widgets, h is a widget, and t is a list of widgets. This is a notation dervied from `ML`, which marks it very clearly as a notation from CS not combinatorics, but we have to call it something. When writing out lists in detail, we use square brackets and semicolons. For example $\mathtt{hd}([5;4;7]) = 5$, and $\mathtt{tl}([5;4;7]) = [4;7]$.

$\mathtt{len}(l)$ is the length of the list l.

$\mathtt{butlast}(l)$ is the list l minus $\mathtt{last}(l)$, its last element.

A stream is an infinite list. Lists can be thought of as finite sequences, and streams as infinite (ω-) sequences.

[*HOLE must decide what do do about consing stuff on the ENDS of lists... SNOC???*]

Lists and streams may have multiple occurrences of individual items, and the number and location of these multiple occurrences matter: $[1;1;1] \neq [1;1]$ and $[2;1] \neq [1;2]$. Multisets are like lists in that the *number* of occurrences of an item matters but unlike them in that the *location* doesn't. ("Throw away the order information"). What is left is **multiplicity** information: the number of times an element appears. The general theory of multisets is obscure[1] and there is not even a generally agreed notation. Fortunately all our multisets have "finite multiplicity".

The square brackets will be overloaded, since we will still use them, as is traditional, for denoting closed intervals in total orderings. [1,n] is $\{1, 2, \ldots n\}$.

lambda notation?

\

colex ordering [*HOLE might have to be careful here, and spell things out. When comparing to w things under the colex oprdering we compare *last* elements first, then look at penultimate elements, then antepenultimate and so on. In fact just like lex only doing it right-to-left instead of left-to-right.*]

A binary structure $\langle X, R\rangle$ consists of a **carrier set**, X, associated with a relation. Given two binary structures $\langle A, R\rangle$ and $\langle B, S\rangle$, we say $\langle B, S\rangle$ is an **end-extension of** $\langle A, R\rangle$ if $A \subseteq B$ and $R \subseteq S$, and whenever $y \in A$ and xSy then $x \in A$ too.

Partitions for us are not the same as partitions in number theory. A partition of a set X is a set is a set $\Pi \subseteq \mathcal{P}(X)$ s.t. $\bigcup \Pi = X$ and $(\forall \pi_1, \pi_2 \in \Pi)(\pi_1 \cap \pi_2 = \emptyset)$. The members of Π are **pieces**.

If X is a set, $\mathcal{P}(X)$ is the power set of X; $[X]^n$ is the set of unordered n-tuples from X. $[X]^{<n}$ is $\bigcup\limits_{j<n} [X]^j$. (Thus $X^{<\omega}$ is the set of finite lists of members of X). [*HOLE the exponent can be infinite: is it a cardinal or an ordinal??*]

$\{i < j < k\}$ will be the triple $\{i, j, k\}$ accompanied by the information that $i < j < k$,

Here n is a cardinal not an ordinal, and where infinite cardinals are involved we will sometimes write '$\mathcal{P}_\kappa(X)$' rather than '$[X]^{<\kappa}$'. X^n is of course the set of *ordered* n-tuples from X and this notation is extended transfinitely, so that X^α is the set of α-sequences from X. (Contrast $X^{<\omega}$ with $X^{<\aleph_0}$: the first superscript is an ordinal so the whole expression denotes the set of finite sequences, whereas the second is a

[1] For example it is not generally agreed what the axioms should be of a multiset version of ZF

cardinal so we get the set of finite subsets!) Beware! Some people will write 'X^ω' when they intend to denote the set I would denote by '$[X]^{\aleph_0}$'. Prömel and Voigt **??** write $\binom{X}{\omega}$ for the set of infinite subsets of X. 'X^Y' and '$Y \to X$' both denote the set of all functions from Y to X. There are three very good reasons for preferring the second notation (i) the first notation overloads the exponential; (ii) the second notation makes it typographically easy to iterate (try writing '$(A \to B) \to ((B \to C) \to (A \to C))$' in the first style!); (iii) it reflects typographically the **Curry-Howard correspondence**.

Expain arrow notation for Ramsey theory here. We won't prove Ramsey's theorem here. we will speak of 'monochromatic' sets not 'homogeneous' sets. ('Homogeneous' is too overloaded already)

Quasiorders and posets

A binary relation that is transitive and reflexive will here be called a **quasiorder**. The expressions "quasi-order", "quasiordering" and "preorder" are also to be seen in the literature. A **partial order** is a quasiorder $\leq$ that is **antisymmetric**: that is to say, $(\forall xy)(x \leq y \wedge y \leq x \to x = y)$.

The intersection of a quasiorder with its converse is an equivalence relation and will be called the **corresponding** equivalence relation. The quotient is a partial order and will be called the **corresponding** partial order. Notice that equivalence relations are quasiorders: in particular the identity relation on any set is a quasiorder. However no nontrivial equivalence relation can be a *partial order*.

Although to most readers the symbol '$\leq$' probably connotes a partial order we will here use it for quasiorders as well. The reader should make a mental note not to assume antisymmetry!. The relation $x \leq y \not\leq x$ is **the strict part** of $\leq$ and will be written '$x < y$'. (That is to say, delete one line from the pair of diagonal lines in the symbol used to denote a quasiorder to obtain a notation for the strict part of that relation.) In what follows we will use '$<$' for both the (strict part of the) quasiorder relation *and* inequality on $\mathbb{N}$. The reader is warned! We will write '$\geq$' and '$>$' for the converses of the relations denoted by '$\leq$' and '$<$' without further comment.

A **strict partial order** is a relation that is transitive and irreflexive. Alternatively a strict partial order is $(R\setminus$ identity$)$ $|$ domain(R) where R is a partial order, that is to say, the strict part of a partial order. Indeed if $<$ is a strict partial order then there is a unique partial order of which it is the strict part: if R is a partial order, with $R \setminus R^{-1}$ the strict part of it, then one can recover R from $R \setminus R^{-1}$ by unioning it with the identity relation.

With quasiorders—but not with partial orders—one needs to distinguish **indifference** ($p \leq q \leq p$) from **indecision** ($p \not\leq q \not\leq p$). I've actually just made this notation up but it might come in useful. [*HOLE How old is this joke?*] Notice also that the neat 1-1 correspondence between partial orders and strict partial orders breaks down in the case of quasiorders in general: if a quasiorder believes $a \leq b$ and $b \leq a$, then the strict part believes that $a \not< b$ and $b \not< a$. But it would also believe $a \not< b$ and $b \not< a$ if the original quasiorder had had $a \not\leq b$ and $b \not\leq a$, so in the general case of quasiorders information gets lost in the passage from quasiorders to their strict part. If we take the strict part of a quasiorder and attempt to recover the original quasiorder by forming the union with the identity relation we cannot expect to get back the original quasiorder. To take an extreme example: equivalence relations are quasiorders, and if we perform this two-step process to an equivalence relation then all we get back is the identity relation.

A realistic example of a quasi-order that is not a partial order but is nevertheless natural is the relation $|A| \leq |B|$ between cardinals ("A is empty or there is a surjection from B onto A"). A further natural example is provided by the embedding quasiorder between linear order types. This is a quasiorder but is not a partial order: consider the half-open intervals $(0, 1]$ and $[0, 1)$. Each embeds in the other, but they are not isomorphic. One does not want to study these order types through the corresponding equivalence relation, since it would identify these two significantly distinct order types and thereby expunge too much structure.

Two further examples of quasiorders that are not obviously antisymmetrical are the graph minor and subgraph relations on the class of infinite graphs.)

Directed posets and total orderings

A relation R is **connected** if $(\forall x, y)(R(x, y) \vee R(y, x))$. The abstract noun is **connexity**. There is no special designation for quasiorders that are connected, but a partial order that is connected is said to be a **total** or **linear** order.

If $\langle X, \leq_X\rangle$ is a poset, a subset $X' \subseteq X$ of X is a **directed** subset if $(\forall x_1 x_2 \in X')(\exists x_3 \in X')(x_1 \leq_X x_3 \wedge x_2 \leq_X x_3)$. A poset $\langle X, \leq_X\rangle$ is a directed poset if X itself is directed. (miniexercise: $\langle X, \leq_X\rangle$ is a total order iff every subset of X is directed).

In a directed poset all pairs of elements have sups. What is a poset in which pairs of elements never have a sup? Two comparable elements in a poset always have a sup—trivially: it will be one or the other, so our question should be: what about posets where a pair of incomparable elements never have a sup? Such a poset is called a tree, (at least if it has a bottom element, and sometimes even if it doesn't. Trees without bottom elements are sometimes called *forests* though we will not be using that terminology here). The bottom element is a **root**. Such trees are *upward-branching* trees. There are also *downward-branching* trees. We will encounter both.

Trees can also be thought of as digraphs, particularly if they have only countably many vertices, and edges of finite degree. However, for us, trees (be they upward or downward branching) will usually be posets.

Complete posets and fixed point theorems

$\langle Q, \leq_Q \rangle$ is a **complete partial order** ('CPO' for short) if every subset has a least upper bound. $\langle Q, \leq_Q \rangle$ is a **chain-complete poset** if every chain has a least upper bound. An antichain (in a poset) is set of elements no two of which are $\leq$-comparable.

A function $f : X \to Y$ is a **monotone** map from the quasiorder $\langle X, \leq_X \rangle$ to the quasiorder $\langle Y, \leq_Y \rangle$ if $(\forall x, x' \in X)(x \leq_X x' \to f(x) \leq_Y f(x'))$

The difference between CPOs and chain-complete posets will matter to us. The collection of quasiorders on a fixed arbitrary set is a complete poset under inclusion (thinking of a quasiorder as a set of ordered pairs and ordering those sets by inclusion) whereas the collection of partial orders on a fixed arbitrary set is merely chain-complete since a pair of partial orders that disagree has no upper bound.

2

Other background: Ramsey theory, Wellfoundedness, ordinals etc

[*HOLE If we assume DC and KL from the start, we can use the descending-chain account of wellfoundedness throughout. That makes the excluded substructure story nicer. We do need to say a little bit about induction and recursion on rectypes beco's of trees lists and Kruskal. Perhaps we can just say that the reader is assumed to be familair with structural induction? No. There is an issue about how much detail we can go into concerning induction and rectypes.*]

[*HOLE This may be the point to talk about excluded substructre characterisations*]

μ for minimum.

We will need to know some fixed point theorems: every monotone function from a complete poset into itself has a fixed point (Tarski-Knaster), and every inflationary function from a chain-complete poset into itself has a fixed point (Witt) However we do not need to know the proofs.

Explain pre-fixed points and post-fixed points.

People from a combinatorial tradition will be familiar with the practice of mathematical induction, but perhaps less familiar with the general idea of which it is a special case: structural induction.

Only theoretical computer scientists are familiar with coinduction.

State and Prove T-K and Witt. (Poss make connection with QO of linear orders unde embedding? Cococo theorem?)

Structural induction is associated with inductively defined sets. An inductively defined set is the $\subseteq$-least set containing certain things and closed under certain operations. That is to say, it is the least fixed point (in the chain-complete poset of all subsets of the universe) for the function that takes a set X and adds to X the result of doing each operation once to everything in X. This gives rise to a principle of structural induction.

Now what about the greatest fixed point? Normally any proof or construction applicable to a least fixed point can be dualised to obtain a construction or proof pertaining to a greatest fixed point. So what happens to the principle of structural induction?

A: we get coinduction.

[*HOLE explain coinduction here*]

The collection of X-lists (lists whose elements are Xs) is inductively defined, and is a least fixed point: it is the $\subseteq$-least class containing the empty list and containing x::l whenever (i) it contains l and (ii) $x \in X$.

The collection of X-*streams* is *co*-inductively defined, and is a greatest fixed point: it is the $\subseteq$-greatest class Y such that every member of Y is of the form x::l where (i) $l \in Y$ and (ii) $x \in X$.

[*HOLE someone will surely ask why this gives us ω-sequecnes not $\omega^* + \omega$-sequences.*]

For any set X we have the concepts of finite and infinite trees over X. The trees in this section will be **upward-branching**, and all of them can also be thought of as digraphs with labelled vertices. Finite trees resemble lists and form a recursive datatype: infinite trees resemble streams and form a co-recursive datatype.

First we have a definition of a recursive datatype of Q-trees. A Q-tree is either (i) a singleton from Q; or (ii) an ordered pair of an element of Q with a list of Q-trees.

'$\mathtt{rt}(t)$' will denote the root of t—the label of the bottom vertex. The subtree corresponding to a child of the root will be called a **child** of the tree (inversely a **parent**). The *list* of children of a tree t will be $\mathtt{children}(t)$. By abuse of notation this will also denote the *set* of children of t.

Thus one would write:

$\mathtt{rt}$(tree(q,treelist) =: q; and $\mathtt{children}$(tree(q, treelist)) =:treelist

Correspondingly the corecursive datatype of infinite Q-trees is the $\subseteq$-largest collection Y such that everything in Y has a root in Q and a stream of children that are all in Y.

[*HOLE more detail?*]

The children could form a list or a stream!

LEMMA 1 *Let $\langle D, \leq_D \rangle$ be a wellfounded directed poset, and $\{\langle A_d, R_d \rangle : d \in D\}$ a family of wellfounded structures, with, for each $d, d' \in D$ a* *map* $i_{d,d'} : \langle A_d, R_d \rangle \hookrightarrow \langle A_{d'}, R_{d'} \rangle$ *making* $\langle A_{d'}, R_{d'} \rangle$ *isomorphic to en end-extension of* $\langle A_d, R_d \rangle$*, such that the* $i_{d,d'}$*s commute, then the direct limit* $\langle A_D, R_D \rangle$ *is also wellfounded.*

"A direct limit of wellfounded structures under end-extension is wellfounded"

Proof:

[*HOLE Supply proof*]

(I don't think we need D to be wellfounded. Consider a family of structure with increasing domain but empty relation indexed by the rationals.)

(see page 23.)

Wellorderings and ordinals. Explain α^* notation for reversals of wellorderings.

Must define prewellordering, just to get things straight

A **prewellordering** is a wellfounded transitive relation whose complement is transitive and connected (or—less quixotically—a wellordered partition).

We will continue with the widespread bad habit of referring to a reflexive relation as wellfounded when we really mean that its strict part is wellfounded (when this is obvious from context). Notice that altho' adding ordered pairs to a wellfounded relation does not normally give you a wellfounded relation, adding ordered pairs to a "wellfounded" quasiorder to obtain a new quasiorder does preserve wellfoundedness.

Exercise 1

Product of finitely many wellfounded relations is wellfounded, but give an example to show that the set of finite strings over an ordered alphabet, ordered lexicographically, is not wellfounded.

Wellfounded part of a relation. (needed for Buchholtz's proof)

Explain how to extract a long wellordering from a wellfounded relation.

It is customary to capture the idea of ranks of wellfounded relations by means of recursively defined maps onto the ordinals. This uses Mostowski collapse and replacement. The desire to have a presentation of these ideas that doesn't use complex set-theoretic notions is not by itself a sufficient reason for seeking a presentation using simpler machinery. However, in this context we will be trying to make sense of allegations that are commonly made in slangy shorthand as "If Kruskal's theorem is true, then the ordinals below Γ_0 are wellfounded". Γ_0 is an ordinal, as the reader will discover. And are not the ordinals below any ordinal wellfounded? Well, yes. What is really being claimed is that if Kruskal's theorem holds then there is a definable wellordering which can be proved to be of length at least Γ_0.[1]

2.0.1 defining rank from first principles

Let R be a wellfounded relation. Quasiorder the carrier set of R by the following recursion:

$x \leq y$ if R^{-1}"$\{x\} \leq^+ R^{-1}$"$\{y\}$

[*HOLE + not defined yet. Do we need to explain* R^{-1}"$\{x\}$]

We will establish that $\leq$ is (i) reflexive, (ii) transitive, (iii) connected and (iv) wellfounded.

(i) Evidently $\leq$ is reflexive.

(ii) We prove that it is transitive by R-induction. Let x be R-minimal so that there are y and z such that $x \leq y$ and $y \leq z$ but $x \not\leq z$. Having picked such an x, pick y R-minimal so that $x \leq y$ and there is a z such that $y \leq z$ but $x \not\leq z$, and having picked such a y pick z R-minimal so that $x \leq y$ and $y \leq z$ but $x \not\leq z$.

Because $x \leq y$ it follows that for every $x' \in R^{-1}$"$\{x\}$ there is $y' \in y$ with $x' \leq y'$. Similarly for every $y' \in R^{-1}$"$\{y\}$ there is $z' \in z$ with $y' \leq z'$. But by minimality of x y and z as counterexamples to transitivity we must have that for all such x' there is a suitable z' so that $x' \leq z'$, which is to say $x \leq z$, and x, y and z are not counterexamples after all.

(iii) We prove that it is connected by R-induction. Suppose $(\forall y)(R(yx) \to (\forall z)(y \leq z \vee z \leq y))$. We will prove $(\forall w)(x \leq w \vee w \leq x)$. Consider R^{-1}"$\{x\}$ and R^{-1}"$\{w\}$. By induction everything in R^{-1}"$\{x\}$ is comparable with everything, and so, in particular, with everything in R^{-1}"$\{w\}$. S if $x \not\leq w$ there is $y \in R^{-1}$"$\{x\}$ s.t. $y \not\leq z$ for any $z \in R^{-1}$"$\{w\}$. But then, by comparability, we have $(\forall z \in R^{-1}$"$\{w\})(z \leq y)$ so $w \leq x$.

(iv) To prove that $\leq$ is wellfounded we prove by induction on R that $(\forall Y)(\forall x' \leq x)(x' \in Y \to (\exists y \in Y)(\forall z \in Y)(y \leq z))$.

[1] The significance of this fact is that constructions which need wellorderings of such extreme length for their successful recursive declaration can actually be carried out. One such construction is the elimination of cuts from proofs in certain formulations of elementary arithmetic. This enables us to prove the consistency of arithmetic, and thereby establish (by appeal to the second incompleteness theorem) that Kruskal's theorem (or did i mean FFF?) is not provable in elementary arithmetic.

Suppose this is true for all $w \in R^{-1}``\{x\}$. We want to show that any set Y containing anything $\leq x$ has a $\leq$-minimal element. If Y contains something $\leq$ any R-predecessor of x then Y has a minimal element by induction hypothesis, so we need only consider the case where every member of Y is $>$ every R-predecessor of x. We want to infer from this that if $y \in Y$ then $x \leq y$, so that x is itself the minimal element that we seek.

For this we need a result to the effect that if $a >$ every R-predecessor of b, then $a \geq b$.

Let a be R-minimal so that there is $b < a$ with $b >$ all R-predecessors of a. Since $b < a$ it cannot be the case that every R-predecessor of $a \leq$ an R-predecessor of b, so there is a' an R-predecessor of a which $>$ every R-predecessor of b and yet is $< b$. We repeat the construction to obtain $b' < a'$ with $b' >$ all R-predecessors of a. But now a' is not R-minimal.

Since $\leq$ is wellfounded and connected quasiorder, the corresponding partial order is a wellorder.

We comment, without proof, that this construction can be executed in very weak systems of set theory using only Δ_0 separation and no replacement, such as the systems Mac and KF.

The rank preorder is the least fixed point. What is the greatest fixed point.

3

Quasiorders

3.1 Lifts

A quasiorder on a set X can give rise to quasiorders on $\mathcal{P}(X)$, on the set of partitions of X, on the set of α-sequences af members of X, multisets of members of X, trees decorated with members of X and so on. Let us use the word 'lift' to describe constructors that take a quasiorder of a set X and return a quasiorder on one of these sets derived somehow from X. A natural general question to ask about a given lift is what properties of its argument are also posessed by the value? If we feed a connected (say) quasiorder to our lift L, do we get back a connected quasiorder on the set derived from X? If it does we say that L **preserves** connexity. We might expect there to be nontrivial relations of the following kind. A lift L preserves F-ness at least for arguments that are also G. In this way *clusters* of properties can emerge as important: a lift L might preserve neither F-ness nor G-ness on their own, but might preserve F-ness-and-G-ness. This makes the property of being F-and-G an interesting cluster of properties.

(See Marcone[2001] for a discussion of lifts).[*HOLE perhaps mine it for material*]

First come three ways to lift a quasiorder from a set to its power set.

Definition 2

Let $\langle Q, \leq, \rangle$ be a quasiorder. For Y and Z subsets of Q say

$Y \;\leq^+\; Z$ *iff* $(\forall y \in Y)(\exists z \in Z)(y \leq z)$;

$Y \;\leq^*\; Z$ *iff* $(\forall z \in Z)(\exists y \in Y)(y \leq z)$;

$Y \;\leq_{1-1}\; Z$ *iff there is an injection* $f : Y \hookrightarrow Z$ *such that* $(\forall y \in Y)(y \leq f(y))$.

Among computer scientists the first is commonly known as the **Hoare** powerdomain construction. The second is the dual construction called the **Smyth** powerdomain construction.

[*HOLE Find a suitable place for this observation [HOLE Barwise approximants here??]*

Notice that for any quasiorder $\langle Q, \leq \rangle$, the quasiorder $\langle \mathcal{P}(Q), \leq^+ \rangle$ is what one might call a **complete quasiorder**, *which is to say that every subset of $\mathcal{P}(Q)$ has suprema and infima that are unique up to the equivalence relation corresponding to $\leq^+$. However it turns out that this is not the notion of complete quasiorder that we need.*]

The operations in definition 2 are all monotone operators on the CPO of all quasiorders of V.

Exercise 2 *Characterise the least and greatest fixed points of +.*

[*HOLE So far we've used + to talk about relations on a fixed set, by "pulling down". Do we want to talk about $V(X)$ *here*?*]

3.2 Lifts to other structures

Power set is not the only constructor we want to lift quasiorders to. We can also define lifts from quasiorders on X to quasiorders on—for example—the set of partitions of X. This doesn't seem to have attracted much interest: X^α seems to have caught the public imagination rather better. A general definition which encompasses this is as follows:

Definition 3 *Let $\langle Q, \leq_Q \rangle$ be a quasiorder, and $\langle I, \leq_I \rangle$ be an (ordered) index set. If f and g are two elements of $I \to Q$ we say $f \leq_I g$ iff there is an order-preserving map $h : I \to I$ s.t.* $(\forall i)(f(i) \leq_Q g(h(i)))$

Notice that we don't ask for h to be injective or anything fancy, so this is just the same as saying $f \leq_I g$ iff $(\forall i \in I)(\exists j \in I)(f(i) \leq g(j))$.

This perhaps reads like a gratuitously complicated way of introducing a way of lifting quasiorders on X to quasiorders of X^α (with α an ordinal) but the extra generality will be needed. For one thing, the definition it gives us of a quasiorder on Q^ω given one on Q can be tweaked into one on $Q^{<\omega}$, and this we will need later. For another, one can take the index set to be a quasiorder (specifically a WQO or even a BQO) and lemmas using this construction are used in Laver's proof of Fraïssé 's conjecture in section 6.1.

3.3 Lists and streams

The case that will be of most concern to us is where I is an initial segment of $\mathbb{N}$ and where we require f to be **injective**. In this case the relation given by definition 3 is written $\leq_l$. We met ω-sequences from Q on page 6 where they were called Q-**streams**, and finite sequences similarly were Q-**lists**. I shall write the set of Q-streams as Q^ω and the set of Q-lists as $Q^{<\omega}$. Following Mathias (oral tradition) we use the word **stretching** to denote the relation that holds between two Q-lists (or Q-streams) l_1 and l_2 if there is a 1-1 increasing map f from the addresses of l_1 to the addresses of l_2 such that for all addresses a, $a \leq f(a)$. We write this '$l_1 \leq_l l_2$' with a subscript 'l' for '$l$ist', and we say l_1 **stretches** into l_2.

As well as the direct definition of stretching for lists and streams given by definition 3 there are inductive and coinductive definitions.

The stretching relation on Q-lists is inductively defined as the $\subseteq$-smallest set of ordered pairs of Q-lists containing $\langle \texttt{nil}, \texttt{nil} \rangle$ and containing $\langle l_1, l_2 \rangle$ if it contains $\langle l_1, \texttt{tl}(l_2) \rangle$, or if $\texttt{hd}(l_1) \leq \texttt{hd}(l_2)$ and it contains $\langle \texttt{tl}(l_1), \texttt{tl}(l_2) \rangle$.

The stretching relation on Q-streams is coinductively defined as the $\subseteq$-largest relation $R \subseteq Q^\omega \times Q^\omega$ such that $R(l_1, l_2) \longleftrightarrow ((\texttt{hd}(l_1) \leq_Q \texttt{hd}(l_2) \wedge R(\texttt{tl}(l_1), \texttt{tl}(l_2))) \vee R(l_1, \texttt{tl}(l_2)))$.

EXERCISE 3 *Prove that the recursive definition of stretching for lists is equivalent to the direct definition of definition 3.*

Prove that the coinductive definition of stretching for streams is equivalent to the direct definition of definition 3. [HOLE flag injectivity somehow]

(I mentioned earlier (definition 2) a way of engendering a quasiorder on $\mathcal{P}(X)$ given a quasiorder on X, a way that looks for injections from one subset of X to another. This was notated '$\leq_{1-1}$'. The list-embedding we have just introduced is notated '$\leq_l$' and this might be thought to be confusing, but the list version is the obvious transformation of the set-version to the sequence version, and they could be thought of as the same operation.)

No list stretches into its tail: for all lists l over a quasiorder, $l >_l \texttt{tl}(l)$. $l \geq_l \texttt{tl}(l)$ by the second clause in the recursive definition of $\leq_l$, and we prove by induction that no list can $\leq_l$ something strictly shorter than itself. In contrast to lists, a stream might stretch into its tail.

3.3.1 Multisets

Multisets. In fact we will consider only finite multisets here (i think!) The **multiset ordering** of finite multisets of a quasiordered set Q is defined as follows $X \leq_m Y$ iff there is a finite sequence $X = X_0, X_1, X_2, \ldots X_n = Y$ where for each n, X_{n+1} is obtained from X_n by replacing finitely many $a_1 \ldots a_k$ by a single b such that $b >$ each a_i.

THEOREM 4 *If $\leq$ is a wellfounded quasiorder of a set X, then the multiset order on the set of finite multisubsets of X is wellfounded.*

Proof:

(This proof is due to Wilfried Buchholz and is included with his permission.)

`http://www.cis.upenn.edu/~bcpierce/types/archives/current/msg00032.html`

[*HOLE can we get any mileage out of Boffa's representation of hereditarily finite multisets? Ordering them by the multiset ordering make it total or something? Compare with the ordering on V_ω by iterating $+$?*]

EXERCISE 4 *Think about the ordinals below ϵ_0, the smallest α such that $\alpha = \omega^\alpha$. By the Cantor normal form theorem every ordinal α below ϵ_0 is a finite sum $\omega^{\alpha_1} + \omega^{\alpha_2} + \omega^{\alpha_3} \ldots$ where the exponents are nonincreasing. That is to say, α codes the multiset $[\alpha_1, \alpha_2 \ldots]$. Every ordinal below ϵ_0 codes a unique multiset of other—smaller!—ordinals below ϵ_0.*

[*HOLE So do we prove CNF in this book or not?*]

3.4 Finite and infinite trees

There are also finite and infinite trees over X, and we need to think how to lift quasiorders of X to trees over X.

There are (at least) two ways of thinking of trees as mathematical objects. (i) A tree can be a special kind of poset, and (ii) a tree can be a special kind of graph (a thing with vertices and edges). Labelled trees can be thought of as naked trees (be they posets or graphs) eqipped with maps to a set of labels. However they can also be thought of as a distinctive kind of mathematical entity as in. [*HOLE as in what??*]

But apart from these questions of *how* we are to think of trees, there are also different kinds of trees. Is the set of children ("litter") of a node equipped with an ordering or not? If littters are ordered, then morphisms between trees must respect the ordering of each litter, and positive results about the existence of embeddings become harder to prove. We will consider both kinds of tree, and where there is a choice about which flavour of tree to assert the result for, we will assert and prove the harder version.

and it is common to equivocate between them.

[*HOLE With these trees the collection of children of a node is a LIST of trees not a SET of trees. We do really need to sort this out. Do we want to consider both sorts of tree?? What worried me is that if the direct definition and the (co)recursive definition are the same, we *must* be dealing with the set version not the list version.*]

There is a fairly close parallel between lists/streams and finite/infinite trees. There is a tree-embedding which corresponds to stretching, and, like stretching, it has both a direct definition analogous to definition 3 and an inductive (definition (for finite trees) or coinductive definition (for infinite trees).

First the direct definition analogous to definition 3 of stretching.

If Q is a quasiorder, a Q-tree can be thought of as a kind of lower-semilattice where no two incomparable points have a common upper bound, decorated with labels from Q. Think of the-thing-that-carries-the-decorations as the **skeleton** of the tree.

Definition 5 *Say $T_i \leq_t T_j$ if there is an* **injective** *lower-semilattice homomorphism f from the skeleton of T_i to the skeleton of T_j such that the label at any node n of T_i is $\leq_Q$ the label at node $f(n)$ of T_j.*

[*HOLE Tidy this up: It would be nice if these two definitions 6 and 5 were to agree on the recursive datatype of definition 2. However this isn't quite true: trees as posets have no left-to-right structure of the kind exploited in the definition in definition 6. If we changed the declaration of the rectype of Q-trees by replacing 'list' by 'multiset' and 'stretching' by 1-1-embedding then the two would be equivalent. I won't bother with the details beco's (i) the definition using lists is stronger (few embeddings between trees) and (ii) We can prove that the stronger version is wellfounded (and WQO, BQO as appropriate, see later) and we don't bother with proving weak results when we can prove strong ones.*]

Definition 5 makes sense for infinite trees as well.

There is of course an inductive definition for tree-stretching for finite trees.

Definition 6 *$T_a \leq_t T_b$ if*

- *Both are singleton trees* $\{a\}$ *and* $\{b\}$ *with* $a \leq b$*; or*
- $T_a \leq_t$ *some child of* T_b*; or*
- *The root of* $T_a \leq$ *root of* T_b *and the list of children of* $T_a \leq_l$ *list of children of* T_b.

The list constructor has finite character. We shall see later that it preserves wellfoundedness but we can see already that it does not preserve connexity: $\langle \mathbb{N}, \leq \rangle$ is a total ordering but neither of of the two-membered lists $[1, 2]$ and $[2, 1]$ stretch into the other. Nor do either of the two streams $\langle 1, 1, 1, \ldots \rangle$ and $\langle 2, 0, 0, 0, 0, \ldots \rangle$ stretch into the other. The three constructors of definition 2 preserve reflexivity and transitivity and thus lift quasiorders to quasiorders. The $\leq^+$ and $\leq^*$ constructors additionally preserve connexity but $\leq_{1-1}$ does not: consider the natural numbers in their usual (connected!) ordering. $\{1, 2, 3\} \not\leq_{1-1} \{4, 5\} \not\leq_{1-1} \{1, 2, 3\}$. It is usually fairly straightforward to check whether or not a lift preserves symmetry, transitivity or connexity. It can be much harder to check whether or not a lift preserves wellfoundedness. It turns out that there is a connection between that question and the idea of finite character, and it is to this that we now turn.

3.5 Finite character

Although this expression is a piece of mathematical slang, it is at least *mathematical* slang, in the sense that the phenomenon it denotes plays a rôle in mathematics. Being an incompletely formalised notion, it is best illuminated by illustration. The following lifts (among others) have finite character:

(i) $Q \mapsto Q$-lists under stretching;
(ii) $Q \mapsto$ finite Q-trees under $\leq_t$;
(iii) $Q \mapsto \langle \mathcal{P}_{\aleph_0}(Q), \leq^* \rangle$;
(iv) $Q \mapsto \langle \mathcal{P}_{\aleph_0}(Q), \leq^+ \rangle$.
(v) The Lexicographic path ordering
(vi) The recursive path ordering

The idea is that lifts of finite character preserve wellfoundedness, in contrast to lifts of infinite character which tend not to. The claims that the lifts listed above preserve wellfoundedness range from easy-and-obvious to downright false.

Proposition 7 *If $\langle Q, \leq \rangle$ is a wellfounded quasiorder, then Q-lists are wellfounded under stretching.*

Proof: Suppose not, and we had an infinite descending sequence of Q-lists under stretching. They can get shorter only finitely often, so without loss of generality we may assume that they are all the same length. But the entries at each coefficient can get smaller only finitely often, so they must eventually be constant. ■

Proposition 8 *If $\langle Q, \leq \rangle$ is a wellfounded quasiorder, then finite Q-trees are wellfounded under tree-embedding.*

Proof: Suppose $\langle Q, \leq \rangle$ is a wellfounded quasiorder and let $\langle t_i : i < \omega \rangle$ be a descending $>_t$-sequence of Q-trees. We will derive a contradiction. The number of children of t_i is a nonincreasing function of i and must be eventually constant: indeed the trees will be of eventually constant shape, and we can delete the initial segment of the sequence where they are settling down. Because the shape is eventually constant there are unique maps at each stage, so for any one address the sequence of elements appearing at that address gets smaller as i gets bigger. ■

Proposition 9 $\langle \mathcal{P}_{\aleph_0}(Q), \leq^* \rangle$ *is not always wellfounded even if $\langle Q, \leq \rangle$ is a wellfounded quasiorder.*

Proof: Consider the identity quasiorder on $\mathbb{N}$. Then if we set $Q_i =: [1, i]$ we find that $Q_i >^* Q_{i+1}$ for all i. ■

Proposition 10 *If $\langle Q, \leq \rangle$ is a wellfounded quasiorder then $\langle \mathcal{P}_{\aleph_0}(Q), \leq^+ \rangle$ is wellfounded.*

Proof: Suppose we have an infinitely descending sequence $\langle Q_i : i \in \mathbb{N} \rangle$ of finite subsets of Q under $<^+$. Without loss of generality we can assume that all the Q_i are antichains, by throwing away from each Q_i all elements that are not maximal. This will ensure that any x that appears in both Q_i and in Q_j with $j > i$ must appear in all intermediate levels: if $x \in Q_j$ then it must be $\leq$ something in Q_{j-1} and so on up to Q_i. Since Q_i is an antichain this thing can only be x itself (or something equivalent to it, which will do!) So any x that appears in infinitely many Q_i must appear in cofinitely many of them. But then it can be deleted altogether. So we can assume that each q appears in at most finitely many Q_i.

For each $x \in Q_0$ we can build a tree whose paths are sequences s where the ith representative comes from Q_i and for all i, $s(i+1) \leq s(i)$. We need to show that all these paths are finite. If they were not, they would have to be eventually constant, and we have just seen that we can assume that each q can be assumed to appear only finitely often. So the tree whose paths are these sequences is a finite branching tree all of whose paths are finite, so it has only finitely many levels. But there are only finitely many things in Q_0, so eventually the Q_i are empty. ■

PROPOSITION 11 *Wellfoundedness of RPO*

PROPOSITION 12 *Wellfoundedness of LPO*

[*HOLE Prove here that $\leq_n$ is wellfounded if $\leq$ is. (I think i meant: the nth Barwise approximant)*]

EXERCISE 5 *We could try to prove that finite trees over a wellfounded quasiorder are wellfounded as follows. If S is wellfounded, so is $S^{<\omega}$. Therefore if Q and S are wellfounded quasiorders, then so is $Q \times S^{<\omega}$. So if Q is a wellfounded quasiorder so are all of $Q \times Q^{<\omega}$, $Q \times (Q \times Q^{<\omega})^{<\omega}$ and so on. Each embeds by end-extension into the next, so the direct limit (which is finite Q-trees) is also wellfounded, by lemma 1.*

What does this construction actually prove?

3.6 How do these operations affect rank?

Prove some theorems about lifts that preserve wellfoundedness, like tree-of-bad-sequence, etc. And find some exercises on rank.

4

WQOs

It seems that wellfounded quasiorders without infinite antichains are going to be objects of interest, since it seems that—and we will prove this in remark 19—it is the absence of infinite antichains in a wellfounded quasiorder $\langle Q, \leq \rangle$ that enables us to show that $\langle \mathcal{P}(Q), \leq^{+} \rangle$ is wellfounded.

This motivates the following definition:

DEFINITION 13 $\langle Q, \leq \rangle$ *is a* **wellquasiorder** *(hereafter "WQO") iff whenever* $\langle x_i : i \in \mathbb{N} \rangle$ *is an infinite sequence of elements from* Q *then there are* $i < j \in \mathbb{N}$ *s.t.* $x_i \leq x_j$.

DEFINITION 14 *A* **bad sequence** *(over* $\langle Q, \leq \rangle$*) is a sequence* $\langle x_i : i \in \mathbb{N} \rangle$ *such that for no* $i < j$ *is it the case that* $x_i \leq x_j$. *A sequence that is not bad is* **good**. *A sequence* $\langle x_n : n \in \mathbb{N} \rangle$ *is* **perfect** *if* $i \leq j \rightarrow x_i \leq x_j$.

Finite sequences $\langle x_i : i < k \in \mathbb{N} \rangle$ *too will sometimes be said to be* **bad** *as long as they satisfy the remaining condition:* $i < j < k \rightarrow x_i \not\leq x_j$.

Thus a wellquasiorder is a quasiorder with no bad sequences. With the help of Ramsey's theorem we can prove that in a WQO not only is every sequence good but that it must have a perfect subsequence. (Notice that this is not the same as saying that in any quasiorder every good sequence has a perfect subsequence!)

LEMMA 15 *In a WQO every sequence has a perfect subsequence.*

Proof: Although this theorem is very easy to prove, the usual (indeed only) proof using Ramsey's theorem is so natural and idiomatic, and so important *qua* prototype for so many other applications of Ramsey's theorem, that it is worth doing in full.

Let $\langle Q, \leq_Q \rangle$ be a WQO, and $f : \mathbb{N} \to Q$ a sequence. Partition $[\mathbb{N}]^2$ into the two pieces $\{\{i < j\} : f(i) \leq_Q f(j)\}$ and $\{\{i < j\} : f(i) \not\leq_Q f(j)\}$. An infinite subset monochromatic for the first piece would give us a bad sequence, contradicting the assumption that $\langle Q, \leq_Q \rangle$ was a WQO, and the set monochromatic for the second piece is a perfect subsequence. ■

A quibble: a set monochromatic for the first piece would be an infinite subset $X \subseteq \mathbb{N}$ such that whenever $i < j$, both in X, we have $f(i) \not\leq_Q f(j)$. Now this is not *literally* a bad sequence, since a bad sequence is *inter alia* a function defined on $\mathbb{N}$ not on an infinite subset of it. What we have just seen is a situation where we have to do a bit of renumbering of elements of an index set in order to make a claim literally true. This particular case is so trivial that one hardly notices one is doing it, and there is nothing to be gained at the time by flagging it, but we will find later examples where we really have to be explicit about it.

A quasiorder is a WQO iff the strict version of the corresponding partial order is wellfounded and has no infinite antichains. (miniexercise) Notice this does *not* mean that for each x in a WQO there are only finitely many things incomparable with x, nor even that there are only finitely many equivalence classes of things incomparable with x. What it does say is that if there are infinitely many things incomparable with x, some of them will be comparable with some others.

This gives rise to an excluded substructure characterisation of WQO: a quasiorder $\langle Q, \leq_Q \rangle$ is a WQO iff $\langle \mathbb{N}, <_{\mathbb{N}} \rangle$ cannot be embedded into the complement of $\langle Q, \leq_Q \rangle$ (the binary structure whose carrier set is Q and whose binary relation is $(Q \times Q) \setminus (\leq_Q)$).

A quasiorder $\langle Q, \leq \rangle$ has the **finite basis property** iff whenever $Q' \subseteq Q$ then there is a finite $Q'' \subseteq Q$ such that $(\forall x \in Q')(\exists y \in Q'')(y \leq x)$.

LEMMA 16 *WQOs have the finite basis property.*

Proof: If $\langle Q, \leq \rangle$ is WQO the relation $x \leq y \not\leq x$ is wellfounded, so if we want a finite basis for $Q' \subseteq Q$ consider the subset of Q' consisting of elements minimal under this relation. This set may be infinite of course, but we may consider its quotient under the corresponding equivalence relation and this will be finite. Pick one element from each equivalence class to obtain Q''. (There are only finitely many equivalence classes so we don't need AC) ■

The finite basis property of WQOs has been very significant in complexity theory. The existence of a finite basis can mean that there are only finitely many cases to check, and can result in the existence of algorithms for checking properties that one might *prima facie* expect not to be decidable. See [] on this.

The converse is of course true as well: a quasiorder with the finite basis property is a WQO. This is not the same as saying that every subset Q' has only finitely many minimal elements: it may have infinitely many, but they must belong to finitely many equivalence classes under the corresponding equivalence relation.

Now some basic facts about WQO's, some with an algebraic flavour.

PROPOSITION 17

(i) Substructures of WQOs are WQO;

(ii) Homomorphic images of WQOs are WQO;

(iii) The pointwise product of finitely many WQOs is WQO;

(iv) The intersection of finitely many WQOs is WQO;

(v) Disjoint unions of finitely many WQO are WQO.

(vi) If $\leq_1$ and $\leq_2$ are both quasiorders of a set Q, and the graph of $\leq_1$ is a subset of the graph of $\leq_2$, and $\leq_1$ is a WQO, then so is $\leq_2$.[HOLE explain overloading of 'graph']

Proof:

(i). Any bad sequence in a substructure is a bad sequence in the whole structure.

(ii). If $f : \langle Q, \leq\rangle \to \langle X, \leq\rangle$ is a quasiorder homomorphism then f^{-1}"S of a bad sequence S in X will be a bad sequence in Q.

For (iii) (iv) and (v) it is clearly sufficient to deal with the case of two WQOs. The proofs of all three use Ramsey's theorem with exponent 2, or the perfect subsequence lemma (lemma 15). For (iii) consider the product of two WQOs $\langle Q, \leq_Q\rangle$ and $\langle X, \leq_X\rangle$, and suppose we have a bad sequence $\langle\langle x_i, q_i\rangle : i \in \mathbb{N}\rangle$. By the perfect subsequence lemma there must be an infinite $I \subseteq \mathbb{N}$ such that for $i < j$ both in I we have $x_i \leq_X x_j$. Now consider the sequence of q_i for $i \in I$. This must be a good sequence, since $\langle Q, \leq_Q\rangle$ is WQO, so there are $i < j$ both in I with $q_i \leq_Q q_j$. So $\langle\langle x_i, q_i\rangle : i \in \mathbb{N}\rangle$ was not bad.

The proofs of (iv) and (v) are almost exactly the same.

Finally (vi) is obvious, but it will be generalised later so a bit of detail may be helpful. A quasiorder is a WQO if the complement of its graph does not contain a copy of $\langle\mathbb{N}, <_{\mathbb{N}}\rangle$. This property of (the graph of) a relation is clearly preserved under superset. ■

[*HOLE Talk here about excluded substructures?*]

(vi) has the rather bizarre consequence that the WQOs on a fixed domain form a filter in the complete lattice of quasiorders on that domain. (miniexercise) However the more immediate significance of (vi) is to be found in the greedy algorithm in the proof of proposition **??**. Let R_1 be $\{\langle 2n, 2n+1\rangle : n \in \mathbb{N}\})$ and R_2 be $\{\langle 2n+1, 2n+2\rangle : n \in \mathbb{N}\})$ Neither of these is WQO but their union is, so the filter is not prime. If R and S are quasiorders on a set X, define a quasiorder $R \leq S$ iff$_{df}$ $(\exists Q \subseteq X^2)(Q$ a WQO $R \cap Q \subseteq S$. The corresponding equivalence relation gives us a quotient.

However there are negative results as well.

Proposition 18 *The class of WQOs is not closed under direct limits or projective (inverse) limits.*

Proof:

(i) Direct limits. Consider the following sequence of Hasse diagrams of WQOs:

```
                                                .   x
                                                 \ /
                                .   x         .   .
                                 \ /           \ /
                   .   x      .   .         .   .
                    \ /        \ /           \ /
       .   x     .   .      .   .         .   .
        \ /       \ /        \ /           \ /
x        .         .          .             .
```

Each WQO in this this sequence is obtained from its neighbour to the left by replacing an 'x' by a

```
.   x
 \ /
  .
```

The direct limit is not a WQO. This example shows that closure under direct limits fails, even if we consider end-extensions only.

This is in contrast to the situation with wellfounded structures, the class of which *is* closed under direct limits of end-extensions, as we saw in lemma 1.

(ii) Inverse limits. Let A_n be $\{0, 1, \ldots n\}$ in their natural order, and

$\lambda m.(m \dot{-} 1) : A_{m+1} \twoheadrightarrow A_m$: then the inverse limit is of order type $1+\omega^*$. Not only is the inverse limit not WQO, it isn't even wellfounded ■

In fact no way of tinkering with the definition of WQO will ever give us a class of structures that is closed under direct limits and inverse limits and we will not worry about these constructors further. However, as we shall see, the class of WQOs is not closed under power set either, and in contrast this does lead to interesting new definitions.

We have already checked that if $\langle Q, \leq \rangle$ is a quasiorder, so is $\langle \mathcal{P}(Q), \leq^+ \rangle$. We are now in a position to come clean on the authorial omniscience with which this chapter began.

Lemma 19

Let $\langle Q, \leq \rangle$ be a quasiorder. Then the following are equivalent

(i) $\langle Q, \leq \rangle$ *is WQO;*

(ii) $\langle \mathcal{P}(Q), \leq^+ \rangle$ *is wellfounded;*

(iii) $\langle \mathcal{P}_{\aleph_1}(Q), \leq^+ \rangle$ *is wellfounded.*

Proof:

(ii) obviously implies (iii).

(iii) $\to$ (i)

If $\langle q_i : i \in \mathbb{N} \rangle$ is a bad sequence then set $Q_i =: \{q_j : j > i\}$ for each $i \in \mathbb{N}$. Then $Q_1 >^+ Q_2 >^+ Q_3 \ldots$ is a $<^+$-descending sequence of countable subsets of Q.

(i) $\to$ (ii)

We will actually prove something slightly more refined, namely that if $\langle Q, \leq \rangle$ is wellfounded, and that $Q_0 >^+ Q_1 >^+ \ldots$ is a $>^+$ descending chain of subsets of Q, there is an infinite antichain $\subseteq Q$. This will be sufficient to establish (i) $\to$ (iii).

For each i pick $q_i \in Q_i \not\leq$ anything in Q_{i+1}. So in particular, we immediately have $q_i \not\leq q_{i+1}$. But since $Q_j <^+ Q_i$ for $j > i$ it follows that if $j > i$ we cannot have $q_i \leq q_j$ since q_j must be less than something in Q_{i+1}, and q_i has been chosen not to be $\leq$ anything in q_{i+1}. An application of Ramsey's theorem to the set $\{q_i : i \in \mathbb{N}\}$ gives either a set of representatives which form an infinite descending sequence under $<$, which is impossible by wellfoundedness, or an antichain, which was what we wanted.

■

Do not allow the ease with which this lemma can be proved to lull you into thinking that it is a triviality. It is the key to WQO and BQO theory. It touches the following themes:

(i) By showing how a fiddly combinatorial property (namely being a WQO) of one structure (to wit $\langle \mathcal{P}(Q), \leq^+ \rangle$) can come to be equivalent to the posession by some other structure (namely $\langle Q, \leq \rangle$) of another (less fiddly) property (to wit: wellfoundedness) it will enable us eventually to express refinements of WQO-ness—some of them very sophisticated—purely in terms of wellfoundedness.

(ii) The set-of-countable-subsets constructor behaves like the (full) power-set constructor. This is an important fact which will be very useful to us later, as it will enable us to substitute this constructor (which *does* have fixed points) for the power set constructor (which famously does *not* have fixed points). But this is actually a special case of:

(iii) All constructors of infinite character seem to behave in this way. The exercise which follows invites the reader to prove that the $\leq^*$ constructor behaves in the same way, as will all infinitary constructors. Indeed we shall see later how the stream constructor behaves like the three constructors of lemma 19, in that we could have added a fourth equivalent to the list, namely:

(iv) $\langle Q^\omega, \leq_l \rangle$ is wellfounded.

However, establishing the equivalence of (iv) with (i)—(iii) needs lemma 23, and is on hold for the moment. (It will be proposition 24.)

EXERCISE 6

Prove that the equivalent conditions of lemma 19 are also equivalent to

(v) $\langle \mathcal{P}_{\aleph_0}(Q), \leq^* \rangle$ *is wellfounded; and*

(vi) $\langle \mathcal{P}(Q), \leq_l \rangle$ *is wellfounded.*

(vii) $\langle \mathcal{P}(Q), \leq^{\aleph_0} \rangle$ *is wellfounded, where* $X_1 \leq^{\aleph_0} X_2$ *iff for cofinitely many* $x_1 \in X_1$ *there are infinitely many* $x_2 \in X_2$ *such that* $x_1 \leq x_2$.

REMARK 20 *(Nash-Williams [?] [1965]) If P and Q are WQO then the set of functions with finite support from $P \to Q$ quasiordered in the style of definition 3 is a WQO.*

Proof: Let $\leq_P, \leq_Q$ be the WQO's on P and Q and let $\leq_*$ be the relation on $P \to Q$ induced in the style of definition 3 on the functions in $P \to Q$ with finite support.

Let $\langle f_i : i < \omega \rangle$ be a sequence of functions in $P \to Q$ with finite support. We will show that it is $\leq_*$-good.

By the usual arguments there is an infinite set I of indices such that for $i < j$ both in I we have support(f_i) $(\leq_P)^+$ support(f_j). Because of this we can suppose without loss of generality that we started off with a sequence satisfying this property.

Now for $i < j$ we have support(f_i) $(\leq_P)^+$ support(f_j) but, unless the sequence is good we also have $(\exists x \in \text{support}(f_i))(\forall y \in \text{support}(f_j)(f_i(x) \not\leq_Q f_j(y))$.

There are infinitely many $j > i$ but only finitely many things in the support of f_i so there is an $x \in \text{support}(f_i)$ such that for infinitely many $j > i$ $(\forall y \in \text{support}(f_j))(f_i(x) \not\leq_Q f_j(y))$.

Pick such an x_0 for f_0 and discard all f_j such that $\exists y \in \text{support}(f_j)(f_i(x_0) \leq_Q f_j(y))$. By choice of x_0 there will be infinitely many left. Renumber the survivors and repeat by picking an x_1 in the support of f_1 similarly, and so on for all f_n.

We now have an infinite sequence $\langle x_i : i < \omega \rangle$ of elements of P. We don't know that $i < j$ implies $x_i \not\leq_P x_j$ but we do know that if $x_i \leq_P x_j$ with $i < j$ then $f_i(x_i) \not\leq_Q f_j(x_j)$.

Finally we invoke Ramsey's theorem to obtain a bad sequence in P or a bad sequence in Q.

(References in the literature (like Laver's Fraïssé conjecture article p 92) suggest that this is first proved in

H I A T U S

a few thms and exercises on rank

REMARK 21 *If Q is a WQO of rank α, Q^ω under stretching is of rank α^ω.*

Proof: If R is a relation let R' be that relation whose domain is the graph of R and is defined by $\langle x, y\rangle \; R' \; \langle u, v\rangle$ iff $\langle y, u\rangle \in R$.

Notice that R' is wellfounded iff R is, and has the same rank.

Now let $<$ be the strict part of the quasiorder of Q^ω by stretching. Think about $<'$. Each ordered pair $\langle l_1, l_2\rangle$ in its domain can be replaced by the initial segment of l_1 in virtue of which the greedy algorithm discovered that l_1 does not stretch into l_2. $<'$ on the graph of Q^ω under stretching is obviously isomorphic to the strict part of stretching on $Q^{<\omega}$.

But what is the rank of $Q^{<\omega}$ under stretching—given that $\rho(Q) = \alpha$? By considering Q^2, Q^3, ... we see that it must be at least α^n for all $n \in \mathbb{N}$. But it cannot be more than the rank of the set of finite sequences ordered first by length and then by product (all the singletons first, then all the pairs ...) since it is a subset of that ordering. So it must be exactly α^ω. ■

[*HOLE Presumably we will later be able to prove that: If Q is a β-good WQO of rank α, Q^β under stretching is of rank α^β.*]

Finite character again

Now we can return to the constructors of finite character and ask whether or not they preserve WQO-ness. [*HOLE Make these claims more explicit, and set them out properly*]

(i) $Q \mapsto Q$-lists under stretching;
(ii) $Q \mapsto$ finite Q-trees under $\leq_t$;
(iii) $Q \mapsto \langle \mathcal{P}_{\aleph_0}(Q), \leq^* \rangle$;
(iv) $Q \mapsto \langle \mathcal{P}_{\aleph_0}(Q), \leq^+ \rangle$.
(v) The Lexicographic path ordering
(vi) The recursive path ordering

First $\leq^+$. Let $\langle Q, \leq \rangle$ be a WQO and suppose $\langle Q_i : i \in \mathbb{N} \rangle$ is a bad sequence of finite subsets under $\leq^+$. For each $i > 0$ there is $x_{0,i} \in Q_0$ with $x_{0,i} \not\leq^+ y$ for all $y \in Q_i$. But because Q_0 is finite, infinitely many of these $x_{0,i}$ are the same. Pick x_0 from $\{x_{0,i} : i > 0\}$ such that for some infinite $J \subseteq \mathbb{N}^+$ we have $x_0 \not\leq^+ y$ for all $y \in Q_j$ and all $j \in J$. Discard all the Q_i whos subscripts are not in J and renumber, giving the first one the subscript 1. Now repeat what we have just done to obtain x_1, x_2 and so on. Then we use DC to construct a bad sequence $\langle x_i : i \in \mathbb{N} \rangle$. So this constructor preserves WQO-ness. (This was proved by Higman [23].)

(iii) Finite subsets under $\leq^*$. Suppose $\langle Q_i : i \in \mathbb{N} \rangle$ is a $>^*$-descending chain of finite subsets of Q, where $\langle Q, \leq \rangle$ is WQO. Let x_0 be anything in Q_0 and thereafter pick $x_i \in Q_i$ s.t. $(\forall y \in Q_{i-1})(x_i \not\geq y)$. The x_is then form a bad sequence. This shows that if $\leq$ is WQO, then $\leq^*$ is at least wellfounded. (We proved the converse to this by considering the identity quasiorder on a countable set). Jančar [1999] shows that RADO (which we shall see later) is a counterexample that shows that it need not be WQO.

This shows that if the power set isn't wellfounded under $>^*$ then even the finite sets aren't.

4.1 The minimal bad sequence construction

Any quasiorder that is wellfounded but is not WQO has bad sequences, and has some that are in some sense minimal. This "minimal bad sequence" is a key idea. A precise definition will be given later[1] : for the moment our approach is a two pronged one: (i) How do we make one? (ii) What can it do for us once we have got it?

How do we make one?

Let $\langle X, \leq \rangle$ be a wellfounded quasiorder that is not WQO. Let x_0 be a minimal member of $\{x :$ there is a bad sequence whose first member is $x\}$. Let x_{n+1} thereafter be a minimal member of $\{x :$ there is a bad sequence whose first n members are $\langle x_0 \ldots x_{n-1} \rangle$ and whose $n + 1$th member is is $x\}$. This is a kind of greedy algorithm. Let us say that a sequence constructed by it is an **MBS** ("minimal bad sequence").

[*HOLE Fuss about how it uses DC and* Π^1_1 *comprehension*]

The following remark is not needed until chapter 7 but crops up naturally here. If we topologise Q^ω in the usual way by giving Q the discrete topology and Q^ω the product topology, we find that if Q is a quasiorder that is not WQO then the set of MBSs is a closed subset of Q^ω in the product topology. The set of bad sequences is closed too.

Beware: altho' a subsequence of a bad sequence is a bad sequence, it's not obvious that a subsequence of an MBS is an MBS.

Exercise 7 *A MBS over* $\langle Q, \leq_Q \rangle$ *is a special kind of* Q*-stream. We have seen various ways of lifting* $\leq_Q$ *to* $\mathcal{P}(Q)$ *and* Q^ω*. Use these lifts to characterise the way in which the output of the greedy algorithm is minimal.*

What can it do for us once we have got it?

The significance of MBSs is that any sequence derived from objects in some suitable sense "below" an MBS must be good. This will enable us to elaborate proofs that certain obviously wellfounded structures are

[1] By the reader!

additionally WQO by "minimal counterexample" arguments—which is to say, proofs by induction.

For example:

LEMMA 22

Let $\langle Q, \leq_Q \rangle$ be a wellfounded quasiorder, and let $\langle x_n : n \in \mathbb{N} \rangle$ be a minimal bad sequence. Consider $Q' = \{q \in Q : \exists x_i\ q <_Q x_i\}$. Then Q' is WQO by $\leq$.

Proof:

Let $\langle q_n : n \in \mathbb{N} \rangle$ be a bad sequence of elements of Q'. We prove by induction on n that $(\forall i)(q_i \not< x_n)$.

$n = 1$. We cannot have $q_i < x_1$ because the (tail of tail of ...) subsequence $\langle q_j : j \geq i \rangle$ is certainly bad, being a subsequence of a bad sequence, but its `hd` is below the `hd` of the MBS $\langle x_n : n \in \mathbb{N} \rangle$.

Now assume the induction hypothesis for all $n' < n$, and suppose $q_i < x_n$. Consider now the sequence consisting of those q_j with $j > i$, preceded by $\langle x_1 \ldots x_{n-1} \rangle$. This is a bad sequence, for where could a "good pair" be found? It can't be an $x_i \leq x_j$ because the x's are bad, and it can't be a $q_i \leq q_j$ because the q's are bad. Can we have $x_i \leq q_j$? By hypothesis $q_j < x_k$ for some x_k, and by induction hypothesis this k is at least n so we would have $x_i \leq x_k$ with $i < k$ contradicting badness of $\langle x_i : i \in \mathbb{N} \rangle$, so: no, the sequence is bad. The first $n - 1$ coordinates of this bad sequence are the result of the greedy algorithm. What about the nth? It is q_j, and $q_j < x_n$. But then the greedy algorithm could not have chosen x_n, as it would not have been minimal. So $\langle q_n : n \in \mathbb{N} \rangle$ is not bad and there is no such bad sequence. ■

These properties of MBSs will be exploited in the the proof of Kruskal's thm. (theorem 25)

EXERCISE 8 *Let $\langle Q, \leq \rangle$ be a WQO with an automorphism σ and consider Q with the relation $x \leq \sigma(y)$. Is this a WQO? Prove or give a counterexample.*

EXERCISE 9 *Give counterexamples to the assertion that if $\langle Q, \leq \rangle$ is a WQO the corresponding strict partial order is embeddable in the pointwise product On^n for some finite n.*

howabout on^{on}?

EXERCISE 10 *Can (the strict partial ordering corresponding to) a*

WQO always be embedded in the set of ω-sequences of ordinals with finite support and the pointwise product ordering?

[*HOLE I've now forgotten why i ever tho'rt this was a sensible question....Provide an answer to this*]

4.2 Kruskal's theorem

Next we show that (finite) lists over a WQO are WQO.

Lemma 23 *If $\langle X, \leq \rangle$ is a WQO, so is $\langle X^{<\omega}, \leq_l \rangle$.*

Proof: We use *reductio ad absurdum*. Suppose that $\langle X, \leq \rangle$ is a WQO but that $\langle X^{<\omega}, \leq_l \rangle$ is not. We know by now from proposition 7 that $\langle X^{<\omega}, \leq_l \rangle$ is wellfounded, so let us construct a minimal bad sequence $\langle a_i : i \in \mathbb{N} \rangle$ of lists. Look at the heads of the lists in the minimal bad sequence. These are WQO by hypothesis so (by lemma 15) there must be an infinite subsequence $\langle b_i : i \in \mathbb{N} \rangle$ of $\langle a_i : i \in \mathbb{N} \rangle$ such that for $i < j$, $\mathtt{hd}(b_i) \leq \mathtt{hd}(b_j)$. Throw away all the other lists in this bad sequence. We now have a bad sequence of lists whose heads, at least, form an increasing sequence. Now consider the tails. We want to show that the tails are WQO as well, for that will complete the proof for us by using the third clause of definition 3.3. We know from page 18 that $\mathtt{tl}(l) <_l l$ always, so these tails belong to a collection of things *below* this minimal bad sequence, $\langle a_i : i \in \mathbb{N} \rangle$, in the sense of lemma 22. Therefore the sequence of tails of elements of $\langle b_i : i \in \mathbb{N} \rangle$ is not a bad sequence. So there are $i < j$ such that $\mathtt{tl}(b_i) \leq_l \mathtt{tl}(b_j)$. Therefore (by the third clause in the inductive definition of $\leq_l$) $b_i \leq b_j$, so $\langle b_i : i \in \mathbb{N} \rangle$ is not a bad sequence, and $\langle a_i : i \in \mathbb{N} \rangle$ is not bad either. ■

Before we complete the proof of Kruskal's theorem (the last step of which is analogous to the proof of lemma 23 that we have just seen) let us make a brief digression to complete the agenda set up by lemma 19.

Proposition 24 *The equivalent assertions of lemma 19 are equivalent to the assertion that streams over Q are wellfounded under stretching.*

Proof:

(iv) $\to$ (i)

If Q is not WQO, then it has a bad sequence $f_0 =: \langle q_i : i \in \mathbb{N} \rangle$. Set $f_{i+1} =: \mathtt{tail}(f_i)$. This is a descending sequence in $\langle Q^\omega, \leq_l \rangle$.

(i) $\rightarrow$ (iv)

Let $\langle f_i : i \in \mathbb{N}\rangle$ be a strictly descending ω-sequence of elements of Q^ω under stretching, where $\langle Q, \leq_Q\rangle$ is a WQO. We will derive a contradiction.

There is an obvious greedy algorithm for seeking a map in virtue of which one Q-stream g will stretch into another Q-stream f, and if there is a 1-1 increasing map $\mathbb{N} \rightarrow \mathbb{N}$ in virtue of which $g \leq_l f$ then the greedy algorithm will find it.

$h(0) =: \mu k.(f(k) \geq_Q g(0))$
$h(n+1) =: \mu k > h(n).f(k) \geq_Q g(n+1)$

Thus if f and g are infinite lists and f does not stretch into g there is a finite initial segment of f that doesn't stretch into g.

For each i set g_i to be the initial segment of f_i in virtue of which f_i does not stretch into f_{i+1}. But if g_i doesn't stretch into f_{i+1} it doesn't stretch into any later f either, since f_{i+k} stretches into f_i so the g_i form a bad sequence of finite lists. But now we appear to have a bad sequence of (finite) Q-lists, and this we can't have, because Q is WQO, and 23 tells us that lists over a WQO are WQO under stretching. ■

Notice that we do not use DC in getting a bad sequence of Q-lists from a bad sequence of Q-streams. This is in contrast to the case of countable subsets.

Now we can prove

Theorem 25 *(Kruskal) Finite trees over a WQO are WQO.*

Proof: By wellfoundedness of $<_t$, if there is a bad sequence there is a minimal bad sequence, and let $\langle a_i : i \in \mathbb{N}\rangle$ be one. Look at the roots of the trees. Since the roots are from a WQO there must be an increasing ω-subsequence $\langle b_i : i \in \mathbb{N}\rangle$ from $\langle a_i : i \in \mathbb{N}\rangle$ such that if $i < j$ then (root of b_i) $\leq$ (root of b_j). (lemma 15) Let l_i be the list of children of a_i.

We know that the roots of the a_i form a strictly increasing sequence. What we now have to look at is a countable sequence of lists of children of the trees we started with. Because of lemma **??** these trees form a collection of trees below (in the sense of lemma 22) the minimal bad sequence we started with. So, by lemma 22 they are WQO, so lists over them are WQO as well. Therefore there are $i < j$ with $l_i \leq_l l_j$, so (by the third clause in the definition of $\leq_t$) it follows that $a_i \leq_t a_j$. Thus $\langle a_i : i \in \mathbb{N}\rangle$ is not bad.

■

[*HOLE this needs to be revised in the light of the treatment of MBSs*]

(This proof of Kruskal's theorem is based on a *précis* by Laver [35] of a proof by Nash-Williams. [*HOLE must get the NW reference*])

There is also the possibility of defining stronger quasiorders on the set of finite trees. Stronger means fewer ordered pairs means more and bigger antichains means that the tree of bad finite sequences partially ordered by reverse end-extension has higher rank. But of course this works only if we can show that the quasiorder is a WQO.

The idea of the following quasiorder is Friedman's.

DEFINITION 26 *An n-***labelled** *tree is a finite tree whose vertices have been labelled with natural numbers from 1 to n, represented as an ordered pair of a tree and a labelling function. We say f is a* **gap-embedding** *from $\langle T_1, l_1\rangle$ into $\langle T_2, l_2\rangle$ if*

(i) $(\forall x \in T_1)(l_1(x) = l_2(f(x)))$

(ii) *If $y \in T_1$ is an immediate successor of $x \in T_1$ then* $(\forall z \in T_2)(f(x) < z < f(y) \to l_2(z) \geq l_2(f(y)))$

The proof that it is in fact a WQO is due to Křiž (1989). [*HOLE need to motivate this definition.*]

See Tzameret [57].

4.3 Ranks again

[*HOLE This section needs to be completely rewritten.*] We now digress briefly to return to topics related to rank functions. Recall that the rank of a wellfounded structure is the *least* ordinal α s.t. there is a homomorphism from that structure onto the ordinals below α. (By 'homomorphism' we mean a map π s.t. $xRy \to \pi(x) < \pi(y)$. See exercise **??**.) Given a wellfounded $\langle X, R\rangle$ we can ask: for which ordinals α is there a homomorphism $\pi : X$ onto ordinals below α? There are easy examples to show that α and π need not be unique, and α might be much bigger than the rank of R. The pointwise product order $\mathbb{N} \times \mathbb{N}$ can be mapped 1-1 onto the ordinals below ω^2 by $\langle n, m\rangle \mapsto \omega \cdot n + m$ ("depth-first") and can be mapped 1-1 onto the ordinals below ω by refining the product order to $\langle n, m\rangle < \langle n', m'\rangle$ if $(n + m < n' + m') \vee ((n + m = n' + m') \wedge (n < n'))$ ("breadth-first"). Notice that the rank of the pointwise product $\mathbb{N} \times \mathbb{N}$ is ω because every element has only finitely many predecessors.

We can give a slightly more high-powered example using finite trees over $\mathbb{N}^+$. (that is to say, finite trees decorated with natural numbers > 0) [*HOLE of the kind where the children form a list*] Each tree will be sent to an ordinal as follows. Send each singleton tree to its label. Thereafter, send a tree T to the sum of $\omega^{c_1} + \omega^{c_2} + \omega^{c_3} + \cdots \omega^{c_i} \cdots$ where $c_1, c_2, \cdots c_i \cdots$ are the ordinals given to the children of T taken in list order. That way every finite tree gets a label that is an ordinal $< \epsilon_0$, and every ordinal is hit. Again, although every tree is of finite rank, we have succeeded in giving them labels stretching far into the transfinite, and without leaving any ordinal out.

A more dramatic example has just fallen into our lap with the proof of Kruskal's theorem.

[*HOLE Explain* Γ *numbers here*]

Now, for any α, if there is a homomorphism $\pi : X \twoheadrightarrow$ ordinals below α s.t. $xRy \to \pi(x) < \pi(y)$, then there is a structure of rank $\geq \alpha$ which arises from $\langle X, R\rangle$ in a natural way.

[*HOLE Stuff missing here: there are ideas to be worked out. Consider the tree of descending R-sequences ordered by reverse end-extension. What is its rank??? Presumably, for each hom π, we can prove that the rank of any sequence is at least π of its last member. And for each π we prove by R-induction that the rank of descending sequence is at least π of its last element.*

But if R is WQO we can do even better

]

Remark 27 *Let $\langle X, R\rangle$ be a WQO. Consider the (downward-branching) tree $\langle B, \succcurlyeq\rangle$ of bad sequences ordered by reverse end-extension (each bad sequence is above all its bad end-extensions). $\langle X, R\rangle$ is a WQO (no infinite bad sequences) so $\langle B, \succcurlyeq\rangle$ is wellfounded and therefore has a rank. Now suppose that there is a surjection $\pi : X \twoheadrightarrow$ the set of ordinals below α such that $xRy \to \pi(x) < \pi(y)$. Then the rank of $\langle B, \succcurlyeq\rangle$ is at least α.*

Proof: Every decreasing sequence of ordinals below α pulls back (*via* π) to a bad sequence. Now it's easy to check that the set of decreasing sequences of ordinals below α—ordered in the style of $\langle B, \succcurlyeq\rangle$—has rank α. (In the partial order of decreasing sequences of ordinals inversely p.o.'d by end extension the rank of each decreasing sequence is its smallest (i.e. last) member.) ■

[*HOLE At this point we really need a discussion of some illuminating examples*]

Exercise 11 *Check also that the ordinal we get from the WQO of lists in the sense of remark 27 is indeed (at least) ω^ω.*

Exercise 12

Let $\leq_\alpha$ and $\leq_\beta$ be wellorderings of $\mathbb{N}$ of length α and β respectively. Their intersection is a WQO because of lemma **??**. *What is the rank of the tree of bad sequences in this new WQO?*

[*HOLE Provide answer*]

[*HOLE If we have two wellorderings of the same domain, one to length α and the other to length β, how complicated can their intersection be? Think about a bad sequence in this new, intersection, ordering. Each point has two labels, one from each ordering, and thus a new label that is an ordered pair. In a bad sequence these labels from a bad sequence in the product order $\alpha \times \beta$.*

]

The tree of bad sequences in $\mathbb{N} \times \mathbb{N}$ is of length ω^2, despite the fact that this product order has rank only ω The fact that $\mathbb{N} \times \mathbb{N}$ has the ordinal ω^2 associated with it will prepare us for exciting examples still to come of WQOs where every element is of finite rank but the tree $\langle B, \not\succeq \rangle$ of bad sequences ordered by reverse end-extension has truly *enormous* rank.

This is the appropriate unravelling of the intuition that one can extract an ordinal from a WQO by worrying about how its width increases with height. One can of course use the fact that infinite antichains in X give rise to infinite descending sequences in $\mathcal{P}(X)$, and that width information in X turns into height information in $\mathcal{P}(X)$. The advantage of this approach is that set-of-bad-sequences-ordered-by-reverse-end-extension is a constructor of finite character, and those sequences are finite not infinite. This will enable us to code this development in arithmetic not analysis.

4.4 Topics for discussion

It is very important that we can actually extract a concrete example of a long wellordering from Kruskal's theorem. The wellordering will be a wellordering of sets of bad sequences.

[*HOLE Expand this*]

Think of WQOs as a generalisation of (reflexive closures of) wellorderings, and give definitions of the generalisation of *club* and *stationary*. Also Neumer's theorem etc. etc.

There is no good concept of clubset beco' of the 2-ladder. Let $A = \{a_i : i \in \mathbb{N}\}$ and $B = \{b_i : i \in \mathbb{N}\}$. Then for x, y in $A \cup B$, say $x \leq y$ iff $x = y$ or x's subscript is strictly less than y. Then A and B are disjoint clubsets.

4.5 Some more exercises

EXERCISE 13 *Give an easy proof that the lexicographic product of two WQOs is WQO.*

EXERCISE 14 *Consider the relation "$x \in TC(\{y\})$" on the hereditarily finite sets (also known as V_ω). Is it a WQO?*

EXERCISE 15 *An* **incline** *is a structure with two associative and commutative binary operations $+$ and $\cdot$ satisfying*

(i) $(\forall xyz)(x \cdot (y + z) = x \cdot y + x \cdot z)$;
(ii) $(\forall x)(x + x = x)$;
(iii) $(\forall xy)(x + x \cdot y = x)$.

We define a relation $\leq$ by $x \leq (x + y)$.

Prove that $\leq$ is a quasiorder.

Let $\langle I, +, \cdot \rangle$ be a finitely generated incline. Show that $\langle I, \geq \rangle$ is a WQO.

5

Good quasiorders of finite exponent

[*HOLE fit in somewhere the observation that 'finite exponent' refers not just to the exponent on the power set operation but also to the exponent in Ramsey's theorem...*]

We saw in remark 19 that the '+' operation preserves reflexivity and transitivity. [*HOLE It wasn't in lem 19 that we saw this. We don't seem to have set out clearly that the lifts of infinite character preserve transitivity and reflexivity an d poss connexity*]

It would be nice if in addition it were to preserve the condition on ω-sequences so that $\langle \mathcal{P}(A), \leq^+ \rangle$ is WQO as long as $\langle A, \leq \rangle$ is. We shall see a counterexample due to Rado which will show that the Hoare ordering of the power set of a WQO is not always a WQO. It is natural to ask what extra conditions one has to add to those comprising WQOness to get a property that is preserved under this construction.

First let us suppose that $\langle \mathcal{P}(Q), \leq_Q{}^+ \rangle$ is not WQO. So there is a bad sequence $\langle Q_i : i \in \mathbb{N} \rangle$ where for $i < j$, $Q_i \not\leq_Q^+ Q_j$.

It would be nice if for each i we could pick a member q_i in Q_i to get a bad sequence on Q, but there is no reason to suppose we can. After all, each q_i would have to "do infinitely many things". Later we will see examples where we definitely cannot pick a single q_i in this way.

However, we can at least do the following. $Q_i \not\leq^+ Q_j$ for $i < j$ which is to say that $\neg(\forall q \in Q_i)(\exists q' \in Q_j)(q \leq q')$ so for each pair i, j with $i < j$ we can pick an element $q_{i,j} \in Q_j$ s.t. $(\forall q \in Q_j)(q_{i,j} \not\leq_Q q)$. So, using countable choice we can pick a family of elements of Q indexed by pairs of distinct natural numbers, such that $(\forall i < j < k)(q_{i,j} \not\leq_Q q_{j,k})$. This isn't exactly a bad sequence: it's a thing which we will call a bad *array*, and the definition of 'array' will emerge later. With hindsight, the (bad) sequences we have just encountered will come to be seen to have been merely a special kind of (bad) array. Just as a sequence (of widgets) is a map from $\mathbb{N}$ to widgets, so an array (of widgets) will be a map from a **block** to widgets. We will see the exact definition of block later. For the moment an operational understanding will have to do, and we take as our current interesting example of a block the structure whose carrier set is $\{\{i, j\} : i, j \in \mathbb{N}\}$ equipped with a binary relation $\lhd$ which for all $i < j < k$ relates $\{i, j\}$ to $\{j, k\}$ and to nothing else. The reader should try and think of the bad array that we constructed at the start of this paragraph not as a family of elements of Q with bizarre subscripts but as a map f from the block $\{\{i < j\} : i, j \in \mathbb{N}\}$ to Q such that $(\forall b, b' \in B)(b \lhd b' \to f(b) \not\leq f(b'))$.

Sequences are special kinds of arrays, and the structure $\langle \mathbb{N}, < \rangle$ is a special kind of block. The block we saw in the previous paragraph is the first nontrivial example of a block, and it's a block of a kind that one might call *quadratic*: it is a set of ordered *pairs*, and in the lexicographic order it is of length ω^2. (One can think of *arrays* as quadratic as well, when they are functions defined on quadratic blocks). One could think of the block $\langle \mathbb{N}, < \rangle$ as a *linear* block and take note that it is of length ω in the lexicographic order, but these italicised *aides memoires* are not used formally and I mention them only to help the reader see that $\langle \mathbb{N}, <, \rangle$ and $\langle \{\{i < j\} : i, j \in \mathbb{N}\}, \lhd \rangle$ are creatures of the same kind, but of different lengths. When we go up one stage, as we will soon, we shall see *cubic* blocks. However there is another point that needs to be made at this early stage, before we do that.

Given a bad (quadratic) array on Q we can construct a bad sequence on $\mathcal{P}(Q)$ all of whose elements are countable sets: simply set $Q_i =: \{q_{i,j} : j > i\}$. (The idea here "If there is a bad sequence of subsets there is a bad sequence of *countable* subsets" is the first reappearance of the idea first flagged on page 29.)

Now consider the case where $\langle \mathcal{P}(\mathcal{P}(Q)), (\leq^+)^+ \rangle$ is not WQO. We can do exactly what we did in the case where $\langle \mathcal{P}(Q), {\leq_Q}^+ \rangle$ was not WQO to get a bad array $\{X_{i,j} : i < j \in \mathbb{N}\}$, but this time of course the $X_{i,j}$ are subsets of Q, not elements of Q. So we repeat the process. $X_{i,j} \not\leq^+ X_{j,k}$, so there must be something in $X_{i,j}$ which is $\not\leq$ anything in $X_{j,k}$. We will pick one such and call it $X_{i,j,k}$. Thus we get an analogous condition on increasing triples from Q, namely: $(\forall i < j < k < l)(q_{i,j,k} \not\leq_Q q_{j,k,l})$. This is the condition which fails if $\langle \mathcal{P}^2(Q), (\leq^+)^+ \rangle$ is *not* WQO. This gives us our third example of a block: $\{\{i < j < k\} : i, j, k \in \mathbb{N}\}$. Similarly, given a bad array $\{q_{i,j,k} : i < j < k \in \mathbb{N}\}$ of triples we can get a bad sequence $X_0, X_1 \ldots X_n \ldots$ on $\mathcal{P}^2(Q)$ of countable sets of countable subsets of Q. X_i will be $\{X_{i,j} \subseteq Q : j > i\}$ where $X_{i,j} = \{q_{i,j,k} : k > j > i\}$.

Once the reader is entirely happy with the idea of sifting[1] information about bad sequences in $\mathcal{P}^2(Q)$ or $\mathcal{P}^3(Q)$ to information about bad arrays on Q, they should take on board the idea that this can be done for any finite n.

So the development so far can be summarised as follows.

If $\mathcal{P}^n(Q)$ quasiordered by the result of applying the '+' operation n times to a given quasiorder $\leq_Q$ of Q is not a WQO, then there is a bad $(n+1)$-ary array on Q, which is to say a map f from the set of unordered $n+1$-tuples of natural numbers such that

$$(\forall i_0 < \ldots i_n \in \mathbb{N})(f(\{i_0 \ldots i_{n-1}\}) \not\leq_Q f(\{i_1 \ldots i_n\}))$$

which we discover by sifting.[2]

[*HOLE Aside for logicians. On the face of it, saying that $\mathcal{P}^n(X)$ is wellfounded under $\leq^{+^n}$ ought to be nth order in $L_{\omega_1\omega_1}$ but sifting enables us to reveal that it is still only first order. Notice that being WQO is not obviously second order. At some point we will have to Say something about how the descending chain condition is not nth order for any n—not elementary. There is a probably quite a lot one can say about this.*]

Further, that from the bad array on Q one can recover a bad sequence on $\mathcal{P}^n(Q)$ whose elements are countable sets of countable sets of^n elements of Q. This is worth minuting.

PROPOSITION 28 *If there is a bad sequence in $\mathcal{P}^n(Q)$ then there is one consisting entirely of countable sets of ... countable subsets of Q.*

These sets are **hereditarily countable**.

Proof: We sift a bad sequence in $\mathcal{P}^n(Q)$ to a bad array of elements of Q, indexed by increasing $n+1$-tuples from $\mathbb{N}$. Then we obtain successivly bad arrays on $\mathcal{P}^(Q)$, $\mathcal{P}^2(Q)$, and so on by setting Q_s to be $\{Q_t : t = \texttt{butlast}(s)\}$, first for tuples s of length $n-1$, then for tuples s of length $n-2$, and so on up to tuples of length 1, at which point we have a bad sequence of hereditarily countable elements of $\mathcal{P}^n(Q)$. ∎

[1] "I am soft sift in an hourglass, mined with a motion, a drift ... " *The Wreck of the Deutschland*, Gerald Manley Hopkins.

[2] We are assuming $i_k < i_m$ when $k < m$. That is to say, we are thinking of these objects sometimes as unordered tuples, and sometimes as increasing ordered tuples.

Our first example of a block was the quadratic block $\{\{i < j\} : i, j \in \mathbb{N}\}$ with the binary relation $\lhd$ which holds between $\{i < j\}$ and $\{j < k\}$. We saw the cubic block too, and its rather more complex definition. Although I am still not planning to give a precise definition of blocks, the reader can see how the process of pulling down a bad sequence in $\mathcal{P}^{n-1}(Q)$ to a bad array on Q gives rise to a block consisting of unordered n-tuples. This block is the **canonical n-block**. A quasiorder that has no bad arrays whose domain (remember an array is a map from a block ...) is the canonical n-block is said to be ω^n-**good**. The ordinal alludes to the length of the canonical n-block in the lexicographic order. Thus, in particular, a WQO is a quasiorder that is ω-good.

So we have proved

PROPOSITION 29

The following are equivalent

(i) $\langle Q, \leq_Q\rangle$ *is* ω^n*-good;*

(ii) $\langle \mathcal{P}^n(Q), (\leq_Q)^{+^n}\rangle$ *is* ω*-good (i.e., WQO);*

(iii) $\langle \mathcal{P}^n_{\aleph_1}(Q), (\leq_Q)^{+^n}\rangle$ *is* ω*-good (i.e., WQO).*

(The reader is probably becoming impatient for a proper definition of a block: we will postpone this until we want to make sense of the idea of ω^α-good for $\alpha \geq \omega$. Enthusiasts should for the moment master their impatience and redirect their energies into chewing over the 2-block and attempting exercise 16.)

EXERCISE 16 *The canonical n-block is clearly a binary structure with carrier set a set of n-tuples (unordered n-tuples or increasing ordered n-tuples, according to taste) of natural numbers, with a binary relation $\lhd$.*

What is $\lhd$ exactly?

EXERCISE 17

If $\langle X, \leq\rangle$ is a quasiorder, define $\leq^{\aleph_0}$ on $\mathcal{P}(X)$ as in clause (vii) of exercise 6. Show that if $\langle X, \leq\rangle$ is an ω^2-good quasiorder, then $\langle \mathcal{P}(X), \leq^{\aleph_0}\rangle$ is WQO.

Although there are some essentially new ideas in the study of WQOs "of infinite exponent", most of the challenge to the student comes in scaling up for this new endeavour the *old* ideas from the finite exponent case. Accordingly it is a good idea to properly master the finite exponent case before going transfinite.

Exercise 18 *Prove that $\mathbb{N}^{<\omega}$ is not wellordered by the lexicographic order but that the canonical n-block is.*

Prove that the canonical n-block is of length ω in the colex ordering.

[*HOLE must provide answers to these two questions*]

5.1 Rado's quasiorder

So far we haven't seen an actual example of a WQO that is ω-good but not ω^2-good. How might we construct one? What we are looking for is a quasiorder with a bad array from the canonical 2-block. One thing that leaps to mind is: try defining a quasiorder on the domain, $\{\{i < j\} : i, j \in \mathbb{N}\}$, of the canonical 2-block, and make it disjoint from the block relation $\lhd$. That way we can set the array map to be the identity function.

Let us try to build a quasiorder $\leq_{new}$ on $\{\{i < j\} : i, j \in \mathbb{N}\}$, greedily putting in as many ordered pairs as possible, while keeping it disjoint from $\lhd$. We do this by recursion on the lexicographic order. We can allow $\langle 1,2\rangle \geq_{new} \langle 1,2\rangle$; $\langle 1,3\rangle \geq_{new} \langle 1,2\rangle$ is all right as well. and so on for all $\langle 1,n\rangle$. What about $\langle 1,2\rangle \leq_{new} \langle 2,3\rangle$? Clearly not, because $\langle 1,2\rangle \lhd \langle 2,3\rangle$. But then by the same token we can't allow $\langle 1,n\rangle \leq_{new} \langle 2,m\rangle$ because we have already decided $\langle 1,2\rangle \leq_{new} \langle 1,n\rangle$ and transitivity would give $\langle 1,2\rangle \leq_{new} \langle 2,n\rangle$. Continuing in this way, we construct the following quasiorder.

Definition 30 *Quasiorder $\{\{i,j\} : i < j \in \mathbb{N}\}$ by $\{i < j\} \leq \langle i' < j'\rangle$ iff $((i = i') \wedge (j \leq j')) \ \vee \ (j < i')$. Call this structure RADO.*

Each pair $\{i < j\}$ is of rank j, so everything is of finite rank, and the rank of *RADO* itself is ω. For each i the set $\{\{i,j\} : i < j\}$ is the ith **ray**.

Exercise 19 *Draw a Hasse diagram of RADO.*

Theorem 31 *RADO is WQO.*

Proof:

It is possible to give a more direct proof of this elementary fact, but, with an eye to subsequent generalisation, I give a proof using Ramsey's theorem.

Suppose, *per impossibile* that $f : \mathbb{N} \to RADO$ were a bad sequence, Partition $[\mathbb{N}]^2$ by putting $\{i, j\}$ into one of two pieces depending on whether or not $\mathtt{fst}(f(i)) = \mathtt{fst}(f(j))$. A subset monochromatic in one sense gives us a perfect subsequence of a ray, and a subset monochromatic in the other gives us a perfect "sideways" subsequence. ■

THEOREM 32 *RADO injects isomorphically into every quasiorder that is ω-good but not ω^2-good.*

Proof: [*HOLE Need to cite the original source*]

In fact we can show something slightly stronger. If $\langle Q, \leq_Q \rangle$ is ω-good but not ω^2-good, then every bad quadratic array has a subarray isomorphic to RADO.

The idea is to start from such a bad quadratic array and weed out, Michaelangelo-fashion, the parts one doesn't need. In fact the best way to do it is the quick-and-dirty way using Ramsey's theorem. I suspect that one can derive some enlightenment by "eliminating the cuts" from this proof, and proving only that part of Ramsey that one needs, but we must keep an eye on the possibility of proving the same theorem for higher exponents, and there the quick-and-dirty option is the only show in town.

Suppose $f : RADO \to Q$ is a bad quadratic array. Partition $[\mathbb{N}]^4$ by allocating quadruples $\{i < j < k < m\}$ according to the truth-values of $f(\{i, j\}) \leq_Q f(\{k, m\})$, $f(\{i, j\}) \leq_Q f(\{i, k\})$, and $f(\{i, k\}) \leq_Q f(\{j, m\})$. This gives eight pieces, and it will turn out that the only piece that can have an infinite monochromatic set is the piece in which

$f(\{i, j\}) \leq_Q f(\{k, m\})$, $f(\{i, j\}) \leq_Q f(\{i, k\})$, and $f(\{i, k\}) \not\leq_Q f(\{j, m\})$ all hold. The first two happen because otherwise there would be a bad sequence in Q, and the third happens because otherwise we would have things like $f(\{1, 10\}) \leq_Q f(\{5, 15\})$ and $f(\{5, 15\}) \leq_Q f(\{10, 25\})$ which implies $f(\{1, 10\} \leq_Q f(\{10, 25\})$ contradicting the badness of f.

So let I be a set monochromatic for this partition. I claim that $\langle f``([I]^2), \leq_Q \rangle$ is a copy of RADO(2). [*HOLE Don't we want $f|[I]^2$ to be 1-1? Or does this happen automatically?*]

A colouring of a set of course gives (in some sense) a whole boolean algebra of colours. The colours that we see usually are the atoms of a boolean algebra, but if the algebra is free there are generators can be tho'rt of as primary colours ...

Here's a better way to do it. The pieces of a partition are *colours.*

Unions of several pieces we will call **hues** and it's obvious what we must mean by saying that a sets is monochromatic for a hue. Useful obvious fact: if there is no set monochromatic for a hue, there is no set monochromatic for any of its constituent colours. In this case there are three hues, and each corresponds to one of the three primary colours $f\{i,j\} >_Q f\{k,m\}$, $f\{i,j\} >_Q f\{i,k\}$ and $f\{i,k\} >_Q f\{i,m\}$.

Clearly there can be no set monochromatic for the hue $f\{i,j\} >_Q f\{k,m\}$. Similarly there can be no set monochromatic for the hue $f\{i,j\} >_Q f\{i,k\}$. There can be no set monochromatic for the hue $f\{i,k\} >_Q f\{i,m\}$ beco's of transitivity burble.

So far we've considered $\leq^+$ only: the time has come for another look at $\leq^*$.

■

Exercise 20 *Let $\langle X, \leq, \rangle$ be a WQO. Define $\leq^*$ on $\mathcal{P}(X)$ by:*
$Y \leq^* Z$ iff $(\forall z \in Z)(\exists y \in Y)(y \leq z)$
Is $\leq^*$ a WQO on $\mathcal{P}(X)$? Prove or find a counterexample.

Let us remind ourselves of the nice properties *RADO* has. It is ω-good but not ω^2-good, and a quasiorder is ω^2-good iff it does not have *RADO* as a substructure. (cf dfn of wellfoundedness). And it has a very neat bad array: namely the identity map from the canonical 2-block. This is very neat, and it all comes about because of the clever way we engineered *RADO* to be a sort of complement to the canonical 2-block. So is there a cubic version of *RADO*? A quartic? Quintic? Yes, and we find them by developing further the idea of a maximal quasiorder disjoint from a block.

[*HOLE copy this in from BQOpaper3.tex*]

The principle that "everything that happens, happens on a countable set" (see e.g. proposition 28 or the discussion on p. 29) means that all the facts about RADO analogues of finite exponent (the $RADO(n)$ for finite n) and the power set operation $\mathcal{P}$ work also for the $RADO(n)$ and the operation $\mathcal{P}_{\aleph_1}$ that sends an argument to the set of its countable subsets. Not surprisingly much the same goes for the operation sending a quasiorder $\langle Q, \leq_Q \rangle$ to to the quasiorder of Q-streams under stretching.

Theorem 33 *(Marcone-Pouzet (??)) If $\langle Q, \leq_Q \rangle$ is an ω^n-good quasiorder then $Q^{(\omega^n)}$ is WQO.*

Proof: Suppose $\langle Q, \leq_Q \rangle$ is ω^n-good. Let's show that Q^{ω^n} is WQO.

Suppose $\langle f_i : i < \omega \rangle$ is a bad sequence from Q^{ω^n}. Ask yourself, why does $f_i \not\leq_l f_j$? It's because there is a big value of f_i that happens too soon. Notice that the greedy algorithm for finding a map $\alpha \to \alpha$ that witnesses $f \leq_l g$ will find such a map if there is one [*HOLE a point to be made about closed sets again?*] and if it fails there is a first point at which it fails: this point is the *excrescence*. So for each $i < j$ we have $f_{i,j}$ which is an initial segment of f_i with a last element. $f_{i,j}$ is the shortest initial segment of f_i on which the greedy algorithm fails and is the shortest initial segment of f_i which $\not\leq_l f_j$.

The idea is that we now run, on $f_{i,j}$ and $f_{j,k}$, the process we ran on f_i and f_j, and so on, until we get down to sequences of length 1, which is to say, members of Q. Then we argue that the rank of the block indexing this bad array of members of Q will be ω^n at most.

But for this to work we have to be sure that the process of cutting down functions by taking initial segments will halt, and will halt with a member of Q. The process must indeed halt, because ω^n is an ordinal, and so the lengths of these initial segments are also ordinals, and any descending sequence of ordinals is finite. Can we be sure that the process halts with a member of Q at each point in the array? What is to preclude the possibility of an $f_s \not\leq f_t$ where the excresence that prevents $f_s \leq f_t$ is the last member of f? Nothing, apparently. But it turns out that in these circumstances we can discard everything in f_s except the excresence, and still have a bad array on Q. What we have to do now is prove that the rank of the block indexing this array is bounded somehow by ω^n.

Consider the (downward-branching) tree of truncated fs: where f_s is below f_t if s is an end-extension of t. By induction on the rank of this tree we can prove that $\rho(f_s) \leq \text{length}(f_s)$. [*HOLE finish this off properly. The result is goo, by repeated applications of proposition ??? and the fact that $(Q^\omega)^\omega$ has fewer ordered pairs than Q^{ω^2}*] ■

[*HOLE This is actually the version for infinite exponent. Duplicate and simplify*]

[*HOLE notation for length of a sequence*]

One construct that will be of interest later is the operation that takes a WQO and returns the (downward-branching) tree of bad sequences ordered by (reverse) end-extension. The empty sequence is at the top, and the end-extensions of any sequence s come below s. If we started with a WQO the tree that this construction gives us must be wellfounded, and must therefore have a rank. It will be of interest later because some very long ordinals can arise in this way from quite humble beginnings. If we do this to the best-behaved WQOs, like wellorderings, nothing happens. If we consider the WQO of ordinals below α, for example, the bad sequences are just the descending sequences, and in the tree the rank of each descending sequence is simply its last member, so the rank of the tree is just α.

The RADO structure is interesting in this connection because although it has rank ω (every pair has finite rank) the rank of the tree of bad sequences is transfinite.

Given a bad sequence s, what is its rank? The first thing to notice is that the possible ways of extending a bad sequence to a longer bad sequence depend only on the members of the sequence, and not on the order in which they appear. [*HOLE Notice that this doesn't hold for finite bad *arrays*! The order matters there*] So the same goes for the rank. A bad sequence s excludes the ith ray for all $i >$ second component of any pair in s. So we think about the rays that aren't excluded. A ray may be *entirely available* in the sense that any element of it may be placed on the end of s, or it may be only *finitely available*, because some member of it is already in s. As soon as we add to s an element from a ray that is entirely available, it becomes merely finitely available, and will be used up in finitely many steps. But the first time we pick an element from a ray, we can pick it as late as we like, and thereby choose at that stage how large the finite number of steps is to be, So each entirely available ray represents something that raises the rank by ω. So a first stab at the rank of s will be: $\omega \cdot n$ where n is the number of entirely available rays.

So here's how to compute the rank of a bad sequence. Discard from the carrier set of $RADO$ every unordered pair that is $\geq_{RADO}$ a pair in s. We now have finitely many rays (or initial segments of rays) left. Count ω for each ray that is entirely available, and count n for a finitely available ray, where n is the number of pairs available in that ray. Now add up the numbers pertaining to the rays, but *from right to left*. The result is the rank of s.

Like the rank of RADO, the rank of $RADO(n)$ is ω, whatever n is. What is the rank of the tree of bad sequences, ordered by (reverse) end-extension, of $RADO(n)$?

5.2 Finite exponent stuff to be ironed out

Must mention somewhere that the rank of the RADO^n is precisely ω.

The following exercise is strongly recommended as a preparation for the harder analogues of infinite exponent that await us.

Look back at proposition 17.

EXERCISE 21

(i) *Show that substructures and homomorphic images of ω^n-good quasiorders are ω^n-good.*
(ii) *Prove the analogue of proposition 17 part (vi) for ω^n-good quasiorders.*

Notice that the counterexamples of proposition 18 establish also that the class of good quasiorders of finite exponent isn't closed under direct limit or inverse limit either. However it is closed under power set.

EXERCISE 22 *Prove analogues of the perfect subsequence lemma (lemma 15) for ω^n-good quasiorders, and use it to establish the analogues of the later parts of proposition 17, namely (iii) that the product of finitely many ω^n-good quasisorders is ω^n-good; (iv) that the intersection of (the graphs of) two ω^n-good quasiorders on the same carrier set is ω^n-good, and (v) that a disjoint union of finitely many ω^n-good quasiorders is ω^n-good.*

Also minimality of RADO^n.

5.3 Loose ends

The key fact about ω-good quasiorders (WQOs) is lemma 19 that Q is WQO iff our favourite lift of infinite character is wellfounded. We want to generalise this to higher finite exponents, so that we prove something like:

LEMMA 34 *Let K be one of the constructors: power set, set-of-countable subsets, or streams. Then, for all quasiorders Q and all $n, m \in \mathbb{N}$:*

$$K^n(Q) \text{ is } \omega^{m+1}\text{-good iff } K^{n+1}(Q) \text{ is } \omega^m\text{-good.}$$

This is easy for power set and for set-of-countable-subsets, and these two cases are left as a morale-building exercise for the reader. Streams are another matter. We should expect this: stretching on streams is a stricter order than $\leq^+$ on countable subsets so we must expect to have to work harder to show that if Q is ω^{n+1}-good then Q-streams are ω^n good (than we have to work to infer that countable subsets are ...)

However, one direction at least is easy. Suppose f is a bad quadratic Q-array. Then the rays of f form a bad sequence of Q-streams. This shows that Q^ω ω-good implies Q ω^2-good.[3] It's the other direction that we must expect to be hard. Given a bad sequence of Q-streams we get a bad quadratic array of Q-lists, and we need a binary version of lemma 23 to complete the circle. (This shows that lemma 23 is not just a cute fact, but an important structure theorem, that is part of a large theorem that sez that streams behave like countable subsets)

Now how are we to prove the n-ary version of lemma 23? If we are to prove it the way we proved the original theorem we will need a notion of a minimal bad array. This is not as easy as it might sound.

Let us start with the challenge of proving that Q-lists under stretching are ω^2-good if Q is ω^2-good.

We could try establishing a chain of biconditionals:

$Q^{<\omega}$ is ω^2-good iff (by lemma 34)

$\mathcal{P}(Q^{<\omega})$ is ω-good iff

$(\mathcal{P}(Q))^{<\omega}$ is ω-good iff (by lemma 34)

$\mathcal{P}(Q)$ is ω-good iff (by lemma 19)

Q is ω^2-good.

The problem is with the inference from the second line to the third. There is no obvious reason why $\mathcal{P}(Q^{<\omega})$ and $(\mathcal{P}(Q))^{<\omega}$ should be iso, nor even that $(Q^{<\omega})^\omega$ and $(Q^\omega)^{<\omega}$ should be iso, or one be WQO iff the other is. Come to think of it, why should $(Q^\alpha)^\beta$ be isomorphic to $(Q^\beta)^\alpha$? Ordinal multiplication is not commutative, after all. But it turns out that that is not the reason, as $(Q^\omega)^\omega$ and Q^{ω^2} are not even always isomorphic!

(It's a miniexercise to see that although $(Q^\omega)^\omega$ being WQO obviously implies that Q^{ω^2} is WQO there is no reason to believe the converse.[4])

[3] Readers with an eye for constructive proofs will have noticed that we proved this implication by proving its contrapositive. This is typical in WQO theory!—MOVE THIS REMARK EARLIER

[4] Let f, g be ω^2-sequences from Q. f might stretch into g considered as an ω^2 sequence but not if both are considered as streams of streams. Suppose, for each n, that the head of the nth stream fits into the $2n$th stream of g, and the tail of the nth stream from f stretches into the $2n+1$th stream of g. Then f, considered as an ω^2-sequence, stetches into g, considered as an ω^2-sequence, tho' perhaps not when they are considered as streams of streams. There is a failure of currying, so the exponential notation is misleading.

So it looks as if we will have to prove it directly, by finding the correct generalisation of lemma 23: specificially, by establishing an analogue of the minimal bad sequence argument used to prove the original version.

5.3.0.1 Minimal bad arrays of finite exponent

We suppose that $\langle Q, \leq_Q \rangle$ is ω^2-good, and that Q-lists are not. We then start working on a minimal bad array f of Q-lists, where what we mean by 'minimal bad' will emerge later. We have done exercise 22[5] so we are armed with a perfect subarray lemma, which we use to find a subarray of lists whose heads form a perfect array. Let this subarray be g. Then we argue that the array of the tails of the lists in g cannot be bad. We can get a handle of what the correct concept of minimal bad array is by consider the senses in which g is less than f. Certainly every ray g_i of g, considered as an ω-sequence of Q-lists, is less than the corresponding ray f_i of f in the pointwise product order. That is to say, $(\forall n)(g_i(n) <_Q f_i(n))$

So we want a minimal bad array lemma that says that if there are bad arrays of Q-lists then there is a bad array f such that whenever g is an array s.t. every ray of g is $<$ some ray of f in the pointwise product order, then g is not bad. Now can we prove such a lemma?

To do so, we need an analogue of the greedy algorithm for constructing MBSs. How about this? Let the first ray of f be a ray that, among those rays that are the first ray of an infinite bad array of Q-lists, is minimal under pointwise product. Then the second ray is one that among those rays that are the second ray of an infinite bad array of Q-lists beginning with our choice of the first ray, is minimal under pointwise product. And so on.

Now we have to show that g is not a bad array. The obvious thing to try is a generalisation of the proof of lemma 22. We try to prove that if f is an output of the MBA greedy algorithm, and g is an array s.t. every ray of $g <$ some ray of f in the pointwise product, then g is not bad.

Assume g is bad. We attempt to prove by induction on n that no ray of $g <$ (under pointwise product) any of the first n rays of f. OK for $n = 1$, since if the ith ray of $g <$ the first ray of f then the greedy algorithm would have picked the ith ray of g instead. How about later n?

Assume true for $1, 2 \ldots n-1$, and suppose that the ith ray of $g <$ the

[5] And if we haven't we can look up the model answer!

nth row of f under pointwise product. Why did the greedy algorithm not pick the ith ray of g at stage n? Could it be that the array that kicks off with the first $n-1$ rays of f and then continues with g, starting with the ith ray of g, isn't bad? If so, there must be a "good pair", and it must be that one of the fs can see one of the gs. As it might be, $f(\langle 1, 10\rangle) \leq_{Q^{<\omega}} g(\langle 10, 33\rangle)$. But $g(\langle 10, 33\rangle)$ is the tail of $f(\langle 10, 33\rangle)$ so we have $f(\langle 1, 10\rangle) \leq_{Q^{<\omega}} f(\langle 10, 33\rangle)$, contradicting badness of f. So, on the assumption that the ith ray of $g <$ the nth row of f under pointwise product, the greedy algorithm could have picked the ith ray of g at stage n, and should not have picked the nth ray of f. So this assumption was wrong, and we have the contradiction we wanted.

The induction works, so what have we proved? No ray of g is strictly below (under pointwise product) any ray of f. But this contradicts the fact that rays of g are strictly below the corresponding rays in f. So g was not bad after all.

We complete the proof that f was not bad in the same way as in the proof of lemma 23.

Exercise 23 *What would go wrong if in the MBA construction we had used minimality under stretching rather than the pointwise product?*

5.4 Diestel's lemma or something like that

[12]

So far we have been considering the results of lifting quasiorders on a set X to $\mathcal{P}(X)$, or $\mathcal{P}_{\aleph_1}(X)$ and beyond. We obtain somewhat simpler structures by taking not the set of all (countable or not) subsets of X but all *directed* (countable or not) subsets of X—and beyond.

In general we cannot expect a WQO to be a chain-complete quasi-order. Even such a well-behaved WQO as $\langle \mathbb{N}, \leq_{\mathbb{N}}\rangle$ is not. What do we have to do to obtain a chain-complete quasi-order from an arbitrary quasi-order $\langle Q, \leq_Q\rangle$? (Miniexercise: think a bit about how the definition of chain-complete quasiorder might differ from that of a chain-complete poset) Clearly the first thing we must do is add sups to all chains. We can do this concretely by taking the quasiorder of chains in Q and ordering them by $\leq^+$, and it is not hard to see that this operation is idempotent, up to isomorphism. (This is the usual trick used to deduce Zorn from the assertion that every chain-complete poset has a maximal element.) The same goes for the operations involved in the endeavour to find the completion of Q with respect to sups of countable chains, or of countable directed subsets, or of arbitrary directed subsets: all these operations are idempotent up to isomorphism.

Miniexercise: Complete $RADO$ with respect to countable chains. What is the result?

If we think about trying to add elements to an arbitrary quasiorder $\langle Q, \leq_Q \rangle$ to obtain a countably complete quasiorder (one wherein every countable subset has a sup) we notice that the operation is not idempotent.

Miniexercise: verify that to obtain a quasiorder from $RADO$ that is countably complete one has to add sups of countable chains *twice*. And that to to obtain a quasiorder from $RADO(3)$ that is countably complete one has to add sups of countable chains *thrice*.

[*HOLE Careful. If one wants a chain-complete quasiorder one only has to add sups of chains once - i think! A sup for each ray does it i thihk - but check it. to make it complete, yes, you have to do it twice i agree*]

In $RADO$ the rays form a countable family of subsets, each of them directed, and they must all have different sups, so we can find a countable antichain in the countable-chain-completion of $RADO$. Significantly these rays, altho' they are each directed, are not maximal directed subsets: $RADO$ itself is a maximal directed subset: in $RADO$ there is a unique maximal directed subset, namely $RADO$ itself. There is a striking result of Diestel's that says that any quasi-ordering has a unique decomposition into maximal directed subsets.

burble: don't need the whole thing

Let us say that a subset $Q' \subseteq Q$ is **incompatible** if no two members of Q' have an upper bound. Clearly, if $\langle Q, \leq_Q \rangle$ is WQO then there can be no infinite incompatible subset, as every incompatible subset is an antichain. But we can do more than this. Suppose there were incompatible subsets of arbitrarily large finite size, then by countable choice we could pick an ω-sequence of incompatible subsets of strictly increasing size. Now if $\langle Q, \leq_Q \rangle$ is WQO, so is $\langle \mathcal{P}_{\aleph_0}((Q), \leq^+ \rangle$ by lemma **??** so this sequence must be $\leq^+$-good. Without loss of generality we can even take it to be perfect. Now let Q_i and Q_j be incompatible subsets from this perfect sequence, with $i < j$. No two things in Q_i can be below any one thing in Q_j. So if we draw an edge between things in Q_i and things in Q_j by putting an edge between $\leq$-comparable elements we find that for $i < j$ there must always be either (i) something in Q_j $\not\geq_Q$ anything in Q_i or (ii) something in Q_i is dominated by more than one thing in Q_j. By means of Ramsey's theorem we can cut down to a subsequence such that (i) always happens or (ii) always happens. (If (ii) is not blindingly obvious, just remember that the perfect binary tree (upward-branching version) is not WQO.) Either way we can extract an antichain in Q from such a sequence.

So we have proved

REMARK 35 *(Diestel)*

If $\langle Q, \leq_Q \rangle$ is a WQO then there is a finite number n such that if $Q' \subseteq Q$ is incompatible the $|Q'| \leq n$.

One might call this number the **Diestel** number of a WQO.

EXERCISE 24

Let $\langle Q, \leq_Q \rangle$ and $\langle R, \leq_R \rangle$ be WQOs. In terms of the the Diestel numbers of $\langle Q, \leq_Q \rangle$ and $\langle R, \leq_R \rangle$ what are the Diestel numbers of (i) $Q \sqcup R$; (ii) $Q \times R$; (iii) $\mathcal{P}(Q)$; (iv) $\mathcal{P}_{\aleph_0}((Q)$; (v) $\mathcal{P}_{\aleph_1}((Q)$? (vi) What is the Diestel number of $RADO(n)$?

exploit the 2-ladder p 39.

6

BQOs

If a man can build a better quasiorder, the world will beat a path to his door.[1]

Ralph Waldo Emerson *Voluntaries*

That man was Crispin Nash-Williams.

At this point we could direct our attention to the class of quasiorders $\langle Q, \leq \rangle$ such that, for all $n \in \mathbb{N}$, the result of doing $+$ n times to it is a WQO, the class we have been calling "good quasiorders of finite exponent", and notice that this class is closed under $+$, unlike the class of WQOs. This would give us a definition of a distinguished class of WQOs as the largest class of WQOs closed under $+$, and one could hope that this would turn out to be the resting place for this intuition for tidying up the definition of WQO. However we have to wring this idea out a little further, since there remains much to be gained by considering transfinite iterations under $+$. This is because one will then be able to generalise the array condition to something that has no finite bound on the length of the sequences. The class of WQOs thus obtained will have even nicer closure properties than the class of good quasiorders of finite exponent.

[*HOLE It would be nice to have an example of of some constructor of infinite character under which the class of good quasiorders of finite exponent is not closed. I bet its infinite trees*]

However, to do this, we need to consider expressions like $\mathcal{P}^{\alpha}(Q)$ where α is a transfinite ordinal. One can hardly imagine a better reason for stopping with the ideas of the preceding paragraph than the obvious fact that such a notation, *prima facie* at least, simply makes no sense. How

[1] With apologies to Emerson. The correct quotation is "If a man can write a better book, preach a better sermon, or make a better mousetrap than his neighbour, though he build his house in the woods, the world will make a beaten path to his door"

can it, when the $\mathcal{P}^n(Q)$ were all be taken to be formally disjoint? If we wish to iterate $+$ and $\mathcal{P}$ transfinitely we need the $\mathcal{P}^n(Q)$ to be somehow *cumulative* not disjoint. Given X and Y, both subsets of $\bigcup\{\mathcal{P}^n(Q) : n \in \mathbb{N}\}$, how are we to compare them with respect to the quasiorder we will eventually call $\leq_\infty$? We will have to be able to compare everything in X with everything in Y, and this means comparing things from $\mathcal{P}^n(Q)$ and $\mathcal{P}^m(Q)$ for $m \neq n$. To make sense of this it will be sufficient to identify, once and for all, every element of Q somehow with a subset of Q, for then we can propagate this identification up the cumulative hierarchy of sets built up from Q. Now although there are many ways in which this identification can be done, there is one way that is obviously the simplest, namely to identify each $q \in Q$ with its singleton. Objects identical to their own singletons are called **Quine atoms**. Making this identification has the great advantage that when asking whether or not $q \leq q'$ under the new dispensation (in which every $q \in Q$ is simultaneously a subset of Q and a member of Q) it doesn't make any difference whether we think of q and q' as subsets of Q or elements of Q, since for all q and q' we always have $q \leq q'$ iff $\{q\} \leq^+ \{q'\}$. Notice also the important triviality that if f is an injection from $\langle Q_1, \leq_{Q_1}\rangle$ into $\langle Q_2, \leq_{Q_2}\rangle$ then $\lambda x.f``x$ is an injection from $\langle \mathcal{P}(Q_1), (\leq_{Q_1})^+\rangle$ into $\langle \mathcal{P}(Q_2), (\leq_{Q_2})^+\rangle$. Putting these two together enables us to think of $\langle \mathcal{P}^m(Q), (\leq_Q)^{m+}\rangle$ as an end-extension of $\langle \mathcal{P}^n(Q), (\leq_Q)^{n+}\rangle$ whenever $n \leq m$.

It is not customary to write '$\mathcal{P}^\alpha(Q)$' to be the result of applying the power set operation α times to a set Q, taking unions at limits to keep things cumulative. In these circumstances it is customary to use the letter 'V' instead, thus:

DEFINITION 36 $V_0(Q) = Q$; $V_{\alpha+1}(Q) =: \mathcal{P}(V_\alpha(Q))$, *taking unions at limit ordinals.*

Then $V_\Omega(Q)$ is the union of all the V_α. $V_\Omega(Q)$ sometimes called the **Zermelo Cone** over Q.)

$\lambda X.(\mathcal{P}(X) \cup Q)$ is thus a monotone function from the complete poset $\langle V, \subseteq\rangle$ into itself. Theorem **??** now tells us that this operation has a greatest and a least fixed point. The least fixed point is of course $V_\Omega(Q)$. The greatest fixed point we notate '$V(Q)$'. $V_\Omega(Q)$ is the wellfounded part of $V(Q)$.

The $+$ operation on quasiorders now becomes a monotone function from the complete poset of quasiorders of $V(Q)$, partially ordered by inclusion, into itself. This too will have greatest and least fixed points, both of which we will notate '$\leq_\infty$'.

Now why might this be a natural thing to do? And how far should we go, now that we can iterate transfinitely? Over all ordinals, as my invocation of $V_\Omega(Q)$ apparently portends? The legions of the squeamish will complain that $V(Q)$ might not be a set.

It was with just this pending problem in mind that I prepared the ground earlier (p. 29) by making the point that if there is a bad sequence of subsets of X under $\leq^+$, then there is a bad sequence of *countable* subsets of X under $\leq^+$, and indeed, for each n if there is a bad sequence of elements of $\mathcal{P}^n(X)$ there is a bad sequence of (countable sets)^n of members of X. That means that whatever new mathematics we discover by iterating the power set operation, we can discover by iterating instead the set-of-all-countable-subsets operation. This is much less problematic. If we repeatedly apply the function $\lambda X.(\mathcal{P}_{\aleph_1}(X) \cup Q)$, starting at $\emptyset$, and take unions at limits, we will reach a fixed point after ω_1 steps. This (least) fixed point is notated $H_{\aleph_1}(Q)$, and is the **hereditarily countable sets over** Q. Now, by thinking of Q as a set of Quine atoms as before, we can lift $\leq_Q$ transfinitely often by $+$ to be defined on the whole of $H_{\aleph_1}(Q)$. To be a bit more precise, we consider the complete poset of quasiorders of $H_{\aleph_1}(Q)$ that extend $\leq_Q$, ordered by inclusion, and note that $+$ is a monotone function from this poset into itself, and must have a fixed point. It is this fixed point that interests us. The more adventurous can relax and accept the application of this process to the poset of quasiorders of $V(Q)$ ordered by inclusion. We will use the same notation—$\leq_\infty$—for both these quasiorders. (The first is simply the restriction of the second to $H_{\aleph_1}(Q)$.)

(It is at this point—where we claim that $\leq_\infty$ can be defined on the whole of $V(Q)$—that we use the fact that '$+$' is being applied to quasiorders not to partial orders: the collection of partial orders of $V(Q)$ is not a complete lattice under inclusion but only a chain-complete poset, and we cannot appeal to Tarski-Knaster. [**?**])

Armed with the concepts of $H_{\aleph_1}(Q)$, $V(Q)$ and $\leq_\infty$, we can now define a more robust concept than wellquasiordering. A quasi-ordering $\langle Q, \leq_Q\rangle$ was ω^n-good if the result of lifting $\leq_Q$ n times by $+$ to $\mathcal{P}^n(Q)$ was WQO. Or, which is equivalent by lemma 34, if the result of lifting $\leq_Q$ $n+1$ times by $+$ to $\mathcal{P}^n(Q)$ is wellfounded.

We now say

Definition 37

A quasi-ordering $\langle Q, \leq_Q \rangle$ is BQO if $\langle H_{\aleph_1}(Q), \leq_\infty \rangle$ is wellfounded

Of course this is equivalent to $\langle V(Q), \leq_\infty \rangle$ or $\langle V_\Omega(Q), \leq_\infty \rangle$ being wellfounded (or indeed BQO-ed!!)

But we *still* haven't given a proper definition of block!

6.0.1 Blocks and Games

[*HOLE put discussion of games and fixed points in intro chapter?*]

It is nowadays widely understood that there is a connection between greatest fixed points and open games, and we can indeed characterise $\leq_\infty$ by means of a game, and the game will give us the correct definition of block and a combinatorial definition of BQO that is in the same format as the definition of ω^n-good quasiorder.

Things in $V(Q)$ can be thought of as downward-branching trees (possibly with infinite branches) all of whose leaves are labelled with members of Q. (They satisfy various extensionality conditions which it is not illuminating to dwell on here.)

The game $G_{X \leq_\infty Y}$ is played as follows.

`false` picks a member X' of X, `true` picks a member Y' of Y. If their two choices are both in Q, `true` wins if $X' \leq_Q Y'$ and `false` wins if not. If neither of them are in Q they continue, playing $G_{X' \leq_\infty Y'}$. (If one is in Q and the other isn't then we procede as if neither were: since we have identified each $q \in Q$ with $\{q\}$ we can take elements of Q to be subsets of Q when this is necessary—as now.) If the game goes on forever player `true` wins. The game is open so, by **??**, one or the other player has a winning strategy.

If `false` has a winning strategy in $G_{X \leq_\infty Y}$—and plays according to it!—the play will end with player `true` picking a member of Q.

Let us say $X \leq_\infty Y$ iff player `true` has a winning strategy.

We are now going to turn our attention to identifying those WQO's $\langle Q, \leq_Q \rangle$ such that $\langle V(Q), \leq_\infty \rangle$ is also a WQO. It will turn out that they have a nice combinatorial characterisation.

We start out by noticing that if x is an illfounded member of $V(Q)$ then $(\forall y \in V(Q))(y \leq_\infty x)$. This means that $\langle V(Q), \leq_\infty \rangle$ has (up to equivalence) only one more element than $\langle V_\Omega(Q), \leq_\infty \rangle$. This is not going to make one a WQO when the other is not. Accordingly we can restrict our attention to $\langle V_\Omega(Q), \leq_\infty \rangle$.

Now suppose that $\leq$ wellquasiorders Q but $\leq_\infty$ does not wellquasiorder $V(Q)$. Let us see if we can simplify this to something sensible.

We start with a bad sequence $\langle X_i : i \in \mathbb{N}\rangle$ of members of $V(Q)$. Some of these elements might be members of Q. They cannot all be, because Q is WQO by $\leq$, by hypothesis. We are going to leave alone all X_i that are in Q, and elaborate the others until they, too, turn into members of Q. (The complication in this transfinite case is that we do not know in advance how often we are going to have to unwrap each set).

Start off with $\{X_i : i \in \mathbb{N}\}$, and a digraph which initially is simply the usual wellordering on $\mathbb{N}$, so there is an arrow from X_i to X_j iff $i < j$. We will make ω passes.

When we consider x_s we first check to see if it is a member of Q. If it is, it is then **ratified** which means it will never be replaced. If it is not a member of Q life is a bit more complicated. For each X_t such that there is an arrow from X_s to X_t we choose a member of X_s that is not $\leq_\infty$ anything in X_t, and we call it $X_{(s;t)}$ (for the moment at least). We discard X_s and redirect all arrows ending at X_s to $X_{(s;t)}$ (so we replace each old arrow by a host of new ones) and we replace the arrow from X_s to X_t with a new arrow from $X_{(s;t)}$ to X_t.

After ω passes everything has been ratified or discarded. The wellfoundedness of $\langle V_\Omega(Q), \in\rangle$ ensures that there can be no infinite sequence of Xs with later subscripts always end-extensions of earlier subscripts.

The subscripts are a bit of a mess at the moment: every subscript is an ordered pair of earlier subscripts. Notice that at stage one the only new subscripts we construct are pairs of natural numbers where the first component is smaller than the second, and the only new arrows we generate are things like $X_{(1;3)} \not\leq_\infty X_{(3;5)}$. So there must be a member of $X_{(1;3)}$ that $\not\leq_\infty X_{(3;5)}$ and we call it $X_{((1,3);(3,5))}$. Since this is the only way we can invent new things at this level, we might as well rewrite it as '$X_{1,3,5}$' to remove the duplication of the '3'. The second component of the first pair and the first component of the second pair are always the same!

Now for what subscripts s do we know that $X_{1,3,5} \not\leq_\infty X_s$? (All arrows going *into* $X_{1,3,5}$ arose from arrows going into $X_{1,3}$.) The only arrows going *from* $X_{1,3,5}$ go to $X_{3,5}$ in the first instance, and thereafter to things with subscripts that are end-extensions of $\{3, 5\}$ should $X_{3,5}$ not be a member of Q and have to be replaced.

The upshot is that we can take subscripts to be increasing finite sequences of natural numbers, and we only ever arrange for an arrow from X_s to X_t when t is an end-extension of the tail of s.

This will lead us to the correct definition of block.

Now consider a set S of finite sequences from $\mathbb{N}$ that arises from a bad

Q-sequence in this way. We will show that every increasing ω-sequence from $\mathbb{N}$ has a unique initial segment in S. Let $f : \mathbb{N} \to \mathbb{N}$ be increasing. Is $\langle f(0)\rangle$ in S? It will be if $X_{f(0)} \in Q$, and if that happens we are done. If $\langle f(0)\rangle$ is not in S this must be because $X_{f(0)} \notin Q$ and in these circumstances we have discarded $X_{f(0)}$ and replaced it by the infinitely many $X_{f(0),j}$ for $j > f(0)$. In particular we will have done this for $j = f(1)$. So is $\langle f(0), f(1)\rangle \in S$? It will be unless $X_{f(0),f(1)} \notin Q$. In those circumstances we discarded $X_{f(0),f(1)}$ and replaced it by each of $X_{f(0),f(1),j}$ for $j > f(1)$. And so on. Eventually we hit a member of Q and at that point we have an element of s that is an initial segment of f. Notice that we only ever put into S a sequence s if we have already discarded all initial segments of s, so that the initial segment in S of our infinite sequence is unique.

[*HOLE Rearrange:*

(Finally we will need a notation for the subscript we invent when we find something in virtue of which $q_s \not\leq q_t$, and we will write it $q_{s;t}$. Thus, for example, $q_{1,3} \not\leq q_{3,4}$ because of $q_{1,3,4}$.)

At some point must make a fuss about the unique-initial-segment condition. This is what enables us to generalise (using Open Ramsey) all the theorems using finite-exponent-ramsey.

]

This motivates the definitions which follow.

Definition 38 *A* **block** *is a set B of strictly increasing finite sequences of naturals with the property that every strictly increasing ω-sequence of natural numbers has a unique initial segment in B. We write $s \lhd t$ if t is the tail of an end-extension of s.*

Notice also $\lhd$ is not transitive except in the unique case of a block all of whose elements are singletons.

(We really do mean "tail of an end-extension" not 'end-extension of the tail': which would allow $\langle 3\rangle \lhd \langle 2\rangle$.)

Notice that this agrees with our picture of the canonical n-blocks for finite n.

Exercise 25 *Look back at exercise 18. Extend the results of that exercise from n-blocks to arbitrary blocks.*

Exercise 26 *Given the way blocks arise from sifting, there should be a declaration of a recursive datatype of blocks. Give one.*

We also need the notion of a **derivative** of a block. You remember the construction of a bad array on X from a bad sequence of subsets of X? What this construction does is accept as input a bad array on the power set of X indexed by a block B, and returns a bad array on X indexed by the derivative of B. That's what a derivative is. (Thus for example if B contained $\{1, 10\}$ and $\{10, 15\}$, then the derivative would contain $\{1, 10, 15\}$.[2])

Thus $D(B) := \{\mathtt{hd}(b)::b' : b \lhd_B b'\}$.

[*HOLE What is the lexicographic length of $D(B)$ as a function of the length of B??*]

However it is equally natural to wellorder blocks by the lexicographic ordering, and this is more informative, in the sense that blocks can get lengths other than ω under this scheme. For example, the graph of $<$ on $\mathbb{N}$ is a block of length ω^2. For B a block, we write '$|B|$' for the length of B in the lexicographic order. ('$|\langle B, \lhd_B \rangle|$' might be a bit misleading co's it doesn't depend on $\lhd_B$ but only on the internal structure of the carrier set B thought of as a set of tuples of natural numbers. We will wrie it for the length of B where drawing attention to $\lhd_B$ helps.)

A quasiorder that has no bad arrays indexed by blocks of length $< \alpha$ is said to be α-good. This is consistent with our earlier usage of the word 'good' (see p. 43).

[*HOLE Some blocks have the feature that they are iso to all their proper sublocks. $\mathbb{N}$, for example, RADO too. Is this true of all blocks? This has some bearing on the universal embedding property of generalisations of RADO*]

So, in the first instance, a block is a set of increasing sequences of natural numbers with special properties. If we rub out the tuple information and just keep the graph information, so that we think of a block as $\langle B, \lhd_B \rangle$ where the elements of the carrier set have no internal structure, can we recover the tuple information? It turns out that we can, and we do it as follows.

Exercise 27

Whatever else it is, a block $\langle B, \lhd_B \rangle$ is, at the very least a wellfounded binary structure of height precisely ω, so it has a rank function ρ.

(This seems a non-sequitur)

(i) *If you are given a block $\langle B, \lhd_B \rangle$, show how to ascertain from $\lhd_B$ what tuples of natural numbers the elements of B must be.*

[2] Marcone ([**?**] p645 calls this new block B^2.

(ii) *(For logicians only) Why is there no first-order theory of blocks?*

Now we can give the combinatorial definition of BQO, the one that uses blocks:

DEFINITION 39 *Let $\langle Q, \leq_Q \rangle$ be a quasiorder and B a block. A map $f : B \to Q$ is an* **array**. *An array is* **good** *if there are $s \lhd t \in B$ such that $f(s) \leq_Q f(t)$.*

Then $\langle Q, \leq_Q \rangle$ is a **better-quasiorder** *(hereafter "***BQO***") iff for every block B every array $f : B \to Q$ is good.*

THEOREM 40 *The two definitions of BQO, 37 and 39 are equivalent.*

Proof: The definition of block was cooked up precisely to make this true. ∎

EXERCISE 28 *Let $\langle Q, \leq \rangle$ be a quasiorder such that for all $q \in Q$, $\{q' \in Q : q \not\leq q'\}$ is finite. Must $\langle Q, \leq \rangle$ be BQO?*

There are some further equivalences we should take note of, and they will follow from the following observations (all either easy or established previously) that (i) $H_{\aleph_1}(Q)$ is identical to the set of its countable subsets, and (ii) $\langle \mathcal{P}_{\aleph_1}(Q), \leq_Q^+ \rangle$ is wellfounded iff $\langle Q, \leq_Q \rangle$ is WQO. (iii) $\langle H_{\aleph_1}(Q), \leq_\infty \rangle$ is WQO iff it is wellfounded. (CHEXCK THIS: IS THIS WHAT I MEANT??)

THEOREM 41

The following are equivalent for a quasiorder $\langle Q, \leq_Q \rangle$:

(i) $\langle Q, \leq \rangle$ is BQO;

(ii) $\langle V_\Omega(Q), \leq_\infty \rangle$ is wellfounded;

(iii) $\langle H_{\aleph_1}(Q), \leq_\infty \rangle$ is wellfounded;

(iv) The free countable completion is wellfounded. [*HOLE stuff missing here*]

Proof: We have already shown (i) implies (ii), and (ii) implies (iii) because any substructure of a wellfounded quasiorder is wellfounded. It remains only to show that (iii) implies (i). We will prove the contrapositive.

Suppose $\langle Q, \leq \rangle$ is not BQO. Then there is a block S_0 and a bad array $f : S \to Q$. The idea is now to "reverse-sift" this bad array (on Q) into a bad array (on $H_{\aleph_1}(Q)$) on a shorter block. We extend f to a map f^* defined on the set S^* of all initial segments of sequences in S by recursively setting $f^*(s) =: \{f^*(t) : s = \texttt{butlast}(t)\}$. This recursion can fail only if there is an infinite sequence $\{s_i : i \in \mathbb{N}\} \subseteq S$ where for all i, S_i is an initial segment of s_{1+1}. This is impossible by the "unique initial segment' clause in the definition of block.

We then find that if S' is a subset of S^* that is a block, then the restriction of f^* to S' is a bad array on $H_{\aleph_1}(Q)$. In particular f^* restricted to $\mathbb{N}$ is a bad sequence on $H_{\aleph_1}(Q)$. But a bad sequence in $\mathcal{P}_{\aleph_1}(X)$ always gives rise to an infinite descending sequence in X, and $H_{\aleph_1}(Q) = \mathcal{P}_{\aleph_1}(H_{\aleph_1}(Q))$, so $H_{\aleph_1}(Q)$ must be actually illfounded as well as not being a WQO.

■

We have seen this recursion in the proof of proposition 28. It gives us another way of associating a rank with a block. For any block B the recursion will produce a bad sequence in $H_{\aleph_1}(X)$ from a bad array $f : B \to X$, but it produces sequences in $H_{\aleph_1}(X)$ from arrays on X, be they bad or not. One can then ask about the (set-theoretic) rank of the sequence in $H_{\aleph_1}(X)$ that the recursion builds from an array $f : B \to X$. Clearly the rank of the sequence does not depend on the array map f but only on B. We can see this by turning B into a tree by closing under shortening. (see **??**. Order this set of finite tuples) by shortening, so that a sequence is preceded by all its end-extensions. This is wellfounded, because of the "every infinite sequence has a unique initial segment in B" clause in the definition of block. So the tree has a rank.

Miniexercise: what is the rank of the canonical n-block?

In fact we can strengthen theorem 41 further.

THEOREM 42 *If $\langle Q, \leq_Q\rangle$ is BQO, so is $\langle V(Q), \leq_\infty\rangle$.*

Proof:

Suppose there is a bad array over $V(Q)$. We will show how to refine it into a bad array on on Q. This is merely a more developed version of the process we applied to $\mathcal{P}^n(Q)$ earlier on.

Let $\{X_s : s \in B\}$ be a bad array over $V(Q)$. For each pair s, t in B with $s \lhd t$ we have $X_s \not\leq_\infty X_t$. Player `false` has a winning strategy $\sigma_{X_s \not\leq_\infty X_t}$ in the game $G_{X_s \leq_\infty X_t}$.

All the games $G_{X_s \leq_\infty X_t}$ will be played simultaneously. Indeed many *plays* of these games will be going on simultaneously. To be precise, there is a play for each infinite ascending $\lhd$-sequence.

It is convenient to describe what happens in terms of an ω-sequence of what one might as well call *passes*.

At the first pass, in each game $G_{X_s \leq_\infty X_t}$, `false` uses his strategy to pick a member of X_s. This will become $X_{s;t}$. At the second pass (and all subsequent passes) each play of $G_{X_s \leq_\infty X_t}$ multifurcates. At the first pass there was only one play of each game. For `false` to decide what to do as his second move in $G_{X_s \leq_\infty X_t}$ he deems `true`'s move in this game to be `false`'s move in $G_{X_t \leq_\infty X_u}$, for $t \lhd u$. Thus he deems `true` to have played $X_{t;u}$. Since he does this for *each* u such that $t \lhd u$, the one play of $G_{X_s \leq_\infty X_t}$ which was proceeding at pass one has become infinitely many. In each play he continues to use $\sigma_{X_s \not\leq_\infty X_t}$ and—since this strategy is winning—each play will terminate with a win for player `false`. This tells us that after ω passes every play of every game will have terminated in a win for player `false`.

Of course, since there is an entire bad array out there, we must expect to have to deal with $X_t \not\leq_\infty X_u$ for various u as well. For each game $G_{X_t \leq_\infty X_u}$ where $t \lhd u$ player `false` in that game uses his winning strategy to pick $X_{t;u}$. Player `false` in the game $G_{X_s \leq_\infty X_t}$ now has infinitely many replies to contend with, but he uses $\sigma_{X_s \not\leq_\infty X_t}$ to reply to each, and the play multifurcates, but `false` can continue to use $\sigma_{X_s \not\leq_\infty X_t}$ in each.

Since all the strategies $\sigma_{X_s \not\leq X_t}$ are winning for `false`, this process must halt with player `true` picking elements of Q. This gives us a bad array on Q.

■

This implies that $\langle Q, \leq_Q \rangle$ is BQO iff $\langle Q, \leq_Q \rangle$ belongs to the largest class of WQOs closed under the operation taking $\langle X, \leq \rangle$ to $\langle H_{\aleph_1}(X), \leq_\infty \rangle$. (Or, equivalently, to $\langle V(X), \leq_\infty \rangle$ or $\langle V_\Omega(X), \leq_\infty \rangle$.) Indeed, since $\langle Q, \leq_Q \rangle$ is WQO iff $\langle \mathcal{P}(Q), \leq^+ \rangle$ is wellfounded we can strengthen this to the remarkable

COROLLARY 43 *$\langle Q, \leq_Q \rangle$ is BQO iff $\langle Q, \leq_Q \rangle$ belongs to the largest class of wellfounded quasiorders closed under the operation taking $\langle X, \leq \rangle$ to $\langle H_{\aleph_1}(X), \leq_\infty \rangle$. (Or, equivalently, to $\langle V(X), \leq_\infty \rangle$ or $\langle V_\Omega(X), \leq_\infty \rangle$.)*

EXERCISE 29

If $\langle X, \leq \rangle$ is a quasiorder, define $\leq^{\aleph_0}$ on $\mathcal{P}(X)$ as in clause (vii) of exercise 6. Show that if $\langle X, \leq \rangle$ is a BQO, so is $\langle \mathcal{P}(X), \leq^{\aleph_0} \rangle$.

Hint: use the fact that any block, ordered colex, is of length ω.

If the cone above every element of $\langle X, \leq \rangle$ is cofinite, then $\langle X, \leq \rangle$ is a BQO (every point in a bad array has infinitely many things *not* above it) so in particular all (reflexive closure of) wellorderings are BQO; and if σ is an automorphism of a nonstandard model of arithmetic, the relation $\sigma(x) \leq y$ (or its reflexive closure) is also a BQO.

[*HOLE Let's have some more exercises here*]

Theorem 41 says that two definitions of BQO are equivalent. Some facts about BQOs are more easily proved for one definition than another. The definition in terms of blocks and arrays makes it very easy to show that any substructure of a BQO is BQO. The definition in terms of Zermelo cones makes it possible to prove that a disjoint union of two BQOs is BQO.

There are analogues of proposition 17 saying that every substructure or homomorphic image of a BQO is a BQO, and the proofs are exactly analogous.

PROPOSITION 44

.

Substructures of BQOs are BQO.
Refinements of BQOs are BQO.
Homomorphic images of BQOs are BQO.

Notice that the counterexamples of proposition 18 establish also that the class of BQO's isn't closed under direct limit or inverse limit either.

The following is very much harder.

LEMMA 45 *The disjoint union of two BQOs is BQO.*

Let $\langle A, \leq_A \rangle$ and $\langle B, \leq_B \rangle$ be two BQOs. We will take the dfn of BQOs in terms of good sequences on Zermelo cones.

Define by recursion two functions $\mathcal{D}_A : V_\Omega(A \sqcup B) \to V_\Omega(A)$ and $\mathcal{D}_B : V_\Omega(A \sqcup B) \to V_\Omega(B)$ as follows.

- for $a \in A$ set $\mathcal{D}_A(a) =: a$ and $\mathcal{D}_B(a)$ undefined;
- for $b \in B$ set $\mathcal{D}_B(b) =: b$ and $\mathcal{D}_A(b)$ undefined;
- Thereafter $\mathcal{D}_A(s) =: \mathcal{D}_A$“$s$ and $\mathcal{D}_B(s) =: \mathcal{D}_B$“$s$.

We then prove by induction on $\in \times \in$ on $V_\Omega(A \sqcup B)$ that for all s_i and s_j in $V_\Omega(A \sqcup B)$ we have $s_i \leq_\infty s_j \longleftrightarrow (\mathcal{D}_A(s_i) \leq_\infty \mathcal{D}_A(s_j) \wedge \mathcal{D}_B(s_i) \leq_\infty \mathcal{D}_B(s_j))$.

Now let $\langle s_i : i \in \mathbb{N} \rangle$ be an ω-sequence of things in $V_\Omega(A \sqcup B)$. Consider the sequence $\langle \mathcal{D}_A(s_i) : i \in \mathbb{N} \rangle$. This has a perfect subsequence. Consider the indices that appear in that perfect subsequence, and the sequence of values of $\mathcal{D}_B$ applied to s_i for i an index of the perfect subsequence. This sequence is good, so there are two naturals $i < j$ with $\mathcal{D}_A(s_i) \leq_\infty \mathcal{D}_A(s_j)$ and $\mathcal{D}_B(s_i) \leq_\infty \mathcal{D}_B(s_j)$ whence $s_i \leq_\infty s_j$ and $\langle s_i : i < \omega\rangle$ is $\leq_\infty$-good. ■

There is of course a shorter, hi-tech, proof which uses Open Ramsey **??**, and the perfect subarray lemma.

Lemma 46 *Every array has a perfect subarray.*

The "fixed point" characterisation of BQOs enables us to prove the following, for which there is no analogue for WQO's.

Corollary 47 *If Q and S are BQO then $Q \to S$ ordered as in definition 3 is a BQO as long as Q and S are.*

Proof: The (graphs of) the functions in $Q \to S$ are elements of $\mathcal{P}^k(S \sqcup Q)$ for some k and $\leq_\infty$ quasiorders (the extensions of) these functions in precisely this way. Then we use the fact that substructures of BQOs are BQO. ■

Sadly we cannot use the fixed-point characterisation of BQOs to prove an analogue of Kruskal'stheorem for BQOs—true tho' it is. This is beco's—since + preserves connexity—$\leq_\infty$ will be connected if $\leq$ is and $\leq_l$ won't in general—but substructures of connected connected quasiorders are likewise connected.

The (locally) minimal bad array lemma

(This is a standard treatment lifted from the literature—specifically Marcone)

Let $\langle Q, \leq_Q\rangle$ be a quasiorder. We will say of a transitive subset of the graph of $\leq_Q$ that it is **compatible** with $\leq_Q$.

Any compatible relation R induces a relation on Q-arrays thus: say $f \leq^{(R)} g$ if $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $(\forall s \in \mathrm{dom}(f))(\langle f(s), g(s)\rangle \in (R \cup I))$. ($R \cup I$ is of course the reflexive closure of R). Similarly we write $f <^{(R)} g$ if $f \leq^{(R)} g$ and $(\exists s \in \mathrm{dom}(f))(\langle f(s), g(s)\rangle \in R)$.

A Q-array f is **locally minimal bad with respect to** R if it is bad and no $g <^{(R)} f$ is bad.

Lemma 48 The (locally) minimal bad array lemma

Let $\langle Q, \leq_Q \rangle$ be a quasiorder, let R be a wellfounded relation compatible with $\leq_Q$, B a block and $f : B \to Q$ be a bad Q-array. Then there is $g \leq^{(R)} f$ which is locally minimal bad with respect to R.

Proof:

Let T be the set of all finite sequences S of the form $\langle\langle s_0, q_0\rangle \ldots \langle s_{k-1}, q_{k-1}\rangle\rangle$ where the s_i are pairwise distinct elements of B and the q_i are in Q, and S can be extended to a bad array $g <^{(R)} f$.

T has a natural tree structure.

We now define two infinite sequences $\langle s_i : i \in \mathbb{N}\rangle$ and $\langle q_i : i \in \mathbb{N}\rangle$. The intention is that every initial segment t_k of `zip` of these two sequences will belong to T.

Suppose we have got the first k elements of both sequences, so we're trying to find s_k and q_k. We do at least know that there are s and q such that the $k+1$-list consisting of the `zip` so far with $\langle s, q\rangle$ on the end belongs to T. (T contains finite sequences that can be end-extended to bad arrays). We will choose s_k and q_k to be minimal among these in the following sense.

We want $max(s_k)$ [*HOLE in the sense of the lexicographic order, presumably?*] to be minimal among the $s \in B$ such that there is a q with $t_k \frown \langle s, q\rangle \in T$. (Marcone sez: notice that if $i < k$ then $\max(s_i) < max(s_k)$.). Now pick q_k to be R-minimal among the q such that $t_k \frown \langle s_k, q\rangle \in T$.

Let $B^* =: \{s_k : k \in \mathbb{N}\}$ and define an array $g : B^* \to Q$ by setting $g(s_k) =: q_k$. We claim that B^* is a subblock of B and that $g <^{(R)} f$ is locally minimal bad wrt R.

■

6.0.2 Here's how to use minimal bad array and Open Ramsey.

The perfect subarray lemma

First explain the concept of a subblock. You can work out what this must be ...

If B is a block and $X \subseteq \mathbb{N}$ is infinite then the subblock $B|X$ is the set of those finite sequences in B whose components are all in X. That is to say, it's $B \cap X^{<\omega}$

[*HOLE What can we say about the *lengths* of perfect subarrays?*]

To understand how to use Open Ramsey to prove a perfect subarray lemma, think about how we used *binary* Ramsey to prove the perfect subsequence lemma (lemma 15). Of course the same idea works for

ω^n-good quasiorders for any finite n, and we use Ramsey theorem on partitions on unordered n-tuples into two pieces, as we have seen. With BQOs there is no finite bound on the length of the finite sequences in the block, so we cannot use ordinary n-ary Ramsey any more. but the underlying idea is the same.

Now the graph of the $\lhd$ relation of a block gives rise to another block. Think about how $\mathbb{N}$ itself is a block, then the graph of the relation on that is another block, namely RADO! Recall the definition of *derivative* of a block from page 61.

Suppose our array is a function $f : B \to Q^\omega$. We have to partition the set of infinite subsets of $\mathbb{N}$. Each infinite subset X has a unique initial segment which meets the derivative of B, and thereby identifies a unique pair of sequences in B. For example, if X had started off $\{1, 10, 15 \ldots\}$, it would identify the two elements $\{1, 10\}$ and $\{10, 15\}$ from the B considered above.

We then colour X blue if $\mathtt{hd}(f(\{1, 10\})) \leq_Q \mathtt{hd}(f(\{10, 15\}))$ and red if not. This clearly is an open partition, since for any X there is a finite initial segment (namely its intersection with the derivative of B) that determines which colour it gets.

Now let Y be a set monochromatic for this partition. Consider the obvious restriction of B to Y—which will be a block—and the restriction of f to this block. Consider the heads of the values of f. These will form either a perfect array or a bad array. They cannot form a bad array because Q is BQO.

This is the perfect subarray lemma.

6.1 Laver's proof of Fraïssé's conjecture

Simpson sez we need a notion of partial ranking to expolit the minimal bad array lemma. This will come from Hausdorff's recursive characterisation of SCAT.

Definition 49 *A linear order type is* **scattered** *if one cannot embed the rationals in it.*

Alternatively we could adopt a recursive datatype declaration saying that a scattered ordering is either the one-point total order or is a wellordered or reverse-wellordered union of scattered orderings.

Hausdorff proved that these two definitions are equivalent. We do so as follows.

One direction is easy: we prove by induction on Haussdorff's rectype that everything in it is scattered.

Conversely, let L be scattered. define $\sim$ on dom(L) by $x \sim y$ iff the interval $[x, y]$ (open or closed, it makes no difference, and $[y, x]$ will do equally well) is in Hausdorff's rectype. Now think about the quotient. If it is nontrivial then it must be dense, because no element of the quotient can have an immediate neighbour—the sum of two orders in the Haussdorff rectype is also in the Haussdorf rectype. But if it's dense we can use DC to pick a subset of the quotient isomorphic to the rationals[3], and then countable choice again to pick a set of representatives, contradicting the assumption that L was scattered. So the quotient is a single point. So all intervals $[x, y]$ are in Haussdorff's rectype. Now use DC to pick a coinitial and cofinal $\omega^* + \omega$ sequence. This partitions L into pieces each of which are in Haussdorf's rectype, so L itself must be as well.

[*HOLE Can we get round AC by defining scattered as "has no quotient with a dense subset" ("No dense minor!!)*]HIATUS

The proof procedes as follows. There is a generalisation of Kruskal that sez that the set of (wellfounded, upward-branching) trees of height ω is BQO under the usual (meet-preserving) embedding relation. This remains true even when the nodes of the tree are labelled with elements from an arbitrary BQO. (Laver *op cit* thm 2.2.)

Next we show by induction on the recursive datatype of scattered order types that if $\alpha \in \mathtt{scat}$ then $Q \in BQO \rightarrow Q^\alpha \in BQO$.

6.1.1 Laver's theorem is best possible

[*HOLE Best possible in what sense, precisely? Can we do the same thing to the canonical η_1 set instead of the continuum, for example?*]

Lemma 50 *(Sierpinski [1950]) (AC) If $E \subseteq \Re$ and $|E| = 2^{\aleph_0}$ then $\exists H \subseteq E$ $|H| = 2^{\aleph_0}$ and for all strictly increasing $f : E \to E$ $(f``E) \setminus H$ is nonempty.*

Consequently the order type of H is strictly less than the order-type of E. (because there is an order-embedding $H \hookrightarrow E$ but not conversely.)

Proof:

[3] Let Q be the quotient. Let R be the relation on $\mathcal{P}_{\aleph_0}(Q)$ defined by XRY iff there is a point of Y between any two points of X. Then by DC there is an infinite sequence, whose union is a copy of the rationals.

Let us suppose the continuum has a wellordering $<_c$ of length ω_α. The significance of this is that every initial segment of this wellordering will be of size *less than* $\aleph_\alpha$. Since $|E| = 2^{\aleph_0}$ the family of increasing functions $E \to E$ is also of size $2^{\aleph_0}$. (This crucial fact depends on E being a uncountable subset of the *reals*—it doesn't work for the rationals for example! The proof is left as an exercise.) This means there is a wellordering $<_f$ of these strictly increasing functions $E \to E$ of order type ω_α. Now we define sequences $\langle p_i : i < \omega_\alpha \rangle$ and $\langle q_i : i < \omega_\alpha \rangle$ as follows.

p_1 is the $<_c$-first real in E.

q_1 is the $<_c$-first thing in $(f_1 \text{“} E) \setminus \{p_1\}$, where f_1 is the $<_f$ first strictly increasing function $E \to E$.

Thereafter for $\beta < \omega_\alpha$ we make the following recursive definition: given $A_\beta = \{p_i : i < \beta\} \cup \{q_i : i < \beta\}$, set

(A): p_β is the $<_c$-first real in $E \setminus A_\beta$. (There is such a thing because $|A_\beta| < \aleph_\alpha = 2^{\aleph_0}$ and $|E| = 2^{\aleph_0}$.)

(B): q_β is the $<_c$-first thing in $(f_\beta \text{“} E) \setminus \{p_i : i \leq \beta\}$, where f_β is the βth strictly increasing function $E \to E$ (in the sense of $<_f$). (There is such a thing because $|f_\beta \text{“} E| = 2^{\aleph_0}$ and $|\{p_i : i < \beta\}| = |\beta| < \aleph_\alpha = 2^{\aleph_0}$).

Now set $H = \{p_\beta : \beta < \omega_\alpha\}$

By construction the p_β are all distinct so $|H| = \aleph_\alpha = 2^{\aleph_0}$ as desired. Each q_β is in the range of f_β, so we want to know that $q_\beta \notin H$. Can q_β be a p_ζ? It cannot be a p_ζ with $\zeta > \beta$ because of (A). It cannot be a p_ζ with $\zeta < \beta$ because of (B). Therefore any strictly increasing $f : E \to E$ takes at least one value outside H. ■

COROLLARY 51 *The collection of order-types of linear orders of power $2^{\aleph_0}$ is not wellfounded.*

[*HOLE I do not know if corollary 51 can be proved without assuming that the continuum can be wellordered. Presumably it's possible to prove with only minimal choice - DC or less- that the linear order types are not even wellfounded?*]

Well relations

There are some refinements of this due to Marcone [1994]. Note first of all that we can separate the “better” part of being a BQO from the ordering part of being a BQO. Thus a relation R is a **well** relation if there are no bad R-arrays.

We can use this to refine the concept of a well relation. Let us say a relation R is α-well if whenever B is a block of length α then every array $f : B \to dom(R)$ is good. Obviously a WQO is merely a quasiorder that is ω-well. A BQO turns out to be a quasiorder that is α-well for every countable ordinal α.

Might be worth checking for which (if any!) α is it the case that the intersection of two α-well relations is α-well.

OK, at the end we decide that a quasiorder is BQO if when you lift it it remains wellfounded. Now there's nothing about wellfoundedness that ties it to transitive relations. Can we generalise this to arbitrary relations? Not as obviously as one might hope: the point being that the '+' operation adds a certain amount of structure thru' prolonged iteration and you tend to end up with quasiorders anyway.

7

The topological Approach

The Ellentuck topology. Descriptive set theory

Galvin-P.

The key paper here, and the original source of topological ideas in BQO theory, is Simpson's *coda* [52] to Weitkamp and Mansfield, [60]

Part of the definition of block is the unique-initial-segment condition: if B is a block then every increasing function $\mathbb{N} \to \mathbb{N}$ has a unique initial segment in B. This sounds fiddly and *ad hoc* but it is actually very significant. Two points:

- There are lots of facts about good quasiorders of finite exponent that one proves by means of Ramsey's theorem of finite exponent. We prove the analogues for good quasiorders of transfinite exponent by expliting the Open ramnsey theorem.
- A Q-array is a map f from a block to Q. A block is a set of finite tuples. Simpson's idea [*HOLE must check that this is historically correct*] is that f can be thought of as a map from $[\mathbb{N}]^\omega$ rather than from a block. One thing we know about a block B is that every increasing $f : \mathbb{N} \to \mathbb{N}$ has a unique initial segment in B. So if we are to think as f as a map defined on $[\mathbb{N}]^\omega$ (when all along it is really a map defined on B) then whenever f is wondering what value to give to an argument $h \in [\mathbb{N}]^\omega$, it is allowed to look only at that unique initial segment of h that is in B. But this is just to say that f is continuous in some suitable topology.

Open Ramsey

Treatment brazenly nicked from **Bollobas: Combinatorics**, CUP pbk. pp 160ff.

This can be done in terms of infinite sequences from $\mathbb{N}$ or in terms of infinite subsets of $\mathbb{N}$. Here we do it in terms of infinite subsets, tho' i do find myself thinking of them as strictly increasing sequences.

$(A, X)^\omega$ is the set of infinite subsets of $\mathbb{N}$ that have an initial segment in A and the corresponding terminal segment in X.[1] It will be understood when this notation is used that A is finite, X is infinite and $sup(A) < inf(X)$. I think we shall also write 'X^ω' for the set of infinite subsets of X.

We say M **accepts** A ("into Y") if $(A, M)^\omega \subseteq Y$. M **rejects** A if no infinite subset of M accepts A.

There is a standard natural topology on the infinite subsets of $\mathbb{N}$, wherein, for any finite $x \subseteq \mathbb{N}$, the set of its supersets is a basis element. Although this is the one we will use, developing the theory of infinite exponent partition relations makes use of the *Ellentuck* topology which has a basis of elements of the form $(A, X)^\omega$.

Fix Y a set of infinite subsets of $\mathbb{N}$: think of it as a two-colouring of the set of infinite subsets of $\mathbb{N}$. If there is an infinite monochromatic set we say Y is **Ramsey**.

LEMMA 52 *(The Galvin-Prikry lemma) Let $Y \subseteq \mathbb{N}^\omega$ and $M \in \mathbb{N}^\omega$.*

- *If M does not reject the empty set then some infinite subset of M accepts all its finite subsets;*
- *If M does reject the empty set then some infinite subset of M rejects all its finite subsets.*

Proof:

First bullet. Suppose M does not reject the empty set. That is to say that some infinite subset L of M accepts the empty set—then this L accepts all its finite subsets.

Second bullet. Suppose that M rejects the empty set. We will construct inductively a sequence $a_1 < a_2 < a_3 \ldots$ in M which rejects all its finite subsets. More specifically we will construct an increasing sequence of as in M (start counting with 1 the first subscript) and a $\supset$-decreasing sequence of infinite subsets of M (start counting with 0 the first subscript) such that $a_i \in M_{i-1}$ and M_i rejects all subsets of $A_i = \{a_1, \ldots a_i\}$. The desired infinite subset of M is then the set of the a_is.

We can certainly begin this construction because we take $M_0 =: M$ which rejects the empty set (which is A_0—no as yet!)

[1] I don't like the use of 'ω' instead of '$\aleph_0$' but i think this notation is standard.

Now for the recursive step. Suppose we have M_k and A_k but we cannot find M_{k+1} or a_{k+1}. Rename M_k as N_1 (we will be building a whole string of N_i!). Let b_1 be something in N_1 bigger than a_k. Now by hypothesis we couldn't set $M_{k+1} =: N_1$ and $a_{k+1} =: b_1$ so there is a subsequece $N_2 \subseteq N_1$ that accepts some subset F_1 of $A_k \cup \{b_1\}$. Since N_2 rejects all subsets of A_k, F_1 must be $E_1 \cup \{b_1\}$ for some $E_1 \subseteq A_k$.

Now choose $b_2 \in N_2$ and $b_2 > b_1$—as before. Now—as before—setting $M_{k+1} =: N_2$ and $a_{k+1} = b_2$ won't work, so there are N_3 (an infinite subset of N_2 accepting some subset $F_2 = E_2 \cup \{b_2\}$) with E_2 and $b_2 \in N_2$ bigger than b_2 and so on ...

That way we get sequences $a_k < b_1 < b_2 < \ldots$ $M_k = N_1 \supset N_2 \supset N_3 \ldots$ with $b_i \in N_i$ and N_{i+1} accepts $E_i \cup \{b_i\}$. All the E_i are subsets of A_k and so "by passing to a subsequence if necessary" (!!) we may assume they are all the same. But then set $B = \{b_i : i < \omega\}$ is an infinite subset of M_k which accepts $E \subseteq A_k$, contradicting assumption.

So the construction never fails, and the set of all the a_i rejects all its finite subsets. ■

COROLLARY 53 *Open Ramsey Every set open in the Ellentuck topology (and a fortiori the usual topology) is Ramsey.*

8

BQOs and fast-growing functions etc

Abstract:

First present totality proofs that rely on compactness. Then show that some of these results can be proved directly by induction on long wellorderings. Finally try to extract some converses. The extraction of converses has become a little cottage industry called "reverse mathematics".

What is so characteristic of the transfinite is that we then go on iterating the iteration, iterating the iteration of the iterations, and so on, until somehow our apparatus buckles; and the least transfinite number after the buckling of the apparatus is how strong the apparatus was.

W.V.Quine: *Set Theory and its logic*

The reader who has done exercise 11 will have no difficulty thinking of finite trees over $\mathbb{N}$ as notations for ordinals below ϵ_0 and concluding that we can associate ϵ_0 with the WQO of trees over $\mathbb{N}$. Such a reader will be in the right frame of mind to read Cichon's illuminating short note [1983].

However we can contort ourselves into thinking of finite trees over $\mathbb{N}$ as notations for ordinals below bigger things than ϵ_0.

Remark 54 *(Friedman) The rank of the set of all bad sequences in the WQO of finite trees over the one-point WQO (partially ordered by reverse end-extension) is at least* Γ_0*.*

Proof:

We will define a function h from finite trees over the one-point WQO to the ordinals below Γ_0.

- h sends the one-point tree to 0.

- Thereafter we first look at how many successors the root has.
  - If it has only one, so that T has precisely one child, T_1 then set $h(T) =: h(T_1)$.
  - If the tree has precisely two children T_1 and T_2 (with $h(T_1) \geq h(T_2)$) then h will send it to $h(T_1) + h(T_2)$.
  - If the tree T has precisely three children T_1, T_2, and T_3 with $h(T_1) \geq h(T_2) \geq h(T_3)$ and $h(T_1) < \phi(h(T_1), h(T_2))$ send T to $\phi(h(T_2), h(T_1))$. If, on the other hand, $h(T_1) = \phi(h(T_1), h(T_2)$, set $h(T) =: h(T_1) + h(T_2)$.
  - Finally if T has four or more children send it to $\phi(h(T_1), h(T_2))$. where $h(T_1) \geq h(T_2)$ are the two children with largest h.

The normal form theorem for ordinals below Γ_0 ensures that h is onto. We also have to check that $T_1 \leq_t T_2 \rightarrow h(T_1) \leq h(T_2)$.

Remark 27 will now ensure that the tree of bad sequences inversely ordered by end-extension has rank at least Γ_0.

■

(Even this is not best possible. By complicating the definition of the map we can arrange to map the set of all finite trees onto longer initial segments of the ordinals.)

The significance of this is that the fact that the finite trees as WQO means that whenever we have a system of notations (using trees) for some total order types which has the feature that when one ordertype is less than another then the tree corresponding to the second doesn't stretch into the first, then we can show that this family of linear orders is wellfounded.

8.1 FFF

Consider the one-point WQO, and suppose there is an integer k such that for all n there is a bad sequence of length n:

$$T_1^n, T_2^n, T_3^n \ldots T_n^n$$

where T_i^n is a finite tree (over the one-point WQO) with $k + i$ nodes. Then there will be an infinite triangular matrix of trees (one row for each n):

$$\begin{array}{llll}
T_1^1 & & & \\
T_1^2 & T_2^2 & & \\
T_1^3 & T_2^3 & T_3^3 & \\
T_1^4 & T_2^4 & T_3^4 & T_4^4 \\
\vdots & \vdots & \vdots & \vdots
\end{array}$$

Consider the (multi)set $\{T_1^n : n \in \mathbb{N}\}$. Each tree in this set has only $k+1$ nodes, so only finitely many of them can be distinct. So at least one of them must be present with infinite multiplicity. Pick on one such and throw away all the rows that do *not* begin with this tree. Now consider the second column. (We can do this canonically so we don't need DC). Iterate with all subsequent columns. Eventually we will have constructed an infinite bad sequence of trees. But this would contradict theorem 25. Therefore the initial assumption was wrong, so there is no such k, and we have proved

THEOREM 55 $\forall k \exists n$ *if* $T_1 \ldots T_n$ *is a list of trees where* T_i *has* $k+i$ *nodes, then there are* $j < l \leq n$ *s.t.* $T_j \leq T_l$.

■

(This uses KL not full DC, for what it's worth)

Notice that we could have replaced '$k+i$' by '$k+g(i)$' where g is any total function. Worth noticing also that this works only if Q is actually finite—o/w we can't assume that each column contains only finitely many distinct trees. So we assert it only for trees over the 1-pt WQO.

EXERCISE 30 *We have just seen how to extract a* Π_2 *truth of arithmetic from theorem 25. Show how, for each* n, *to extract a similar* Π_2 *truth of arithmetic from the assertion that finite trees over the 1-point BQO form an* ω^n*-good quasiorder. (Hint: use the fact that blocks ordered colex are of length* ω*.)*

This proof is of interest because it proves a result about finite sets which can only be proved by reasoning about infinite sets. The reader may well be puzzled by this: isn't $\mathbb{N}$ defined as the intersection of an infinite family of infinite sets? in which case are not actually-infinite sets involved right from the start?

There is an alternative definition of $\mathbb{N}$ which is sometimes very useful: Let $P(n)$ be $n-1$ if $n > 0$ and 0 otherwise. Define

$$\mathbb{N}^* =: \{m : (\forall Y)((m \in Y \wedge (P``Y \subseteq Y)) \rightarrow 0 \in Y)\}$$

(we can also say $q(m)$ (for "m is a Quine natural") iff $(\forall Y)((m \in Y \wedge (P``Y \subseteq Y)) \rightarrow 0 \in Y)$, in tandem with '$N(m)$' for "$m$ is a natural".

We claim $\mathbb{N}^* = \mathbb{N}$.

Clearly $\mathbb{N}^*$ contains 0 and is closed under S and so $\mathbb{N} \subseteq \mathbb{N}^*$. (i.e., we can prove $q(m)$ for all $m \in \mathbb{N}$ by induction). For the other direction we will justify induction over $\mathbb{N}^*$: this will enable us to prove that everything in $\mathbb{N}^*$ is in $\mathbb{N}$. Suppose (i) that $F(0)$ and (ii) that $(\forall n)(F(n) \rightarrow F(n+1))$, and take $a \in \mathbb{N}^*$. Suppose, *per impossibile*, that $\neg F(a)$. Then $\{m : m < a \wedge \neg F(m)\}$ contains a and is closed under P (by (ii)), and so must contain 0, contradicting (i).

Notice the parallel with the definition of 'regular set' in Set Theory.

This definition of natural number does not involve quantification over infinite sets and so is meaningful even in contexts where we are not assuming the axiom of infinity.

Adrian has a nasty hacky way of avoiding quantifying over infinite sets, namely by defining a finite set as something admitting a total order which is a wellorder both ways up. Interestingly this doesn't need AC. It does however have the drawback that it doesn't justify induction.

To complete the picture we would need to show that FFF cannot be proved in Peano arithmetic. The proof of this is too fiddly to do here, since it involves the use of ordinals in proof theory....

Well, here is a sketch. Any halfway sensible system N of notations for an initial segment of the ordinals involves parse trees for the notations. It's probably possible to do it so the parse trees are naked (undecorated) and that if $N(\alpha) \leq N(\beta)$ (where $N(\alpha)$ is the tree that the notation N gives to the ordinal α) then $\alpha \leq \beta$. Now suppose $\langle \alpha_i : i \in \mathbb{N} \rangle$ were a strictly descending sequence of things which had notations. Then $\langle N(\alpha_i) : i \in \mathbb{N} \rangle$ would be a bad sequence of trees.

(What's at work here is simply the elementary fact that any quasi-order extending a WQO is wellfounded.)

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9

Answers to selected Exercises

Chapter 3

Exercise 2

Patter about $V(X)$

Might be better to treat + applied to arbitrary binary relations.

I shall use the letter 'γ' to range over fixed points and prefixed points and postfixed points.

$x \leq f(x)$ is a prefixed point....

The first point to notice is that if R is reflexive then R^+ is a superset of $\subseteq$. The operation is increasing in the sense that $R \subseteq S \to R^+ \subseteq S^+$. Suppose $R \subseteq S$ and xR^+y. Then for every $z \in x$ there is $w \in y$ $R(z,w)$ whence $S(z,w)$ whence $R^+ \subseteq S^+$.

Now for limits. Suppose $R_\infty = \bigcup_{i \in I} R_i$. Clearly, for all $i \in I$, ${R_i}^+ \subseteq {R_\infty}^+$ so $\bigcup_{i \in I} {R_i}^+ \subseteq {R_\infty}^+$. For the converse

$x{R_\infty}^+y$ iff $(\forall z \in x)(\exists w \in y)(zR_\infty w)$ iff $(\forall z \in x)(\exists w \in y)(\exists i)(zR_i w)$ so it is not cts at limits. (Presumably this is for the same reason that $\mathcal{P}$ is not continuous.)

So + is monotone but not continuous

Remark 56 $\in\ \subseteq$ *the GFP*

Proof: If $x \in y$ then $(\forall z \in x)(\exists w \in y)(z \in w)$ ... and the w is of course x itself. That is to say $\in \subseteq \in^+$: $\in$ is a postfixed point.

[*HOLE The GFP is the union of all postfixed points. Is it that way round...?*]

Obvious questions: does γ extend $\in$? Is it connected? Is it wellfounded? Is γ restricted to wellfounded sets wellfounded? Is it a WQO or a BQO?

There are other way of deriving a rank relation. We could consider sets containing $\emptyset$ and closed under $\mathcal{P}$ and (i) unions or (ii) directed unions or (iii) unions of chains. Then if X is such a set we say $x\gamma y$ if $(\forall Y \in X)(y \in Y \to x \in Y)$. For each of these three we can prove by induction that the least fixed point consists (for any $X \supseteq \mathcal{P}(X)$, entirely of sets in X. We should also prove that if X is a prefixed point under the heading (i) (ii) or (iii) then every wellfounded set is in a member of X.

We need to check that the LFP and the GFP are nontrivial. The identity is a postfixed point and the universal relation is a prefixed point. (Incidentally this shows that the GFP is reflexive) But LFP $\subseteq$ GFP? It is if there is a fixed point.

REMARK 57

The GFP is transitive

Proof: First we show that $\gamma^+ \subseteq \gamma \wedge \gamma'^+ \subseteq \gamma' \to (\gamma \circ \gamma')^+ \subseteq \gamma \circ \gamma'$. Suppose $\langle X, Z\rangle \in (\gamma \circ \gamma')^+$. That is to say, $(\forall x \in X)(\exists z \in Z)(\langle x, z\rangle \in \gamma \circ \gamma')$. This is $(\forall x \in X)(\exists z \in Z)(\exists y)(\langle x, y\rangle \in \gamma \wedge \langle y, z\rangle \in \gamma)$. or $(\forall x \in X)(\exists y)(\langle x, y\rangle \in \gamma \wedge (\exists z \in Z)(\langle y, z\rangle \in \gamma))$. Then for this y we have $\langle X, \{y\}\rangle \in \gamma^+$ and thence $\langle X, \{y\}\rangle \in \gamma$ and $\langle \{y\}, Z\rangle \in \gamma'^+$ and thence $\langle \{y\}, Z\rangle \in \gamma'$ which is to say $\langle X, Z\rangle \in \gamma \circ \gamma'$.

Similarly the set of post-fixed points is closed under composition, which means that the GFP is transitive.

We can prove by $\in$-induction that any fixed point is reflexive on well-founded sets.

REMARK 58 *Any two fixed points agree on wellfounded sets.*

Proof: Let γ and γ' be fixed points. We will show that for all wellfounded x and for all y, $\langle x, y\rangle \in \gamma$ iff $\langle x, y\rangle \in \gamma'$.

We need to show that $\mathcal{P}(\{x : (\forall y)(\langle x, y\rangle \in \gamma \longleftrightarrow \langle x, y\rangle \in \gamma')\} \subseteq \{x : (\forall y)(\langle x, y\rangle \in \gamma \longleftrightarrow \langle x, y\rangle \in \gamma')\}$.

Let X be a subset of $\{x : (\forall y)(\langle x, y\rangle \in \gamma \longleftrightarrow \langle x, y\rangle \in \gamma')\}$. Then for all Y

$\langle X, Y\rangle \in \gamma$ iff

$(\forall x \in X)(\exists y \in Y)(\langle x, y\rangle \in \gamma)$ which by induction hypothesis is the same as

$(\forall x \in X)(\exists y \in Y)(\langle x, y\rangle \in \gamma')$ which is

$\langle X, Y\rangle \in \gamma'$

We will also need to show that for all wellfounded y and for all x, $\langle x, y\rangle \in \gamma$ iff $\langle x, y\rangle \in \gamma'$.

We need to show that $\mathcal{P}(\{y : (\forall x)(\langle x,y\rangle \in \gamma \longleftrightarrow \langle x,y\rangle \in \gamma'\} \subseteq \{y : (\forall x)(\langle x,y\rangle \in \gamma \longleftrightarrow \langle x,y\rangle \in \gamma'\}$.

Let Y be a subset of $\{y : (\forall x)(\langle x,y\rangle \in \gamma \longleftrightarrow \langle x,y\rangle \in \gamma'\}$. Then for all X

$\langle X,Y\rangle \in \gamma$ iff

$(\forall x \in X)(\exists y \in Y)(\langle x,y\rangle \in \gamma)$ which by induction hypothesis is the same as

$(\forall x \in X)(\exists y \in Y)(\langle x,y\rangle \in \gamma')$ which is

$\langle X,Y\rangle \in \gamma'$

Remark 59 *If $\gamma^+ \subseteq \gamma$ then*

$(\forall y \in WF)(\forall x)(\langle x,y\rangle \in \gamma \vee \langle y,x\rangle \in \gamma)$

Proof:

We prove by $\in$-induction on 'y' that $(\forall x)(\langle x,y\rangle \in \gamma \vee \langle y,x\rangle \in \gamma)$. Suppose this is true for all members of Y, and let X be an arbitrary set. Then either everything in Y is γ-related to something in X (in which case $\langle Y,X\rangle \in \gamma^+$ and therefore also in γ) or there is something in Y not γ-related to anything in X, in which case, by induction hypothesis, everything in X is γ-related to it, and $\langle X,Y\rangle \in \gamma^+$ (and therefore in γ) follows.

■

Remark 60

If $\gamma \subseteq \gamma^+$ and $\mathcal{P}(X) \subseteq X$ then $(\forall y \in WF)(\forall x)(\langle x,y\rangle \in \gamma \to x \in X)$.

If $\gamma \subseteq \gamma^+$ and $\mathcal{P}(X) \subseteq X$ we prove by $\in$-induction on 'y' that $(\forall x)(\langle x,y\rangle \in \gamma \to x \in X)$. Suppose $(\forall y \in Y)(\forall x)(\langle x,y\rangle \in \gamma \to x \in X)$ and $\langle X',Y\rangle \in \gamma$. $\langle X',Y\rangle \in \gamma$ gives $\langle X',Y\rangle \in \gamma^+$ which is to say $(\forall x \in X')(\exists y \in Y)(\langle x,y\rangle \in \gamma)$. By induction hypothesis this implies that $(\forall x \in X')(x \in X)$ which is $X' \in \mathcal{P}(X)$ but $\mathcal{P}(X) \subseteq X$ whence $X' \in X$ as desired. ■

Corollary 61 *If $\gamma \subseteq \gamma^+$, $y \in WF$ and $x\ \gamma\ y$ then $x \in WF$*

One obvious conjecture is that if γ is a fixed point then $x \in y \to \langle x,y\rangle \in \gamma$.

There is an obvious proof by $\in$-induction on 'x' that $(\forall y)(x \in y \to \langle x,y\rangle \in \gamma)$ but the assertion is unstratified and so the inductive proof is obstructed, at least in *NF*.

Suppose $\gamma^+ \subseteq \gamma$ and x is an illfounded set such that $y \; \gamma \; x \to y \in WF$. Since x is illfounded it has a member x' that is illfounded. $\neg(x'\gamma \; x)$ because everything related to x is wellfounded. Now suppose $y\gamma x'$. Then $\{y\}\gamma^+ x$ and $\{y\}\gamma x$ (since $\gamma^+ \subseteq \gamma$) and $\{y\}$ is wellfounded. So y is wellfounded as well, and x' is similarly minimal.

Now suppose x is such that $G \circ F(x) \subseteq x$. Then $F(x) \in x$. $G \circ F(x \setminus \{Fx\}) \subseteq G \circ F(x) \subseteq x$ As before, we want '$x \setminus \{Fx\}$' on the RHS. So we want

$z \in G \circ F(x \setminus \{Fx\}) \to z \neq Fx$ which is to say $Fx \notin G \circ F(x \setminus \{Fx\})$. But this follows by monotonicity and injectivity of F and the fact that $F(x \setminus \{Fx\})$ is the largest element of $G \circ F(x \setminus \{Fx\})$.

So $G \circ F(x \setminus \{Fx\}) \subseteq (x \setminus \{Fx\})$ and x was not minimal. ■

WQOs: chapter 4

Exercise 5

The direct limit in question is the direct limit of Q_i where $Q_0 = Q$ and Q_{i+1} is $Q \times (Q_i)^{<\omega}$. How is Q_{i+1} an end-extension of Q_i? any such embedding must correspond to an attempt to think of trees of height i as trees of height $n+1$. So, given an injection from Q_i into Q_{i+1} how are we going to lift it to a map Q_{i+1} into Q_{i+2}? [*HOLE A lot of work to do here!*]

Exercise 6

WRT (v) If $\langle q_i : i \in \mathbb{N} \rangle$ is a bad sequence from Q then the set of (domains of) terminal segments form a descending sequence under the 1-1 embedding. I can't see how to do the other direction offhand; Laver [35] asserts it but doesn't prove it.

Further sketches of material relevant to an answer.

Suppose $\langle X_i : i \in \mathbb{N} \rangle$ is a $>^*$-descending chain of subsets of X. Let x_0 be anything in X_0. Thereafter, once we've cut down to a finite subset $Y_i \subseteq X_i$ pick enough xs from X_{i+1} to ensure that everything we have picked from X_i is $\geq$ one of the xs. Then—just to be sure if we haven't already done it—add something that is $\not\geq$ anything in Y_i. The Y_i now form a $>^*$-descending chain of *finite* sets. This shows that if the power set isn't wellfounded then even the finite subsets aren't. [*HOLE This lastpara could be better put*]

Exercise 7

A **minimal bad sequence** is a bad sequence $\langle x_i : i \in \mathbb{N}\rangle$ with the property that if $\langle y_i : i \in \mathbb{N}\rangle$ is a sequence such that $\forall i \in \mathbb{N}\ \exists j \in \mathbb{N}\ y_i \le x_j$ and $\exists i \in \mathbb{N}\ \forall j \in \mathbb{N}\ x_j \not\le y_i$ then $\langle y_i : i \in \mathbb{N}\rangle$ is not bad.

This is the definition in Laver [35]

Alternatively:

$f : \mathbb{N} \to Q$ is a minimal bad sequence if it is bad and for all $g : \mathbb{N} \to Q$ s.t. $g``\mathbb{N} \le^+ f``\mathbb{N} \wedge f``\mathbb{N} \not\le^* g``\mathbb{N}$ then g is not bad.

Notice that this is *not* minimal w.r.t. $\le^+$. If f is a bad sequence, then $\mathtt{tail}(f) <^+ f$ but the tail is bad too.

Exercise 8

The disjoint union of two copies of $\mathbb{N}$ affords a counterexample.

Exercise 9

We cannot embed $\mathbf{2}^3$, the 8-element boolean algebra, into $\mathbb{N}\times\mathbb{N}$. Suppose we have embedded three atoms a, b and c as $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle$ and $\langle x_3, y_3\rangle$ with $x_1 > x_2 > x_3$ and $y_1 < y_2 < y_3$. Then the element ac above both a and c must be above b, which it shouldn't be! This shows that $\mathbf{2}^3$ doesn't embed in $\mathbb{N}\times\mathbb{N}$. ($\mathbf{2}$ is the two-element boolean algebra.) Similarly we can show that $\mathbf{2}^{k+1}$ doesn't embed in $\mathbb{N}^k$. The killer blow comes from reflecting on the WQO that is the result of concatenating $\mathbf{2}^k$ for all finite k.

Exercise 11

Make every list correspond to an ordinal below ω^ω. We define a function $\mathtt{ord}$ on lists recursively as $\mathtt{ord}(l) =: \omega^{len(l)} \cdot \mathtt{hd}(l) + \mathtt{ord}(\mathtt{tl}(l))$.

Exercise 13

Give an easy proof that the lexicographic product of two WQOs is WQO.

The point is that the lexicographic product contains more ordered pairs that the pointwise product (which we already know to be WQO) and any superset of a WQO is WQO, by proposition 17, part (vi).

Exercise 14

Consider the relation "$x \in TC(\{y\})$" on the hereditarily finite sets. Is it a WQO? No: set $x_n =: \{\iota^n(\emptyset), \iota^{n+1}(\emptyset)\}$, where $\iota^n(x)$ is the n-times singleton of x.

Exercise 15

There is a proof in [5] Cao, Kim, Roush "Incline Algebras and applications" thm 1.1.7 page 7. This is my proof not theirs.

The distributivity law means that any polynomial can be expressed as a sum of monomials (and the coefficients are all 1, because of idempotence of addition). Commutativity and associativity of addition enable us to think of each polynomial as a set of monomials. Suppose we could prove that monomials in a fixed finite number of variables were WQO would that be enough to show that polynomials are WQO under $\geq$? It will be if we can show that the finite subsets of the carrier set of a WQO are wqo under the 1-1 embedding, and this is a consequence of the fact that finite lists over a WQO are WQO: send each list to its carrier set and appeal to the fact that a homomorphic image of a WQO is WQO. However in this case we can give a slightly easier direct proof by exploiting idempotence of $+$.

Let X and Y be finite sums of monomials (tho'rt of as sets) and suppose for each monomial x in X, $f(x)$ is a monomial in Y s.t $x \geq f(x)$. If f is injective we are done, as the 1-1 embedding is WQO. If it isn't, we just put in as many copies of each $f(x)$ as we need to make f injective, and appeal to idempotence of addition to claim that the extra copies don't do anything. That way $X \geq Y$ as polynomials iff $X \geq^+ Y$ as finite sets of monomials.

So all we have to do is establish that monomials are wqo. This of course is where we will need the fact that the incline is finitely generated. Sse we are given a sequence $\langle m_i : i \in \mathbb{N} \rangle$ of monomials. Each monomial is a product of finite powers of generators. For each generator g two-colour the complete graph on $\mathbb{N}$ depending whether the exponent of g in m_i is $\geq$ or $>$ the exponent of g in m_j. By discarding monomials we can end up with a subsequence wherein, for each generator g, the exponents of g in the remaining monomials are nondecreasing. And we know that $x^2 \leq x$.

I think this actually proves that its a BQO. For further details see [5]. This result is on page 7.

Chapter 5

Exercise 16

If s and t are increasing sequences from $\mathbb{N}$ of length n then $s \lhd t$ iff t is an end-extension of $\mathtt{tl}(s)$.

Exercise 17

If $\langle X, \leq \rangle$ is a quasiorder, define $\leq^{\aleph_0}$ on $\mathcal{P}(X)$ as in clause (vii) of exercise 6. Show that if $\langle X, \leq \rangle$ is an ω^2-good quasiorder, then $\langle \mathcal{P}(X), \leq^{\aleph_0} \rangle$ is WQO.

Exercise 18

Prove that $\mathbb{N}^{<\omega}$ is not wellordered by the lexicographic order but that the canonical n-block is.

Prove that the canonical n-block is of length ω in the colex ordering.

Exercise 20

Is $\leq^*$ a WQO on $\mathcal{P}(X)$? Prove or find a counterexample.

RADO is a counterexample. For every n let $B_n = \{\langle i, n \rangle : i < n\}$. $\{B_n : n \in \mathbb{N}\}$ is an antichain in the power set of RADO. If $m > n$ then $\langle n, m \rangle \in B_m$ and is not above anything in B_n. Notice that the B_n are all finite!

However:

Remark 62 *If $\langle A, \leq \rangle$ is a BQO then $\langle \mathcal{P}(A), \leq^* \rangle$ is BQO.*

Petr Jancar [24] proved that RADO embeds in any WQO $\langle A, \leq \rangle$ st $\langle \mathcal{P}(A), \leq^* \rangle$ is not WQO.

(Marcone's sketch of a proof of the remark)

If B is a barrier and $f : B \to \mathcal{P}(A)$ is $\leq^*$-bad, consider the barrier $B(2)$ (defined in Milner's paper on p.494) and define $g : B(2) \to A$ by letting, for every $b_1 \cup b_2 \in B(2)$, $g(b_1 \cup b_2)$ to be an element of $f(b_2)$ which is not above any element of $f(b_1)$. Such an element exists because f is bad and hence $f(b_1) \not\leq^* f(b_2)$. It is immediate to check that g is bad and hence A is not BQO.

Using the "fine analysis" of the notion of bqo (see my paper in Transactions of the AMS 345 (1994), 641-660) we can state this result as follows: if α is a countable indecomposable ordinal and A is α-wqo then

$\langle \mathcal{P}(A), \leq^* \rangle$ is β-wqo for any $\beta < \alpha$. In particular this means that any counterexample to your original statement [that * preserves WQOness] is a WQO which is not ω^2-WQO, and is known (theorem 1.11 in Milner, combined with the results in my paper) that any such wqo contains an isomorphic copy of Rado's counterexample. Thus you were right that that counterexample is in a precise sense the only possible one.

Exercise ??

Not necessarily. Consider $RADO(3)$. Then $\langle 2,4,7 \rangle \not\leq_{new} \langle 4,5,8 \rangle$ beco's $\langle 2,4,5 \rangle$ is an impediment. But so is $\langle 1,4,5 \rangle$. Can there be infinitely many impediments? [*HOLE check this: i thought it was obvious but the definition has changed*]

Exercise 21

If $\leq_1$ and $\leq_2$ are both quasiorders of a set Q, and the graph of $\leq_1$ is a subset of the graph of $\leq_2$, and $\leq_1$ is an ω^n-good quasiorder, then so is $\leq_2$.

This is immediate given the excluded-substructure characterisation of ω^n-good quasiorders as quasiorders whose complements do not contain a copy of $RADO(n)$.

Exercise 22

"Prove analogues of the perfect subsequence lemma (lemma 15) for ω^n-good quasiorders."

We treat the case $n = 2$ only, for the sake of ease of exposition. Let $\langle Q, \leq_Q \rangle$ be an ω^2-good quasiorder, and $\{q_{i,j} : i < j \in \mathbb{N}\}$ an array. What would a perfect subarray be? Well, it must be a set $\{q_{i,j} : i, j \in X\}$ for some infinite $X \subseteq \mathbb{N}$ and $q_{i,j} \leq_Q q_{j,k}$ whenever $i < j < k$, all in X. Now two-colour the triples from $\mathbb{N}$: $\{i < j < k\}$ is red if $q_{i,j} \leq q_{j,k}$ and blue otherwise. Clearly there cannot be an infinite subset of $\mathbb{N}$ all of whose triples are blue, and a set all of whose triples are red gives a perfect subarray.

To show (iv) that the intersection of (the graphs of) two ω^n-good quasi-orders $\leq_1$ and $\leq_2$ on the same carrier set is ω^n-good we procede as follows. First use the perfect subarray lemma that we have just proved to extract a perfect subarray in the sense of $\leq_1$. Then any array on this substructure must be good (with respect to $\leq_2$) so there is a "good" pair as desired.

(iii) and (v) are similar.

Exercise 23

If Q is ω^2-good then Q-streams are wellfounded under stretching, and it seems positively luddite not to attempt to exploit this fact. Using minimality under stretching one would get an MBA f all of whose rays were perfect sequences, which sounds useful, and stretching is a weaker relation than the pointwise product so an MBA constructed according to stretching ought to satisfy more constraints. However minimality under pointwise product neither implies nor is implied by minimality under stretching, since in the definition of R-minimality the R has both positive and negative occurrences.

The problem would come with the proof by induction on n that no ray of g is strictly below any ray of f under stretching.

Chapter 6

Exercise 25

The lexicographic order on $\mathbb{N}^{<\omega}$ is dense. Whenever $s <_{lex} t$ are finite sequences of numbers, any end-extension of s is later than s but earlier than t.

[*HOLE Clarificatory chat about colex ordering somewhere pse*]

Exercise 26

A block is a set of increasing finite sequences of natural numbers and is either $\{\langle n\rangle : n \in \mathbb{N}\}$ or is the result of replacing a single sequence in a block by all its immediate end-extensions. We call this *exploding* an element. If $<$ is the induced strict partial order, and C is a $<$-chain of blocks, we set $\bigvee C$ to be the set of sequences belonging to a terminal segment of the blocks in C. Then $\bigvee C$ is a block too. Almost!

Start with $\mathbb{N}$. Explode 1. Then explode 1,2. Then explode 1,2,2. Then explode 1,2,3,4; and continue. Then at the limit we have an infinite descending seq containing 1,3; 1,2,4; 1,2,3,5; 1,2,3,4,6; 1,2,3,4,5,7; and so on. This sequence of explosions is not permitted beco's it has unbounded height. But notice that if we performed an infinite sequence of explosions in this way, the infinite sequence it defined would have no initial segement in the block. Notice also that an infinite sequence of explosions like that would correspond to an infinite play in $G_{X \leq_\infty Y}$ and there aren't any infinite plays: `false` is playing according to his winning strategy and he wins games of finite length only. So sups of chains are OK as long as there are no infinite sequences of explosions.

Must prove that this is the same as the ordinary definition. Must also prove that all these things are wordered by the lex ordering.

Exercise 27

(i) Easy to check that $\rho(b)$ must be $\mathtt{hd}(b)$. This means that we can read off the first elements of the tuples. What about the second element of a tuple, if there is one? Consider the triple $\{2, 7, 9\}$ for example. We can tell that its first component is 2 beco's we know $\rho(\{2, 7, 9\}) = 2$. What about the second component? The only tuples b s.t. $\{2, 7, 9\} \lhd_B b$ are tuples whose first component is 7, and therefore the only tuples b s.t. $\{2, 7, 9\} \lhd_B b$ are tuples of rank 7. So we can tell that the second component is 7. So in general if $\rho(b') = n$ for all b' s.t $b \lhd_B b'$, the second component of b must be n. If there is more than one n that is $\rho(b')$ for some $b' \rhd b$ then b does not have a second component. In general, the 0th component of b is $\rho(b)$ and thereafter the $n + 1$th component of b is k iff $(\forall b' \rhd b)$(the nth component of b' is k). If this quantity is undefined then b has at most k elements.

(ii) In any block, for every b there are infinitely many b' s.t. $b \lhd b'$, and this fact can be captured in a first-order way. However for every b there are *finitely* many b' s.t. $b' \lhd b$ and this cannot be captured in a first-order way.

Exercise 29

Chapter 8

Exercise 30