Wellquasiorders and Betterquasiorders

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Contents

Chapter 1

From qo to wqo

1.1 Introduction

A relation $\leq$ on a set (or class) Q is a **quasi-order** (often **preorder**, particularly for category theorists) if $\leq$ is reflexive ($a \leq a$ for every $a \in Q$) and transitive ($a \leq c$ if $a \leq b$ and $b \leq c$). We will often abbreviate *quasi-order* by *qo*. A qo Q is **well-quasi-ordered** (**wqo**) if Q has no infinite strictly decreasing sequences and no infinite **antichains** (subsets X such that no two distinct $a, b \in X$ satisfy $a \leq b$). The notion of *wqo* is very natural; it sits nicely between *well-ordered* and *well-founded*, providing a simple easy-to-express notion of well-behaved well-founded order.

Many of the most interesting questions about quasi-orders ask whether a certain class of natural objects is wqo under embeddability. Though the answer is obviously *No* in many of the first cases that come to mind — groups, topological spaces, etc. — other examples have sparked remarkable work in wqo theory. One example is the following conjecture of Fraïssé.

Call a linear order L **scattered** if there is no order-preserving injection $\mathbb{Q} \hookrightarrow L$. In the 1940s Fraïssé conjectured that the class of scattered linear orders was wqo. Laver [6] eventually proved (a generalisation of) Fraïssé's conjecture, but not until after Nash-Williams had revolutionised wqo theory by introducing in [14] the stronger notion of *bqo* (see §2.2 for the definition of *bqo*), on which Laver relied in his proof.

It turns out that — although many finitary operations do — important infinitary operations do not preserve wqoness. One might therefore hope for a notion stronger than wqo (but still weaker than wellorderedness, of course) that is stable under infinitary operations like the powerset operation $\mathscr{P}$. This notion is Nash-Williams' *better-quasi-order* (*bqo*), and we spend Chapter 2 defining it and describing many of bqo theory's central results.

Nash-Williams' main theorem in [14] asserts that *rooted trees of height ω are bqo*, settling a conjecture of Kruskal. Laver's proof of Fraïssé's conjecture relies on a generalisation of this theorem — namely, that *Q-labelled trees are bqo if Q is bqo* — and in §2 of [6] Laver provides instructions for carrying out the generalisation. The primary purpose of this essay is to undertake the generalisation carefully; this is the subject of Chapter 3.

This essay's secondary purpose (hopefully fulfilled in Chapter 2) is to paint a more-or-less complete picture of bqo theory from the topological perspective introduced by Simpson in [18]. In particular, we provide a proof that Nash-Williams' original definition of bqo coincides with Simpson's definition (Proposition 2.3.2). (This is not a difficult result, but no proof appears in the literature.) Many early results in bqo theory were proved using the so-called *minimal bad array technique*. This technique has been promoted to a theorem, the Minimal Bad Array Lemma (2.3.3), and we explore several of its applications in §2.4.

For two reasons, our proof will be slightly less cluttered than the generalisation Laver envisioned. First, we follow a slight simplification of Nash-Williams' proof suggested by Nash-Williams [14] and carried out by Kühn in [5]. Second, our proof uses Simpson's definition of *bqo*, which avoids some of the combinatorial baggage required by Nash-Williams' original definition.

Sources consulted; acknowledgements

It would be difficult to conceal which sources I have consulted in this essay's production, as I have already declared most of it to be modified versions of others' results. The content of Chapter 1 is basic and standard; much of it I have adapted from the Reading Group's discussion, Forster's notes [2], or §1 of Laver's paper [7].

The goals of Chapter 2 are to frame bqo theory from the 'Simpsonian' perspective introduced in [18] and to prove (using Simpson's topological definition) that certain operations preserve bqoness (§2.4). To that end, I have included a proof that Simpson's definition is equivalent to Nash-Williams'. It is an elementary result, but to my knowledge it appears nowhere in the literature. The proof of the Minimal Bad Array Lemma 2.3.3 I extracted from my notes (and my memory, to fill some gaps) of Zachiri McKenzie's Reading Group presentation. It follows closely Simpson's original proof [18]. Many of the proofs in Chapter 2 are modified from proofs in other sources so that they use Simpson's definition. Though the perspective is different, the ideas are essentially the same.

In Chapter 3 I present the proof that Q-labelled trees are bqo. As noted above, this proof differs from Laver's suggested proof in two ways: (1) in the few places where a definition of bqo is required, I use Simpson's definition; and (2) the proof takes a slight shortcut relative to Nash-Williams' proof. Kühn [5] provides an account of Nash-Williams' original proof (the non-Q-labelled version) using this shortcut.

I would like to offer thanks to Dr. Forster, who introduced me to wqo/bqo theory and to David, Zachiri, Oren, Lovkush, and Philipp, who participated in the Reading Group and offered several illuminating comments and presentations as I learned the theory.

1.2 Quasi-orders and conventions

The reader should be familiar with the basic language of sets, functions, ordinals, and partially ordered sets. Write ORD for the class of ordinals and CARD for the class of cardinals. A **sequence** with terms in a set A is simply a function from an ordinal to A. We write $\mathrm{lh}(s)$ for the **length** of a sequence s, which we identify with its domain. The nth term of a sequence s will be written $s[n]$, and a sequence will occasionally appear as a list of its terms, in order, surrounded by angle brackets: $s = \langle s[0], s[1], \ldots \rangle$. The restriction of a function f to a set A is denoted $f{\restriction}_A$, and f **applied to** A is denoted $f''A$. (That is, $f''A = \{f(a) : a \in A\}$.) The preimage $\{a \in \operatorname{dom} f : f(a) \in B\}$ of a set B under a function f will be denoted $f_{-1}B$.

The central objects of our study will be quasi-orders and sequences. Our notation for this is mostly standard. As in the Introduction, $(Q, \leq)$ is a **qo** if $\leq$ is a reflexive transitive relation on Q. Unlike a partial order, a qo can have distinct elements a, b such that $a \leq b$ and $b \leq a$. When this is the case, we say a and b are **indistinguishable** and we write $a \equiv_Q b$ or $a \equiv b$. (The overloaded term **equivalent** is more common in the literature.)

Because much of our study (particularly in Chapter 2) concerns strictly increasing sequences in ω, we will find it useful to identify (to a non-frightening extent, I hope) a subset of ω with the strictly increasing sequence that enumerates it. Thus of subsets $X, Y \subseteq \omega$ we will happily say, e.g., 'X is an initial segment of Y' when the (unique) increasing sequence that enumerates X is an initial segment of the increasing sequence that enumerates Y. Occasionally it will be useful to abbreviate the assertion 's is an initial segment of t'; we will do so by writing $s \preceq t$. And $s \prec t$ will abbreviate 's is a proper initial segment of t'. We also say t is an **end-extension** of s to mean that s is an initial segment of t.

This essay's most controversial conventions are sure to be our notation for sets of sequences and sets of subsets, as there seem to be several popular conventions, none of which is universally liked. I follow most of my sources in writing $[A]^{<\omega}$ for the set of finite subsets of A, $[A]^n$ for the n-element subsets of A, and $[B]^\omega$ for the set of infinite subsets of B. (For us, such a B will always be a subset of ω, so recall we will consider $[B]^\omega$ identical to the set of strictly increasing infinite sequences in B.) Hoping to mark a clear distinction between infinite subsets (= increasing sequences) and *all* sequences, I write ${}^\omega B$ for the set of sequences $\omega \to B$. Similarly, ${}^{<\omega}A$ denotes the set of finite sequences with terms in A.

For a qo Q there are two natural ways of quasi-ordering the set $\mathscr{P}Q$ of subsets of Q. (We use Laver's $\leq_1$ and $\leq_m$ notation, which is also used in other sources.) The first is to declare $A \leq_m B$ iff there is a function $f\colon A \to B$ such that $a \leq_Q f(a)$ for every $a \in A$. (Since we are not concerned about the axiom of choice, this is equivalent to requiring that for every $a \in A$ there is an element $b \in B$ such that $a \leq_Q b$.) The second is to declare $A \leq_1 B$ iff there is an *injective* $f\colon A \to B$ witnessing that $A \leq_m B$. There is also a natural way to quasi-order $\operatorname{Seq} Q$, the class of sequences $\alpha \to Q$, α an ordinal: for sequences $s\colon \alpha \to Q$ and $t\colon \beta \to Q$, say $s \leq t$ iff there is a strictly increasing function $\phi\colon \alpha \to \beta$ such that $s(\eta) \leq_Q t(\phi(\eta))$ for every $\eta < \alpha$. From this ${}^{<\omega}Q$ and ${}^\omega Q$ inherit a quasi-order.

Finally, we quasi-order the cartesian product $Q \times R$ of two (and by extension, finitely many) quasi-orders using the **product order**: $(a_1, a_2) \leq (b_1, b_2)$ iff $a_1 \leq_Q b_1$ and $a_2 \leq_R b_2$.

1.3 Basic properties of wqos

In this section we take a whirlwind tour of wqo theory, seeing only the sights necessary to move on to bqo theory in the next chapter.

Definition. Let Q be a quasi-order. A **Q-sequence** is a function $\omega \to Q$. A Q-sequence $f\colon \omega \to Q$ is **bad** if $f(a) \not\leq_Q f(b)$ for all $a, b \in \omega$ such that $a < b$. A Q-sequence is **good** if it is not bad. A Q-sequence f is said to be **perfect** if $f(a) \leq_Q f(b)$ for all $a < b$.[1]

Q is **well-quasi-ordered** (**wqo**) if there is no bad Q-sequence.

Ramsey's Theorem is fundamental to wqo theory, so we record it here.

Theorem 1.3.1 (Ramsey's Theorem)**.** Suppose $c\colon [\omega]^n \to k$ is a function (a **k-colouring** of $[\omega]^n$). Then there is an infinite set **monochromatic** for c, an $X \in [\omega]^\omega$ such that $c\restriction_{[X]^n}$ is constant.

The following equivalence is an easy application of Ramsey's Theorem.

Proposition 1.3.2. Let Q be qo. The following are equivalent:

(i) Q is wqo; that is, every Q-sequence is good.

(ii) Every strictly decreasing sequence in Q is finite, and every antichain in Q is finite.

(iii) Every Q-sequence has a perfect subsequence.

One other equivalent definition is worth mentioning: Q is wqo iff every linear extension of Q is a wellorder.

A moment's reflection on the definition of wqo reveals that:

(i) every subset of a wqo Q (given the order it inherits from Q) is wqo;

(ii) a homomorphic image of a wqo is wqo;

(iii) the binary product of two wqos is wqo;

(iv) the union of two wqo subsets of a qo Q is wqo; and

(v) if $(Q, \leq)$ is wqo and $\leq'$ is a qo on Q containing $\leq$ (that is, $q_1 \leq q_2$ implies $q_1 \leq' q_2$), then $(Q, \leq')$ is wqo.

The first proper theorem of wqo theory is Higman's Lemma (attributed by many sources to [3]), which we prove here to highlight a striking similarity between wqo theory and bqo theory. The proof of Higman's Lemma is a prototypical example of the *minimal bad sequence technique*, the 'poor man's version' of the *minimal bad array technique*, which forms the backbone of our study of bqos in Chapters 2 & 3. To render completely transparent the analogy between the minimal bad sequence technique and the minimal bad array technique, we (following Laver [7]) state a general result that will assume the role of the minimal bad sequence technique.

[1]It isn't clear to me why authors of wqo theory prefer *perfect* to, e.g., *nondecreasing*; perhaps it helps remind the reader that in a qo an injective nondecreasing sequence needn't be strictly increasing: in fact, an injective sequence can be both nondecreasing and nonincreasing!

Let Q be qo. With an eye toward the $<^*$ relation defined in §2.3, we define, for bad Q-sequences f and g,

$$g \leq^* f \text{ iff } \forall m\, \exists n\; g(m) \leq_Q f(n)$$
$$g <^* f \text{ iff } g \leq^* f \text{ and } \exists m\, \exists n\; g(m) <_Q f(n).$$

(Note that the relation $<^*$ is not the strict part of the relation $\leq^*$!) A bad Q-sequence f is called **minimal bad** if there is no bad g such that $g <^* f$. It is clear from the definition that $\{q \in Q : q < f[n] \text{ for some } n \in \omega\}$ is wqo if f is a minimal bad Q-sequence. A nontrivial observation that we will not use: a subsequence of a minimal bad sequence is minimal bad.

Lemma 1.3.3. If Q is well-founded but not wqo, then there is a minimal bad Q-sequence.

Proof. Define f inductively (using DC) as follows: since Q is well-founded, we can pick $f[0]$ to be minimal among first terms of bad Q-sequences. If we have picked $f[0], \ldots, f[n]$, pick $f[n+1]$ to be minimal among first terms of bad end-extensions of $\langle f[0], \ldots, f[n] \rangle$. Then f is minimal bad:

Suppose for a contradiction that the bad Q-sequence g satisfies $g <^* f$. Let n be least such that $g[m] <_Q f[n]$ for some m. Pass to a terminal segment of g so that $g[0] <_Q f[n]$. By the choice of n and the badness of g, at most n terms of g are $\leq_Q$ one of the first $n-1$ terms of f; so we may assume — by passing to a subsequence of g that excludes these terms — that

$$(\forall j)\, (\exists k \geq n) \;\; g[j] \leq_Q f[k]. \tag{†}$$

(A subsequence of a bad sequence is still bad, so this is a fair assumption to make of g.) Consider the sequence

$$s = \langle f[0], \ldots, f[n-1], g[0], g[1], \ldots \rangle.$$

If it were the case that $f[j] \leq_Q g[n+k]$ for some $j \leq n-1$, our assumption (†) would demand that $f[j] \leq_Q g[n+k] \leq_Q f[m]$ for some $m \geq n$, contradicting the badness of f. Therefore s is bad. But this contradicts the minimality of $f[n]$ arranged for in the construction of f, so no such g exists. We conclude that f is minimal bad. ∎

Theorem 1.3.4 (Higman's Lemma)**.** If Q is wqo, then ${}^{<\omega}Q$ is wqo.

Proof. Suppose f is a bad ${}^{<\omega}Q$-sequence. Certainly ${}^{<\omega}Q$ is well-founded, so we can apply Lemma 1.3.3 to assume that f is minimal bad. Since f is bad, each term of f (a finite sequence in Q) has positive length. Let $q[n]$ be the final term of $f[n]$. Let f_* denote the sequence whose nth term is $f[n]\restriction_{\mathrm{lh}(f[n])-1}$; i.e., $f_*[n]$ is $f[n]$ without its final term $q[n]$. The set $A := \{f_*[n] : n \in \omega\}$, a subset of $\{q \in Q : q < f[n] \text{ for some } n \in \omega\}$, is wqo since f is minimal bad. Therefore $A \times Q$ is wqo, and so is $\{(f_*[m], q[m]) : m \in \omega\}$. Consequently there are m, n such that $m < n$, $f_*[m] \leq_{<\omega Q} f_*[n]$, and $q[m] \leq_Q q[n]$. But this means $f[m] \leq_{<\omega Q} f[n]$, contradicting the badness of f. ∎

The first major theorem in wqo theory is ([4]) Kruskal's theorem — a generalisation of Higman's Lemma 1.3.4 — which asserts that the class of finite trees is wqo (under an embeddability relation similar to the one we define for infinite trees in §3.2). See Nash-Williams' paper [13] for a proof of Kruskal's theorem. It is worth noting that Kruskal proved a 'Q-labelled' version of the theorem; this generalisation is very similar to Laver's 'Q-labelled' generalisation of Nash-Williams' theorem, which is the subject of Chapter 3.

Much more can be said about wqos. The reader unsatisfied with the minimal treatment here should consult Milner's paper [12], §1 of Laver [7], and Forster's notes [2].

Chapter 2

From wqo to bqo

2.1 We need a stronger notion: Rado's example

The operations that preserve wqoness are all finitary. And the following example of Rado [17] proves that this inconvenience is unavoidable. Rado's example demonstrates that neither ${}^{\omega}Q$ nor $\mathscr{P}Q$ need be wqo if Q is.

Let R be $[\omega]^2$ — that is, $R = \{(a, b) \in \omega : a < b\}$ — and quasi-order R in the following way:

$$(a_1, a_2) \leq_R (b_1, b_2) \text{ iff } \begin{cases} a_2 \leq b_1, \text{ or} \\ a_1 = b_1 \text{ and } a_2 \leq b_2. \end{cases}$$

For clarity we will write $(a, b) \in R$ as $\{a < b\}$ in this section.

Proposition 2.1.1. *R is wqo, yet ${}^{\omega}R$ (hence $\mathscr{P}R$) is not wqo.*

Proof. Suppose $f\colon \omega \to R$ is a bad R-sequence. There are three mutually exclusive ways in which $\{a < b\}$ can satisfy $f(a) = \{a_1 < a_2\} \not\leq_R \{b_1 < b_2\} = f(b)$:

(i) $a_1 = b_1$ and $a_2 > b_2$;

(ii) $a_1 < b_1 < a_2 < b_2$;

(iii) $b_1 < a_1$ and $a_2 < b_2$.

Colour $\{a < b\}$ in $[\omega]^2$ according to whether (i), (ii), or (iii) holds of $(f(a), f(b))$. By Ramsey's theorem there is a monochromatic set $\{x_1, x_2, \ldots\}$. A monochromatic set of colour (i) or (iii) cannot exist, since it would give — by considering the first or second coordinates, respectively, of the $f(x_i)$ — a strictly decreasing infinite sequence of natural numbers. Therefore we have a monochromatic set of colour (ii). But this is also impossible, since the sequence of first coordinates of the $f(x_i)$ must be strictly increasing, yet there are for every $\{a_1 < a_2\}$ only finitely many b_1 such that $a_1 < b_1 < a_2$. We conclude that R is wqo.

For $i \in \omega$ let $f_i\colon \omega \to R$ be the sequence whose j^{th} term is $\{i < i + j + 1\}$. (f_i enumerates the i^{th} column of R.) It is easy to see that $i \mapsto f_i$ is a bad ${}^{\omega}R$-sequence, so ${}^{\omega}R$ is not wqo. ∎

For an illuminating account of the motivation for the Rado order R, see Chapter 5 of Forster's notes [2]. We will conclude this section by mentioning that in a strong sense R is the *simplest* example of a wqo whose powerset and set of infinite sequences are not wqo. See 1.7 of Laver's paper [7] for a proof.

Proposition 2.1.2. If Q is wqo but ${}^{\omega}Q$ is not wqo, then Q contains a copy of R.

2.2 Blocks & barriers

The central objects of study in Nash-Williams' definition of bqo are (sets of) finite subsets of ω. Recall $[\omega]^{<\omega}$ denotes the set of finite subsets of ω, and we identify a subset $X \subseteq \omega$ with the unique strictly increasing sequence that enumerates it. We will make extensive use of two bits of notation, of which neither (as far as I know) is standard outside bqo theory.

Definition. For $X \subseteq \omega$, we write ${}_*X$ (read 'butfirst(X)' or $\mathtt{tl}(X)$ for Forster) for the set $X \setminus \{\min(X)\}$. If $s, t \in [\omega]^{<\omega}$, write $s \lhd t$ if ${}_*s$ (which may be empty) is a proper initial segment of t.[1]

Observe that the relation $\lhd$ is not transitive: e.g., $\{0,1\} \lhd \{1,2,3\} \lhd \{2,3,5\}$, yet $\{0,1\} \not\lhd \{2,3,5\}$.

Since Ramsey's theorem 1.3.1 plays an important role in wqo theory, it is sensible to expect a theorem of Ramsey Theory to play a similar role in bqo theory. That theorem is Open Ramsey, Theorem 2.4.1. Following Nash-Williams [14] and Milner [12] we will use the following finitary version till we need the infinitary version 2.4.1 after introducing Simpson's definition of bqo.

Theorem 2.2.1 (Finitary version of Open Ramsey)**.** Let $Y \in [\omega]^{\omega}$. If $B \subseteq [Y]^{<\omega}$, there is $A \in [Y]^{\omega}$ such that either

(i) for every $X \in [A]^{\omega}$ there is $b \in B$ that is a (proper) initial segment of X; or

(ii) $B \cap [A]^{<\omega} = \varnothing$.

Definition. A **block** B is an infinite subset of $[\omega]^{<\omega}$ that contains an initial segment of every infinite subset of $\bigcup B$. We will occasionally refer to $\bigcup B$ as the **base** of B. (Note the base of a block B is denoted $\overline{B}$ in Nash-Williams' papers.) For a qo Q a **Q-pattern** is a function from a block into Q.

A **barrier** B is a block that is also a $\subset$-antichain; that is, distinct elements of B are incomparable under the subset relation $\subset$. For a qo Q, a **Q-array** is a function from a barrier into Q.

Examples.

(i) For every $n > 0$, the set $[\omega]^n$ of n-element subsets of ω is a barrier.

[1] I suggest 's tries at t' as a possible reading of '$s \lhd t$'.

(ii) An important operation for Nash-Williams (that we will not make use of) is the 'exponent' operation for blocks: if B is a block, then B^n (written $B(n)$ by other authors) is the set

$$\{b_1 \cup \cdots \cup b_n : b_1, \ldots, b_n \in B,\ b_1 \lhd b_2 \lhd \cdots \lhd b_n\}.$$

It is easy to verify that B^n is a barrier if B is.

Notice that some authors (e.g., Forster [2] and Marcone [9]) require that a block B own a *unique* initial segment of any given $X \in [\bigcup B]^\omega$. This is the notion of *thin* block in Nash-Williams [14]. This gives an equivalent definition of bqo, since every block contains a thin block. Of course, if B is a barrier, then an initial segment in B of $X \in [\bigcup B]^\omega$ must be unique; we denote this member of B by $X \wedge B$ (written 'X@B' by Forster [2]).

Definition. Let Q be a qo. A Q-pattern (especially a Q-array) $f\colon B \to Q$ is **bad** if $f(s) \not\leq_Q f(t)$ for all $s, t \in B$ such that $s \lhd t$. We say f is **good** otherwise. The qo Q is **better-quasi-ordered** (**bqo**) if there is no bad Q-array.

As an immediate consequence of the definition of bqo, we see that homomorphic images and subsets of a bqo are bqo. Moreover, if $(Q, \leq)$ is bqo and $\leq'$ is a quasi-order on Q containing $\leq$ (i.e., such that $q_1 \leq q_2 \to q_1 \leq' q_2$), then $(Q, \leq')$ is bqo.

An important observation (Lemma 1.3 of Marcone's paper [9]) is that every block is well-ordered by the lexicographic order. In light of this, it is sensible to consider the approximation to bqo given by forbidding bad Q-arrays whose domain has length $\leq \alpha$, for a given ordinal α. In [9] Marcone provides a thorough study of this notion.

Nash-Williams first defined *bqo* to be *there is no bad Q-pattern*, but we will soon prove (as an easy corollary to Theorem 2.2.1) that these definitions are equivalent. (NB. In [12] Milner confusingly uses *Q-pattern* in place of *Q-array*.) Milner [12] offers the following proof, a simplification of Nash-Williams' original.

Proposition 2.2.2. Every block contains a barrier.

Proof. Let B be a block, and let C be the set of $\subseteq$-minimal elements of B. By Theorem 2.2.1 there is $A \in [\bigcup B]^\omega$ such that either

(i) for every $X \in [A]^\omega$ there is an initial segment $c \in C$ of X, or

(ii) $C \cap [A]^{<\omega} = \varnothing$.

Any $b \in B \cap [A]^{<\omega}$ of minimal size must belong to C, so (ii) cannot hold. Therefore (i) holds, and $C \cap [A]^{<\omega}$ is a barrier: (i) guarantees that $C \cap [A]^{<\omega}$ is a block, and the definition of C ensures $C \cap [A]^{<\omega}$ is a $\subseteq$-antichain. ■

Corollary 2.2.3. Let Q be a qo. There is a bad Q-pattern if and only if there is a bad Q-array.

In light of Corollary 2.2.3 we will follow Nash-Williams and take *there is no bad Q-array* as our working definition of bqo, until it is replaced by Simpson's in the next section.

It is immediate from the definitions that every wellorder is bqo and every bqo is wqo. To wit, certainly every block contains an infinite $\lhd$-increasing sequence $b_1 \lhd b_2 \lhd \cdots$; the values of a purportedly bad Q-array at the b_i would give an infinite strictly decreasing sequence in a wellorder Q.[2] And every bad Q-sequence is a bad Q-array, since $[\omega]^1$ is a barrier on which the relation $\lhd$ is simply the usual order $<$ on ω.

More can be said about the combinatorics of blocks and barriers, but we will adopt Simpson's approach to bqos for the remainder of the essay.

2.3 Simpson's insight

In [18] Simpson provides a new topological perspective for bqo theory. His chief insight is that a Q-array f on a barrier B can be thought of as a continuous (wrt the product topology on $[\omega]^\omega$) function $f' \colon [\omega]^\omega \to Q$, because continuity gives finite sequences $s \in [\omega]^{<\omega}$ power to dictate where f' sends their end-extensions. From this perspective the fiddly combinatorial definitions of blocks, barriers, and $\lhd$ are hidden away behind a tidy topological definition of *bad array*.

NB. There are at least two sensible ways to topologise $[A]^\omega$, for $A \in [\omega]^\omega$:

(i) By endowing $[A]^\omega$ with the topology it inherits as a subspace of 2^ω (after sets are identified with their characteristic functions), which has the product topology: This is known as the **product topology**, the **τ-topology**, and the **usual topology** on $[A]^\omega$. This will be our default choice of topology for $[A]^\omega$, and any unqualified use of, e.g., *open* will mean *open wrt the product topology.* In a situation that is likely to cause confusion, we will say, e.g., A is *τ-open.*

(ii) By endowing $[A]^\omega$ with the **$*$-topology** (aka **Ellentuck topology**): The $*$-topology is strictly finer (i.e., has more open sets) than the product topology; its basic open sets are of the form

$$[s, U] := \{X \subseteq [A]^\omega : s \text{ an initial segment of } X \text{ and } X \smallsetminus s \subseteq U\},$$

for $s \in [A]^{<\omega}$ and $U \in [A]^\omega$. We will use *$*$-open*, *$*$-continuous*, etc., to mean *open, continuous, etc., wrt to the $*$-topology.* Notice that this notation allows us to write the basic open sets for the product topology as $[s, A]$ for $s \in [A]^{<\omega}$.

Definition. Let Q be quasi-ordered, and endow Q with the discrete topology. A **Q-Sarray** (*S* for *Simpson*) is a (τ-)continuous function $g \colon [A]^\omega \to Q$ for some $A \in [\omega]^\omega$. A Q-Sarray is **bad** if, for every $X \in [A]^\omega$,

$$g(X) \not\leq_Q g({}_*X).$$

A Q-Sarray that is not bad is **good**.

[2]The reader sensitive to uses of AC may be reassured to know that there is a direct proof that wo $\to$ bqo using the non-DC definition of wellorder.

In light of the following theorem of Mathias, *τ-continuous* can be replaced in the definition of *bad Q-Sarray* by *$*$-continuous* or by *Borel*. Q-Sarrays are Borel maps in Simpson's paper [18], and they are (puzzlingly) $*$-continuous maps in Kühn's paper [5]. This simplified version of Mathias' theorem (6.1 of [11]) is the one provided in Simpson ([18], 9.10).

Theorem 2.3.1 (Mathias)**.** Suppose $A \in [\omega]^\omega$, X is a metric space, and $f\colon [A]^\omega \to X$ is a Borel map. Then There is a $B \in [A]^\omega$ such that $f{\restriction}_{[B]^\omega}$ is continuous.

The proof that this (Simpson's) definition of *bqo* is equivalent to Nash-Williams' is nowhere to be found in the literature, so I provide it here.

Proposition 2.3.2. A quasiorder Q is bqo iff there is no bad Q-Sarray.

Proof. We will prove that there is a bad Q-array iff there is a bad Q-Sarray.

Suppose first that there is a bad Q-array $f\colon B \to Q$ on a barrier B. Put $A = \bigcup B$, the base of B. For each $X \in [A]^\omega$ there is a unique initial segment $X \wedge B$ of X in B. Let $g\colon [A]^\omega \to Q$ be the function that sends X to $f(X \wedge B)$. We need to show that g is continuous and bad. Let $X \in [A]^\omega$ and suppose $Y \in [A]^\omega$ is any end-extension of $X \wedge B$ (i.e., any member of $[X \wedge B, A]$). Then $Y \wedge B = X \wedge B$ by the uniqueness of $Y \wedge B$, so

$$g(Y) = f(Y \wedge B) = f(X \wedge B) = g(X).$$

So every member of $[A]^\omega$ has a basic open neighbourhood on which g is constant; that is, g is continuous. Now we show that g is bad. Let $X \in [A]^\omega$ and recall that ${}_*X$ denotes $X \smallsetminus \{\max X\}$. The fact that B is a barrier forbids ${}_*X \wedge B \subset X \wedge B$, so it must be that $X \wedge B \lhd {}_*X \wedge B$. But now we see that g easily inherits badness from f:

$$g(X) = f(X \wedge B) \not\leq_Q f({}_*X \wedge B) = g({}_*X).$$

For the other direction, assume there is a bad Q-Sarray $g\colon [A]^\omega \to Q$. We can identify A with ω by an increasing enumeration of A, so wlog $A = \omega$. Let $X \in [\omega]^\omega$. Continuity says that g is constant on the set of end-extensions of a proper initial segment s of X. Let $\operatorname{sat}(X)$ denote the shortest such s.[3] Now let B be the set of all such $\operatorname{sat}(X)$: $B := \operatorname{sat}''[\omega]^\omega$. For every $X \in [\omega]^\omega$, certainly $\operatorname{sat}(X)$ is an initial segment of X that belongs to B, so B is a block (not a barrier!). If $\operatorname{sat}(X) = \operatorname{sat}(Y)$, then Y is an end-extension of $\operatorname{sat}(X)$, so $g(X) = g(Y)$. As a result, $f(\operatorname{sat}(X)) := g(X)$ unambiguously defines a Q-pattern $f\colon B \to Q$. To complete the proof, we will show that f is bad. Suppose $b_1 = \operatorname{sat}(X_1)$ and $b_2 = \operatorname{sat}(X_2)$ are members of B such that $b_1 \lhd b_2$. Let $X = b_1 \cup b_2 \cup \{n \in \omega : n > \max b_2\}$. (Any end-extension X of b_1 such that ${}_*X$ is an end-extension of b_2 will do.) Then X is an end-extension of b_1, so $g(X) = g(X_1) = f(b_1)$; and ${}_*X$ is an end-extension of b_2, so $g({}_*X) = g(X_2) = f(b_2)$. We conclude that

$$f(b_1) = g(X) \not\leq_Q g({}_*X) = f(b_2),$$

[3] *sat* abbreviates *shortest approximation to.*

so f is a bad Q-pattern. The observation that there is a bad Q-pattern iff there is a bad Q-array (2.2.3) completes the proof. ∎

One essential method for proving a qo is bqo is the so-called *minimal bad array* technique, whereby one shows that no bad Q-array exists by deducing absurdity from the existence of a *minimal bad* Q-array. Nash-Williams introduced the technique in [14], and it was first synthesised into a theorem by Laver ([8], Theorem 1.9). We prove the theorem here both because proving it by Simpson's approach to bqos is slightly less stressful than proving it using the combinatorial definition, and because it is essential for the next section's results. Our main result — that Q bqo implies Q-labelled trees are bqo — relies on the minimal bad sequence technique, as Nash-Williams' original proof did. Appealing directly to the Minimal Bad Array Lemma will allow us to present a proof that is tidier and more modern in style than a straightforward generalisation of Nash-Williams' proof would be.

Hereafter we will use only the Simpsonian definition of *bqo*, so we will refer to *Q-Sarrays* as *Q-arrays* and use *bqo* to mean *has no bad Q-Sarray.*

The proof presented here of the Minimal Bad Array Lemma is Simpson's proof (9.17 of [18]). (Notice, though, that we require Q-arrays to be continuous, whereas Simpson requires them to be only Borel.) Many thanks to Zachiri McKenzie for presenting it to the Reading Group. In [8] Laver presents a proof (1.9) using Nash-Williams' original combinatorial definition of bqo. Another version of Laver's proof appears as 2.19 in Milner's paper [12].

It is often useful to consider a finer version of a quasi-order on Q. If Q is quasi-ordered by $\leq_Q$, and $\leq'$ is a well-founded partial order on Q 'compatible' with $\leq$ — that is, $q_1 \leq' q_2$ implies $q_1 \leq_Q q_2$ — then we say $<'$ is a **partial ranking** of the qo Q. We will often refer to a quasi-order as **ranked** when there is an understood partial ranking attached to it.

Definition. Let Q be a qo ranked by $<'$. For bad Q-arrays $f\colon [A]^\omega \to Q$ and $g\colon [B]^\omega \to Q$, define $\leq^*$ and $<^*$ thus:

$$f \leq^* g \text{ iff } B \subseteq A \text{ and } g(X) \leq' f(X) \text{ for every } X \in [B]^\omega;$$

$$f <^* g \text{ iff } B \subseteq A \text{ and } g(X) <' f(X) \text{ for every } X \in [B]^\omega.$$

A bad Q-array f is **minimal bad** if f is $<^*$-minimal among bad Q-arrays.

Warning: $<^*$ is not the strict part of the relation $\leq^*$. The relations $<^*$ and $\leq^*$, introduced by Simpson, are the topological analogues of the 'forerun' relations introduced by Nash-Williams in [14]. Note, however, that the forerunning relations are (strangely) backwards: '$f \leq^* g$' corresponds to 'g foreruns f' and '$f <^* g$' corresponds to 'g strictly foreruns f'.

Theorem 2.3.3 (Minimal Bad Array Lemma)**.** Let Q be a ranked qo. Suppose $f\colon [A]^\omega \to Q$ is a bad Q-array. Then there exists a minimal bad Q-array g such that $g \leq^* f$.

Proof. The idea is to assume the theorem is false and obtain a contradiction by producing an unreasonably long strictly decreasing sequence of subsets of ω. Suppose the theorem is false and set $f_0 = f$, $A_0 = A$.

We will define inductively a sequence $\langle f_\alpha \colon [A_\alpha]^\omega \to Q\rangle_{\alpha\in\omega_1}$ such that

(i) each f_α is bad; and

(ii) if $\alpha < \beta$ then $f_\beta \leq^* f_\alpha$ and $A_\alpha \neq A_\beta$ (though $A_\alpha \subset A_\beta$ since $f_\beta \leq^* f_\alpha$).

This is a problem, since choosing a member of each relative complement $A_\alpha \smallsetminus A_{\alpha+1}$ would give $\aleph_1$ distinct members of ω.

Suppose f_α has been defined and f_β for every $\beta < \alpha$ has been defined. Since $f_\alpha \leq^* f_0$, f is not minimal bad; thus there is a bad Q-array $g_\alpha \colon [B_\alpha]^\omega \to Q$ such that $g_\alpha <^* f_\alpha$ and $A_\alpha \smallsetminus B_\alpha$ is infinite. (Shrink B_α as necessary until $A_\alpha \smallsetminus B_\alpha$ is infinite.) The continuity of g_α ensures that there is an initial segment s_α of B_α such that g_α is constant (and takes the value B_α) on $[s_\alpha, B_\alpha]$. (g_α is not quite good enough to be $f_{\alpha+1}$, which will need a domain larger than B_α so that we can handle the limit case.) Define

$$A_{\alpha+1} := B_\alpha \cup \{n \in A_\alpha : n \leq \max s_\alpha\}\,.$$

Observe that $A_{\alpha+1}$ is a proper subset of A_α, since $A_\alpha \smallsetminus B_\alpha$ is infinite. Define $f_{\alpha+1}$ on $[A_{\alpha+1}]^\omega$ as follows:

$$f_{\alpha+1}(X) = \begin{cases} g_\alpha(X) & \text{if } X \in [B_\alpha]^\omega \\ f_\alpha(X) & \text{if } X \in [A_{\alpha+1}]^\omega \smallsetminus [B_\alpha]^\omega. \end{cases}$$

Certainly $f_{\alpha+1} \leq^* f_\alpha$, since $g_\alpha <^* f_\alpha$. Suppose that $f_{\alpha+1}(X) \leq_Q f_{\alpha+1}({}_*X)$ for some $X \in [A_{\alpha+1}]^\omega$. The only possibility is that $f_{\alpha+1}(X) = f_\alpha(X)$ and $f_{\alpha+1}({}_*X) = g_\alpha({}_*X)$, as the other possibilities are immediately ruled out. Then we have

$$f_\alpha(X) \leq_Q g_\alpha({}_*X) <_Q f_\alpha({}_*X),$$

which contradicts the badness of f_α. Therefore $f_{\alpha+1}$ is bad.

Both $[B_\alpha]^\omega$ and $[A_{\alpha+1}]^\omega \smallsetminus [B_\alpha]^\omega$ are open in $[A_{\alpha+1}]^\omega$, and both f_α and g_α are continuous, so $f_{\alpha+1}$ is continuous. This concludes the successor case.

Now suppose α is a limit and for all $\gamma < \beta < \alpha$ we have defined f_γ and f_β so that $f_\beta \leq^* f_\gamma$. Suppose also that we have defined g_β, s_β, and B_β as in the successor case. Define $A_\alpha := \bigcap\{A_\beta : \beta < \alpha\}$.

Claim. A_α is infinite.

Proof of Claim. Suppose for a contradiction that $A_\alpha \subseteq m < \omega$. For every $\beta < \alpha$ let n_β be the least $n \geq m$ such that $n \in A_\beta$. The set $C := \{\beta < \alpha : n_\beta \notin A_{\beta+1}\}$ is infinite: indeed, if cofinitely many β satisfied $n_\beta \in A_{\beta+1}$, then the function $\beta \mapsto n_\beta$ would be eventually constant, and its eventual value would belong to A_α. ($\beta \mapsto n_\beta$ is continuous, by our definition of A_β for β a limit.) This is ludicrous, since we have assumed $A_\alpha \subseteq m$ and $n_\beta \geq m$. Recall that $s_\beta \subseteq A_\beta$. For every $\beta \in C$, $n_\beta \in A_\beta \smallsetminus A_{\beta+1}$; so $n_\beta > \max s_\beta$ by the definition of $A_{\beta+1}$. No member of A_β sits between m and n_β, so we conclude that $\max s_\beta < m$ for every $\beta \in C$. But there are infinitely many $\beta \in C$ and only finitely many possible finite sequences s_β with terms $< m$, so there must be some infinite set $C' \subseteq C$ such that $s_\beta = s_\gamma$ for every pair β, γ of elements of C'. But if $\beta < \gamma$, we have $B_\gamma \subseteq B_\beta$, so $B_\gamma \in [s_\gamma, B_\beta] = [s_\beta, B_\beta]$. But this yields the inequality

$$f_\gamma(B_\gamma) \leq' f_{\beta+1}(B_\gamma) = g_\beta(B_\beta) <' f_\beta(B_\beta).$$

Thus the infinite set C' gives an infinite strictly decreasing $<'$-sequence, which contradicts the well-foundedness of the partial ranking $<'$. We are forced to conclude that A_α is infinite, which was the Claim. $\blacksquare_{\text{Claim}}$

Notice that for every $X \in [A_\alpha]^\omega$ the sequence $\langle f_\beta(X)\rangle_{\beta<\alpha}$ is $\leq'$-decreasing and therefore must be eventually constant. With this in mind, define f_α to be the pointwise limit of the f_β, $\beta < \alpha$:

$$f_\alpha(X) = \lim_{\beta<\alpha} f_\beta(X).$$

The pointwise limit of Borel maps is Borel, so f_α is Borel. By Mathias' theorem 2.3.1 we can restrict A_α further to assume that f_α is continuous. For every $X \in [A_\alpha]^\omega$ the values of $f_\alpha(X)$ and $f_\alpha({}_*X)$ are just the values of $f_\beta(X)$ and $f_\beta({}_*X)$ for some $\beta < \alpha$, so f_α must be bad. One can see that $f_\alpha \leq^* f_\beta$ for $\beta < \alpha$ by choosing γ such that $\beta < \gamma < \alpha$ and $f_\alpha(X) = f_\gamma(X)$ for a given $X \in [A_\alpha]^\omega$. Then $f_\alpha(X) = f_\gamma(X) \leq' f_\beta(X)$, so we see that f_α satisfies all required conditions. This concludes the limit case and the proof. $\blacksquare$

2.4 Bqo-preserving operations

Because infinitary operations rarely preserve wqoness (see §2.1) wqo theory alone failed to provide Nash-Williams with the tools necessary to prove his infinite tree theorem [14]. He therefore found it necessary to introduce the *bqo* notion, since (as it turns out) bqoness is stable under infinitary operations. This section's purpose is to depict the wealth of qo-to-qo operations that preserve bqoness. The title 'Applications of the Minimal Bad Array Lemma' would have been just as accurate: every nontrivial result in this section makes typical use of the Minimal Bad Array Lemma.

Our proof in the next chapter that Q-labelled trees are bqo will rely heavily on results from this section, but we will need 'effective' versions of the results. That is, it won't suffice to know simply that Q bqo implies $\mathbf{X}Q$ bqo for some operation $\mathbf{X}$; our result should provide a suitably well-behaved bad array on Q given a bad array on $\mathbf{X}Q$.

The following theorem (which, according to [1], was proved first by our own Nash-Williams) is a consequence of the Galvin–Prikry lemma.

Theorem 2.4.1 (Nash-Williams [15])**.** Open sets are Ramsey. That is, if $A \subseteq [\omega]^\omega$ is open, then there is $B \in [A]^\omega$ such that either $[B]^\omega \subseteq A$ or $[B]^\omega \cap A = \varnothing$.

Definition. For a qo Q, a Q-array $f\colon [A]^\omega \to Q$ is **perfect** if $f(X) \leq_Q f({}_*X)$ for every $X \in [A]^\omega$.

The following Corollary (a version of the Perfect Subarray Lemma 2.4.3) is proved in several sources using Nash-Williams' original blocks-and-barriers definition; our proof is adapted from the standard one to use Simpson's definition.

Corollary 2.4.2. Suppose $f\colon [A]^\omega \to Q$ is a Q-array. There exists $B \in [A]^\omega$ such that $f{\restriction}_{[B]^\omega}$ is either bad or perfect.

Proof. Let $C = \{X \in [A]^\omega : f(X) \leq_Q f({}_*X)\}$. We need to show that C is open to apply Theorem 2.4.1. Recall the sat function from the proof of Proposition 2.3.2: $\mathrm{sat}(X)$ is the shortest proper initial segment of X such that f is constant on $[\mathrm{sat}(X), A]$. Suppose $X \in C$ and let Y be any end-extension in A of $\mathrm{sat}\, X \cup \mathrm{sat}\, {}_*X$. Then $f(X) = f(Y)$ and $f({}_*X) = f({}_*Y)$, so $Y \in C$. That is, the neighbourhood $[\mathrm{sat}\, X \cup \mathrm{sat}\, {}_*X, A]$ of X is contained in C. Therefore C is open. Since open sets are Ramsey, there is $B \in [A]^\omega$ such that $[B]^\omega \subseteq C$ or $[B]^\omega \cap C = \varnothing$. This is exactly the statement that $f{\restriction}_{[B]^\omega}$ is either bad or perfect. ∎

In particular, we have:

Corollary 2.4.3 (Perfect Subarray Lemma)**.** If $f\colon [A]^\omega \to Q$ is a Q-array and Q is bqo, then there is a $B \in [A]^\omega$ such that $f{\restriction}_{[B]^\omega}$ is perfect.

(Compare wqo theory, where Ramsey's theorem guarantees that every Q-sequence on a wqo Q has a perfect subsequence.)

The first operation we will prove preserves bqoness is the binary union operation. Our proof is simply the Simpsonian version of the standard combinatorial proof, which can be found in many sources (e.g., Nash-Williams' original paper [14]).

Proposition 2.4.4. If $Q = Q_1 \cup Q_2$ and both Q_1 and Q_2 are bqo, then so is Q.

Proof. Suppose $f\colon [A]^\omega \to Q$ is bad. We may assume Q is nonempty, so either $f_{-1}Q_1$ or $f_{-1}Q_2$ is nonempty, and each is open. Wlog $f_{-1}Q_1$ is nonempty. By Open Ramsey 2.4.1 there is an infinite $B \subseteq A$ such that either $[B]^\omega \subseteq f_{-1}Q_1$ or $[B]^\omega \cap f_{-1}Q_1 = \varnothing$. In the second case, we just have $[B]^\omega \subseteq f_{-1}Q_2$, so $f{\restriction}_{[B]^\omega}$ is either a bad Q_1- or Q_2-array. This is a contradiction, so no such f exists. ∎

The assertion that Q, Q' bqo implies $Q \times Q'$ bqo is an easy consequence of Proposition 2.4.5. Recall that we order the product $Q \times Q'$ of two qos Q and Q' by the *product ordering*: $(a, a') \leq (b, b')$ iff $a \leq_Q b$ and $a' \leq_{Q'} b'$.

Proposition 2.4.5. Suppose Q' is bqo and $f = (f_1, f_2)$ is a bad $Q \times Q'$-array with domain $[A]^\omega$. Then there is a $B \in [A]^\omega$ and a bad Q-array $g\colon [B]^\omega \to Q$ such that $g = f_1{\restriction}_{[B]^\omega}$.

Proof. By Corollary 2.4.2, there is $B \in [A]^\omega$ such that $f_2{\restriction}_{[B]^\omega}$ is perfect. Set $g = f_1{\restriction}_{[B]^\omega}$. Since $f{\restriction}_{[B]^\omega}$ is bad, it must be that $f_1(X) \not\leq_Q f_1({}_*X)$ for every $X \in [B]^\omega$. That is, g is bad. ∎

The next theorem is the main theorem of Nash-Williams' 1968 paper [16]. Though Nash-Williams' original proof is fairly involved, Simpson's topological approach and the Minimal Bad Array Lemma 2.3.3 take care of many of the technical complications, leaving us to prove the result with little difficulty. The proof we present here is nearly identical to the proof Simpson provides (9.19 of [18]). A reader looking for a less lucid proof should see [5], Lemma 4.

Write $\operatorname{Seq} Q$ (Simpson writes $\widetilde{Q}$) for the class of sequences $\alpha \to Q$, α an ordinal. Recall from Chapter 1 that we quasi-order $\operatorname{Seq} Q$ as follows: if $s\colon \alpha \to Q$ and $t\colon \beta \to Q$ are sequences, say $s \leq t$ iff there is a strictly increasing function $\phi\colon \alpha \to \beta$ such that $s(\eta) \leq_Q t(\phi(\eta))$ for every $\eta \in \alpha$. As usual we write $\operatorname{lh}(s)$ for $\operatorname{dom}(s)$, the *length of* s, and we write $s \preceq t$ to mean s *is an initial segment of* t.

Lemma 2.4.6. If $s, t \in \operatorname{Seq} Q$ satisfy $s \not\leq t$, then there is a $\theta < \operatorname{lh}(s)$ such that $s{\restriction}_\theta \leq t$ and $s{\restriction}_{\theta+1} \not\leq t$.

Proof. Inductively define h as follows:

$$h(\xi) := \text{ the least } \eta < \operatorname{lh}(t) \text{ such that } s(\xi) \leq t(\eta) \text{ and } \eta > h(\xi') \text{ for every } \xi' < \xi.$$

Let θ be the least ξ such that $h(\xi)$ is undefined. Then $t \circ h$ witnesses that $s{\restriction}_\theta \leq t$, but $s{\restriction}_{\theta+1} \not\leq t$. ∎

Theorem 2.4.7 (Nash-Williams [16])**.** Suppose $g\colon [A]^\omega \to \operatorname{Seq} Q$ is a bad $\operatorname{Seq} Q$-array. There is a set $B \in [A]^\omega$ and a bad $f\colon [B]^\omega \to Q$ such that $f(X)$ is a term of the sequence $g(X)$ for every $X \in [B]^\omega$.

Proof. Define an ordering $\leq'$ on $\operatorname{Seq} Q$ as follows: for $s\colon \alpha \to Q$ and $t\colon \beta \to Q$, say $s \leq' t$ iff $\alpha \leq \beta$ and $s = t{\restriction}_\alpha$ (i.e., s is an initial segment of t). It is easy to see that $\leq'$ is well-founded and a partial order, so $\operatorname{Seq} Q$ is ranked by $\leq'$. By the Minimal Bad Array Lemma 2.3.3 we may assume that the bad $\operatorname{Seq} Q$-array g is minimal bad (wrt $\leq'$).

For every $X \in [A]^\omega$ we have $g(X) \not\leq g({}_*X)$. Thus Lemma 2.4.6 gives for every X a $\theta(X)$ such that

$$g(X){\restriction}_{\theta(X)} \leq g({}_*X) \text{ and } g(X){\restriction}_{\theta(X)+1} \not\leq g({}_*X).$$

Note that the following inequality of $\operatorname{Seq} Q$-arrays holds:

$$(X \mapsto g(X){\restriction}_{\theta(X)}) <^* g.$$

(Indeed, certainly $g(X){\restriction}_{\theta(X)} \leq' g(X)$ for every $X \in [A]^\omega$. And g is bad, so we cannot have $g(X) \leq' g(X){\restriction}_{\theta(X)} \leq g({}_*X)$. Hence $g(X){\restriction}_{\theta(X)} <' g(X)$.) Because g is minimal bad, $(X \mapsto g(X){\restriction}_{\theta(X)})$ cannot be a bad $\operatorname{Seq} Q$-array; so it has a perfect subarray (Corollary 2.4.2). That is, there is a $B \in [A]^\omega$ such that $g(X){\restriction}_{\theta(X)} \leq g({}_*X){\restriction}_{\theta({}_*X)}$ for every $X \in [B]^\omega$. Consequently,

$$g(X){\restriction}_{\theta(X)} \leq g({}_*X){\restriction}_{\theta({}_*X)} \text{ but } g(X){\restriction}_{\theta(X)+1} \not\leq g({}_*X){\restriction}_{\theta({}_*X)+1}. \qquad (*)$$

Writing f for the Q-array that maps $X \in [B]^\omega$ to $g(X)[\theta(X)]$, the $\theta(X)$th term of the sequence $g(X)$, we conclude from $(*)$ that f is a bad Q-array. Certainly $f(X)$ is a term of the sequence $g(X)$, so the proof is complete. ∎

It is immediate from Theorem 2.4.7 that Q bqo $\to$ $\operatorname{Seq} Q$ bqo. Another important corollary to the theorem is the fact that Q bqo $\to \mathscr{P}Q$ bqo, ordered either by $\leq_1$ or by $\leq_m$. We will need effective versions of these corollaries in our proof that Q-labelled trees are bqo.

The following corollary is essentially Lemma 28 of Nash-Williams' paper [14], and it also appears as Corollary 5 in Kühn's paper [5]. (Note $\mathscr{P}Q$ quasi-ordered by $\leq_1$ is denoted by $\mathfrak{S}^{\#}(Q)$ (ugh) in [14] and by $\mathscr{S}^{\#}(Q)$ in [5].)

Corollary 2.4.8. If $f\colon [A]^\omega \to \mathscr{P}Q$ is a bad $(\mathscr{P}Q, \leq_1)$-array, then there is a $B \in [A]^\omega$ and a bad Q-array $g\colon [B]^\omega \to Q$ such that $g(X) \in f(X)$ for every $X \in B$.

Proof. Let E assign to each $A \in \mathscr{P}Q$ a sequence enumerating A. If f is a bad $(\mathscr{P}Q, \leq_1)$-array, then $E \circ f$ is a bad $\operatorname{Seq} Q$-array (since $A \not\leq_1 B \to EA \not\leq_{\operatorname{Seq} Q} EB$). By Theorem 2.4.7 there is a $B \in [A]^\omega$ and a bad Q-array $g\colon [B]^\omega \to Q$ such that for every $X \in [B]^\omega$ the element $g(X)$ is a term of the sequence $E \circ f(X)$. But '$g(X)$ is a term of $E \circ f(X)$' just means $g(X) \in f(X)$. ∎

The following corollary is a trivial consequence of Corollary 2.4.8, but we state it explicitly since we will need it in our proof that Q-labelled trees are bqo.

Corollary 2.4.9. If $f\colon [A]^\omega \to \mathscr{P}Q$ is a bad $(\mathscr{P}Q, \leq_m)$-array, then there is a $B \in [A]^\omega$ and a bad Q-array $g\colon [B]^\omega \to Q$ such that $g(X) \in f(X)$ for every $X \in B$.

The following result is not necessary for our proofs in Chapter 3, but any list of applications of the Minimal Bad Array Lemma would be incomplete without it.

We have shown (2.4.8) that $\mathscr{P}Q$ (under $\leq_1$ or $\leq_m$) is bqo, but what if we iterate $\mathscr{P}$? Define $\mathscr{P}^\alpha Q$ inductively as follows: $\mathscr{P}^0 Q = Q$, $\mathscr{P}^{\alpha+1} Q = \mathscr{P}^\alpha Q$, and $\mathscr{P}^\beta Q = \bigcup_{\alpha<\beta} \mathscr{P}^\alpha Q$ for β a limit. Write $\mathscr{P}^\Omega Q$ for the class $\bigcup_{\alpha \in \mathrm{ORD}} \mathscr{P}^\alpha Q$, which we quasi-order as follows. We may assume for simplicity $Q \cap \mathscr{P}^{\alpha+1} Q = \varnothing$ for all α. For $x \in \mathscr{P}^\alpha Q$ and $y \in \mathscr{P}^\beta Q$, we define inductively a quasi-order $\leq_m$ on $\mathscr{P}^\Omega Q$:

(i) if $\alpha = \beta = 0$, then $x \leq_m y$ iff $x \leq_Q y$;

(ii) if $\alpha = 0$ and $\beta > 0$, then $x \leq_m y$ iff $x \leq_m z$ for some $z \in y$;

(iii) if $\alpha > 0$ and $\beta > 0$, then $x \leq_m y$ iff there is $h\colon x \to y$ such that $w \leq_m h(w)$ for every $w \in x$.

We obtain an order $\leq_1$ on $\mathscr{P}^\Omega Q$ using the same definition, except we require that the function h in (iii) be injective.

The following proof is essentially Milner's [12], though I have modified it to use Simpson's approach to bqos. For a proof that $\mathscr{P}^\Omega Q$ ordered by $\leq_1$ is bqo, see 1.11 of Laver [8].

Theorem 2.4.10. If Q is bqo, then $(\mathscr{P}^\Omega Q, \leq_m)$ is bqo.

Proof. For $x \in \mathscr{P}^\Omega Q$ write $\rho(x)$ for the least α such that $x \in \mathscr{P}^\alpha Q$. Define a partial order on $\mathscr{P}^\Omega Q$ by declaring $x \leq' y$ iff $\rho(x) \leq \rho(y)$. It is clear that $<'$ is a partial ranking of $\mathscr{P}^\Omega Q$.

Suppose $f\colon [A]^\omega \to \mathscr{P}^\Omega Q$ is bad. By the Minimal Bad Array Lemma 2.3.3 we may assume f is minimal bad. Recalling that Q is given the discrete topology and that f is continuous, we observe that, by Open Ramsey 2.4.1, there is $B \in [A]^\omega$ such that either $[B]^\omega \subseteq f_{-1}Q$ or $[B]^\omega \cap f_{-1}Q = \varnothing$. We cannot have $B \subseteq f_{-1}Q$; else $f\restriction_{[B]^\omega}$ would be a bad Q-array. Therefore f maps every member of $[B]^\omega$ into $\{x \in \mathscr{P}^\Omega Q : \rho(x) > 0\}$. Fix $X \in [B]^\omega$. Since f is bad, $f(X) \not\leq_m f({}_*X)$. This and the fact that $\rho(f(X)) > 0$ give a

$$g(X) \in f(X) \text{ such that } g(X) \not\leq_m y \text{ for every } y \in f({}_*X).$$

Because the value of f at every end-extension of $\operatorname{sat}(X)$ is just $f(X)$, we can pick $g(X)$ to depend only on $\operatorname{sat}(X)$ so that g is a continuous map $[B]^\omega \to \mathscr{P}^\Omega Q$. Notice $g(X) <' f(X)$ and that $g(X) \not\leq g({}_*X)$ since $g({}_*X) \in f({}_*X)$. That is, g is a bad $\mathscr{P}^\Omega Q$-array and $g <^* f$. This contradicts the minimal badness of f, so our assumption that a bad f exists is false. ∎

To conclude the chapter we provide a(n incomplete) list of operations that preserve bqo.

Theorem 2.4.11. Let $(Q, \leq)$ be bqo.

(i) If R is qo and $f\colon Q \to R$ is an order-preserving map, then $f''Q$ is bqo.

(ii) If $Q' \subseteq Q$, then Q' is bqo.

(iii) If $\leq'$ is a quasi-order on Q containing $\leq$, then $(Q, \leq')$ is bqo.

(iv) If Q' is bqo, then $Q \times Q'$ is bqo.

(v) If $Q \subseteq R$ and $R \smallsetminus Q$ is bqo, then R is bqo.

(vi) $\operatorname{Seq} Q$ is bqo;

(vii) $\mathscr{P}Q$, ordered by $\leq_m$ or by $\leq_1$, is bqo;

(viii) $\mathscr{P}^\Omega Q$ is bqo.

Chapter 3

Q bqo $\to \mathcal{T}_Q$ bqo

3.1 Laver's proof of Fraïssé's conjecture

Though this essay presents in full only one piece of Laver's proof of Fraïssé's conjecture, this section aims to outline briefly the rest of Laver's proof. Recall from the introduction that Fraïssé's conjecture is the assertion that

Scattered linear ordertypes are wqo under embeddability. (Fraïssé)

Laver [6] settles this conjecture by proving a vast generalisation of it: 'If Q is bqo, then the class of *Q-labelled σ-scattered* ordertypes is bqo under embeddability'. As a corollary of this theorem, Laver observes that the σ-scattered (hence the scattered) ordertypes are bqo by taking Q to be the one-point bqo in the theorem. Before proceeding, we will define the terms in Laver's theorem.

Definition. The ordertype of a linear order L is **σ-scattered** iff L can be expressed as the countable union of scattered orders. The class of σ-scattered ordertypes is $\mathscr{M}$.

For Q a qo, a **Q-labelled** linear order (L, l) is a linear order L together with a labelling function $l\colon L \to Q$. Quasi-order the class of Q-labelled linear orders in the following way:

> $(L_1, l_1) \leq (L_2, l_2)$ iff there is a strictly increasing function $f\colon L_1 \to L_2$ such that $l_1(x) \leq_Q l_2(f(x))$ for every $x \in L_1$.

Say Q-labelled linear orders (L_1, l_1) and (L_2, l_2) are **Q-isomorphic** iff there is an order-isomorphism $f\colon L_1 \to L_2$ such that $l_1(x) = l_2(f(x))$ for every $x \in L_1$. The **Q-type** $\operatorname{tp}(L, l)$ of a Q-labelled linear order is its Q-isomorphism-type. For a class $\mathscr{C}$ of linear ordertypes, write $Q^{\mathscr{C}}$ for the class of Q-types $\operatorname{tp}(L, l)$ such that $\operatorname{tp} L \in \mathscr{C}$.

Now we can state Laver's theorem properly:

Theorem 3.1.1 (Theorem 4.8 of [6])**.** If Q is bqo, then $Q^{\mathscr{M}}$ is bqo.

Though technically complicated, Laver's proof avoids fiddly block-and-barrier computations. Remarkably, it relies entirely on bqo-preservation results, without ever mentioning

Nash-Williams' combinatorial definition of *bqo*. This might be seen as evidence that bqo is a robust notion. (It might also suggest that the bqo notion could be characterised category-theoretically by some universal property.)

Next we describe Laver's strategy for proving 3.1.1. The chief task is to build a class $\mathscr{H}(Q)$ of *hereditarily regular unbounded* Q-types with the following properties:

(i) $\mathscr{H}(Q) \subseteq Q^{\mathscr{M}}$;

(ii) Q bqo $\to \mathscr{H}(Q)$ bqo;

(iii) every member of $Q^{\mathscr{M}}$ can be expressed as a finite sum of members of $\mathscr{H}(Q)$.

From the existence of such an $\mathscr{H}(Q)$ the theorem follows:

$$Q \text{ bqo} \to \mathscr{H}(Q) \text{ bqo} \to {}^{<\omega}\mathscr{H}(Q) \text{ bqo} \to Q^{\mathscr{M}} \text{ bqo}.$$

The remaining difficulty — and the point where Laver relies on the generalisation of Nash-Williams' theorem — is to establish (ii), that $\mathscr{H}(Q)$ is bqo if Q is. Laver overcomes this difficulty by defining a suitable extension Q^+ of Q such that Q bqo $\to Q^+$ bqo, and $\mathscr{H}(Q)$ is a homomorphic image of $\mathcal{T}_{Q+}$ (the class of Q^+-labelled trees). Then an appeal to the fact that Q-labelled trees are bqo completes the proof.

We conclude by remarking that — as Laver explains on page 41 of [8] — Theorem 3.1.1 can be obtained by appealing to the Minimal Bad Array Lemma rather than the fact that Q-labelled trees are bqo. Thus his theorem provides yet another example of an operation that can be proved bqo-preserving using the Minimal Bad Array Lemma.

3.2 Q-labelled trees are bqo

Fix a bqo Q for the remainder of this chapter.

In this section we, as promised, will prove that $\mathcal{T}_Q$, the class of Q-labelled trees, is bqo. The trees of interest to us will be *rooted trees of height* ω.

Definition. A **tree** is a partially ordered set $(T, \leq_T)$ such that $\{y \in T : y <_T x\}$ is well-ordered (by $\leq_T$) for every node $x \in T$. A **rooted tree of height** ω is a tree such that

(i) T has a unique **root**, a node x such that $x \leq_T y$ for every $y \in T$; and

(ii) for every node $y \in T$, the ordertype of (the wellordered chain) $\{z \in T : z <_T y\}$ is at most ω.

Let $\mathcal{T}$ denote the class of rooted trees of height ω.

Some familiar terminology and notation will help us discuss trees more easily. For us, the term *successor* will always mean *immediate successor*; that is, in a tree, a node y is a **successor** of a node x iff $y \geq x$ and no z satisfies $y > z > x$. Write $\mathrm{S}(x)$ for the set of successors of x (the tree of which x is a node should be understood).

The class $\mathcal{T}$ is quasi-ordered in the following way: for $T_1, T_2 \in \mathcal{T}$,

$$T_1 \leq T_2 \text{ iff there is an injection } f\colon T_1 \to T_2 \text{ such that } \forall x, y \in T_1\ f(x \wedge y) = f(x) \wedge f(y).$$

We call such an f an **embedding** of T_1 into T_2. (Of course, $x \wedge y$ denotes the greatest lower bound of x and y.) The condition $f(x \wedge y) = f(x) \wedge f(y)$ is more memorably expressed as

'f sends distinct successors of $x \in T_1$ above distinct successors of $f(x) \in T_2$'.

More precisely, if y and z are distinct members of $\mathrm{S}(x)$ in T_1, then there are distinct w and v in $\mathrm{S}(f(x))$ such that $f(y) \geq_{T_2} w$ and $f(z) \geq_{T_2} v$.

The **branch** $\mathrm{br}(x)$ at x is the set $\{y \in T : y \geq_T x\}$ of nodes **above** x. A branch $\mathrm{br}(x)$ of T is **strict** iff $T \not\leq \mathrm{br}(x)$; a branch $\mathrm{br}(x)$ is **regular** if $T \equiv \mathrm{br}(x)$.

The main theorem of Nash-Williams' paper [14] is that $\mathcal{T}$ under our notion of embedding is bqo. To provide our piece of Laver's proof we need to generalise this theorem to the class of members of $\mathcal{T}$ 'decorated'[1] by elements of a bqo Q.

Definition. For a qo Q (especially a bqo), a **Q-labelled tree** is a tree $T \in \mathcal{T}$ equipped with a labelling (or colouring) function $l\colon T \to Q$ of its nodes by members of Q. We will often use an ordered pair (T, l) to denote the tree $T \in \mathcal{T}$ with labelling function $l\colon T \to Q$. Write $\mathcal{T}_Q$ for the class of Q-labelled trees.

Of course, we quasi-order $\mathcal{T}_Q$ by using embeddings that 'respect the Q-labelling': for $(T_1, l_1), (T_2, l_2) \in \mathcal{T}_Q$,

$$(T_1, l_1) \leq (T_2, l_2) \text{ iff } T_1 \leq_{\mathcal{T}} T_2 \text{ by a function } f \text{ such that } l_1(x) \leq_Q l_2(f(x)) \text{ for all } x \in T_1.$$

There are three major ideas in Nash-Williams' proof that $\mathcal{T}$ is bqo. The first is to reduce the problem to proving that the class of so-called 'descensionally finite' trees is bqo. To accomplish this, Nash-Williams proves that the set of descensionally finite branches of any descensionally infinite tree must fail to be bqo. (Our Q-labelled version of this result is Proposition 3.2.1.) This proof provides a prototype of the 'approximation technique' later used to exploit the Minimal Bad Array Lemma.

The second idea is to use the minimal bad array technique (here we will appeal explicitly to the Minimal Bad Array Lemma 2.3.3) to obtain a contradiction under the assumption that there is a bad $\mathcal{T}$-array.

But how is a contradiction reached? This is the third idea. To prove that every bad $\mathcal{T}$-array is $>^*$ another bad $\mathcal{T}$-array, we rely on the infinitary operations that preserve bqoness. (And this is where the effective versions of results in §2.4 earn their keep.) We approximate each tree T by $\Theta(T)$, an element of a suitable extension of $\mathcal{T}$, and then use the effective results from §2.4 to reduce a bad array on the $\Theta(T)$s to a bad $\mathcal{T}$-array $<^*$ our given bad array. The Minimal Bad Array Lemma then provides the contradiction.

Our proof that Q-labelled trees are bqo will follow this general plan. Our first task is to reduce the problem to showing that $\mathcal{F}_Q$, the class of descensionally finite Q-labelled trees,

[1] TF's term.

is bqo. The proof of Proposition 3.2.1 (Lemma 32′ in §2 of [6]) should give the reader a taste of how to approximate a (Q-labelled) tree in a way that allows us to appeal to the stability of bqoness under operations like $\mathscr{P}$. The Φ 'approximation' is similar in spirit to the Θ operation we will use to prove that every $\mathcal{T}_Q$-array is $>^*$ some other $\mathcal{T}_Q$-array.

Definition. Let (T, l) be a Q-labelled tree. If there is a sequence $x_1 < x_2 < \cdots$ of nodes of T such that $\mathrm{br}(x_1) > \mathrm{br}(x_2) > \cdots$, then we say (T, l) is **descensionally infinite**. Otherwise (T, l) is **descensionally finite**. Write $\mathcal{F}_Q$ for the class of descensionally finite Q-labelled trees. Let $\mathrm{F}(T, l)$ denote the set of descensionally finite branches of (T, l).

Of course, we want all Q-labelled trees to be descensionally finite, and we will eventually prove this.

Proposition 3.2.1. Let $(T, \ell) \in \mathcal{T}_Q$. If $\mathrm{F}(T, \ell)$ is bqo, then (T, ℓ) is descensionally finite. That is, $\mathrm{F}(T, \ell)$ bqo $\rightarrow (T, \ell) \in \mathcal{F}_Q$.

We will need to introduce some auxiliary functions to describe (T, l). For nodes x of T, define

$$\begin{aligned} \mathrm{dfba}(x) &:= \{\mathrm{br}(y) : y \in \mathrm{S}(x) \text{ and } \mathrm{br}(y) \text{ descensionally finite}\}\,; \\ \mathrm{diso}(x) &:= \{y \in \mathrm{S}(x) : \mathrm{br}(y) \text{ descensionally infinite}\}\,. \\ \Delta_{(T,l)}(x) &:= \langle \mathrm{dfba}(x), |\mathrm{diso}(x)|\,, \ell(x)\rangle.^2 \end{aligned}$$

(NB. $\mathrm{dfba}(x)$ is a set of branches of (T, l), whereas $\mathrm{diso}(x)$ is a set of nodes of T.) Finally,

$$\Phi_{(T,l)}(x) = \left\{\Delta_{(T,l)}(y) : x \leq_T y\right\}.$$

Thus $\Phi_{(T,l)}(x)$ is a member of the class $\mathscr{P}(\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q)$, which we quasi-order in the following way: $\mathscr{P}\mathcal{F}_Q$ is quasi-ordered by the ordering $\leq_1$ inherited from the quasi-order on $\mathcal{F}_Q$; $\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q$ is given the usual product ordering; and $\mathscr{P}(\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q)$ is given the $\leq_m$ ordering it inherits from the ordering on $\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q$.

Proof of 3.2.1. Since it shouldn't cause any ambiguity, we will abbreviate $\Delta_{(T,l)}$ by Δ and $\Phi_{(T,l)}$ by Φ in this proof.

By assumption $\mathrm{F}(T, l)$ is bqo, so $\mathscr{P}(\mathscr{P}(\mathrm{F}(T, l)) \times \textsc{card} \times Q)$ is bqo, hence wqo. Suppose $(T, l) \notin \mathcal{F}_Q$ and choose a node $x \in T$ such that $\mathrm{br}(x) \notin \mathcal{F}_Q$. We will obtain a contradiction by exhibiting a strictly decreasing sequence in the wqo $\mathscr{P}(\mathscr{P}(\mathrm{F}(T, l)) \times \textsc{card} \times Q)$.

Claim 1. There is a sequence $x <_T x_1 <_T x_2 <_T \cdots$ of nodes such that

$$\Phi(x) > \Phi(x_1) > \Phi(x_2) > \cdots.$$

Proof of Claim 1. Suppose for a contradiction that there is no $x_1 >_T x$ such that

[2] $\mathrm{dfba}(x)$ — the set of *descensionally finite branches above* x — is Laver's $L(x)$, and $\mathrm{diso}(x)$, the set of *descensionally infinite successors of* x, is Laver's $M(x)$.

$\mathrm{br}(x) \notin \mathcal{F}_Q$ and $\Phi(x) > \Phi(x_1)$. Thus every $z \geq_T x$ satisfies $\Phi(z) \leq \Phi(x)$. In particular,

$$(\forall u \geq_T x)\ \mathrm{br}(u) \notin \mathcal{F}_Q \to \Phi(x) \equiv \Phi(u). \tag{$\star$}$$

Because $\mathrm{br}(x) \notin \mathcal{F}_Q$, there is a node $y >_T x$ such that $\mathrm{br}(y) \notin \mathcal{F}_Q$ and $\mathrm{br}(y) < \mathrm{br}(x)$. (Take, e.g., y to be the second node in any sequence witnessing that $\mathrm{br}(x)$ is descensionally infinite.) Apply $(\star)$ to this y to see that $\Phi(x) \equiv \Phi(y)$.

Claim 2. There is an embedding $f\colon \mathrm{br}(x) \hookrightarrow \mathrm{br}(y)$; that is, $\mathrm{br}(x) \leq \mathrm{br}(y)$.

Proof of Claim 2. We will construct f iteratively. Since $\Phi(x) \leq_m \Phi(y)$ in $\mathscr{P}(\mathscr{P}\mathcal{F}_Q \times \text{CARD} \times Q)$, we can pick a $z \geq_T y$ such that $\Delta(x) \leq \Delta(z)$. Put $f(x) = z$. The fact that $\Delta(x) \leq \Delta(z)$ gives $l(x) \leq_Q l(z) = l(f(x))$. Use the fact that $\mathrm{dfba}(x) \leq_1 \mathrm{dfba}(z)$ in $\mathscr{P}\mathcal{F}_Q$ to extend f to embed the members of $\mathrm{dfba}(x)$ into distinct members of $\mathrm{dfba}(z)$. For $x' \in \mathrm{diso}(x)$ and $y' \in \mathrm{diso}(y)$ we have $\Phi(x') \equiv \Phi(y')$ by $(\star)$, so there is a $z' \geq_T y'$ such that $\Delta(x') \leq \Delta(z')$. Repeat the process for (x', z') as for (x, z). This process produces an embedding $f\colon \mathrm{br}(x) \hookrightarrow \mathrm{br}(y)$. $\blacksquare_{\text{Claim 2}}$

Claim 2 gives a contradiction, since y was chosen so that $\mathrm{br}(y) < \mathrm{br}(x)$. We are forced to conclude that our assumption at the beginning of the Claim 1 was false, so we obtain $x_1 >_T x$ such that $\mathrm{br}(x) \notin \mathcal{F}_Q$ and $\Phi(x) > \Phi(x_1)$. Continuing in this way gives a sequence as required in Claim 1. $\blacksquare_{\text{Claim 1}}$

This completes the proof of the proposition. $\blacksquare$

For completeness we record here that we have reduced the problem of showing $\mathcal{T}_Q$ is bqo to showing $\mathcal{F}_Q$ is bqo.

Corollary 3.2.2. If $\mathcal{F}_Q$ is bqo, then $\mathcal{F}_Q = \mathcal{T}_Q$, so $\mathcal{T}_Q$ is bqo.

Proof. Suppose $\mathcal{F}_Q$ is bqo and let $(T, l) \in \mathcal{T}_Q$. Certainly $\mathrm{F}(T, l) \subseteq \mathcal{F}_Q$, so $\mathrm{F}(T, l)$ is bqo. By Proposition 3.2.1 $(T, l) \in \mathcal{F}_Q$, so $\mathcal{T}_Q = \mathcal{F}_Q$. $\blacksquare$

To prove that $\mathcal{F}_Q$ is bqo, Laver ([6], §2) suggests that we follow Nash-Williams and reduce the problem to showing that every *closed*, *well-branched* subset of $\mathcal{F}_Q$ is bqo. We have managed to avoid this additional reduction because our effective version of 'Q bqo $\to$ $(\mathscr{P}Q, \leq_1)$ bqo' (2.4.8) is better suited to the problem than Nash-Williams' Lemma 28 of [14]. Nash-Williams suggests this alternative approach in a Note on page 710 of [14], and Kühn employs it in [5] to provide a less cluttered version of Nash-Williams' proof.

Our next task is to produce from a purportedly bad $\mathcal{F}_Q$-array f another bad $\mathcal{F}_Q$-array g such that $g <^* f$ (for Nash-Williams, f *strictly foreruns* g). Following Laver ([6], §2) and imitating our approach in the proof of Propisition 3.2.1, we first lift f to a bad $\Theta''\mathcal{F}_Q$-array for a suitable bqo-preserving operation Θ. Then we will use the effective results from §2.4 to reduce this lifted version of f to the required g.

The following result is Lemma $29'$ of [6], §2; it is the Q-labelled version of Nash-Williams' Lemma 29 ([14]) or Lemma 6 of [5].

As we did for Proposition 3.2.1, we will need several auxiliary functions. Fix $(T, l) \in \mathcal{T}_Q$. Recall that a branch $\mathrm{br}(x)$ of (T, l) is said to be *strict* iff $\mathrm{br}(x) < (T, l)$; that is, (T, l) does not embed into $\mathrm{br}(x)$. For a node $x \in T$, define:

$$\begin{aligned}\mathrm{sba}(x) &:= \{\mathrm{br}(y) : y \in \mathrm{S}(x) \text{ and } \mathrm{br}(y) < (T,l)\}\\ \mathrm{rs}(x) &:= \{y \in \mathrm{S}(x) : \mathrm{br}(y) \equiv (T,l)\}\\ \Gamma_{(T,l)}(x) &:= \langle \mathrm{sba}(x), |\mathrm{rs}(x)|, l(x)\rangle.^3\end{aligned}$$

As in Proposition 3.2.1, $\mathscr{P}\mathcal{T}_Q$ — of which each $\mathrm{sba}(x)$ is a member — is given the $\leq_1$ ordering. Finally, put

$$\Theta(T,l) := \left\{\Gamma_{(T,l)}(x) : x \in T\right\}.$$

Each $\Theta(T,l)$ is a member of the class $\mathscr{P}(\mathscr{P}\mathcal{T}_Q \times \mathrm{CARD} \times Q)$, which we give the $\leq_m$ ordering. (The use of $\leq_1$ for $\mathscr{P}\mathcal{T}_Q$ and $\leq_m$ for $\mathscr{P}(\mathscr{P}\mathcal{T}_Q \times \mathrm{CARD} \times Q)$ will be important in the proof of Proposition 3.2.4.)

Proposition 3.2.3. If $\Theta(T_1,l_1) \leq_m \Theta(T_2,l_2)$ in $\mathscr{P}(\mathscr{P}\mathcal{T}_Q \times \mathrm{CARD} \times Q)$, then $(T_1,l_1) \leq (T_2,l_2)$. That is, $\Theta\colon \mathcal{T}_Q \to \mathscr{P}(\mathscr{P}\mathcal{T}_Q \times \mathrm{CARD} \times Q)$ preserves the relation $\not\leq$.

Proof. It will be convenient for Γ_1 to abbreviate $\Gamma_{(T_1,l_1)}$ and Γ_2 to abbreviate $\Gamma_{(T_2,l_2)}$.

Suppose $\Theta(T_1,l_1) \leq_m \Theta(T_2,l_2)$. We inductively construct an embedding $(T_1,l_1) \hookrightarrow (T_2,l_2)$. Suppose that for $u \leq_{T_1} v$ in T_1 the value $f(u)$ has been defined and that a partition $A_1(u) \cup A_2(u)$ of $\mathrm{rs}(f(u))$ has been chosen such that

(i) $\mathrm{sba}(u) \leq_1 \mathrm{sba}(f(u)) \cup \{\mathrm{br}(x) : x \in A_1(u)\}$, and

(ii) $|\mathrm{rs}(u)| \leq |A_2(u)|$.

We will show how to define f on $\mathrm{S}(v)$. (It's essential here that T be a tree of height $< \omega+1$.) Use functions witnessing (i) to extend f so that f embeds distinct members of $\mathrm{sba}(v)$ into distinct members of $\mathrm{sba}(f(v)) \cup \{\mathrm{br}(x) : x \in A_1(v)\}$. Now we need to define f on $\mathrm{rs}(v)$. Of course, f must send each member of $\mathrm{rs}(v)$ to a node of some $\mathrm{br}(z)$, some $z \in A_2(v) \subseteq \mathrm{rs}(f(v))$. If y and y' are distinct members of $\mathrm{rs}(v)$, and $f(y) \in \mathrm{br}(z)$ and $f(y') \in \mathrm{br}(z')$ for $z, z' \in A_2(v)$, then z and z' must be distinct. Since by (ii) $|\mathrm{rs}(v)| \leq |A_2(v)|$, it will suffice to show that a suitable $f(y) \in \mathrm{br}(z)$ can be chosen for *any* given $z \in A_2(v)$. Then $f(y)$ can be defined by choosing $z = i(y)$ for any injection $i\colon \mathrm{rs}(v) \hookrightarrow A_2(v)$.

Claim. For any $y \in \mathrm{rs}(v)$ and $z \in A_2(v)$, we can choose $f(y) \in \mathrm{br}(z)$ that satisfies the inductive hypothesis and a partition $A_1(y) \cup A_2(y)$ of $\mathrm{rs}(f(y))$ satisfying (i) & (ii).

Proof of Claim. Because $z \in A_2(v) \subseteq \mathrm{rs}(f(v))$, there is an embedding $j\colon (T_2,l_2) \hookrightarrow \mathrm{br}(z)$. And $\Theta(T_1,l_1) \leq \Theta(T_2,l_2)$, so there is, in particular, a node $w \in T_2$ such that

$$\Gamma_1(y) \leq \Gamma_2(w). \tag{$*$}$$

[3] $\mathrm{sba}(x)$ — the set of *strict branches above* x — is Laver's $J(x)$; $\mathrm{rs}(x)$ — the set of *regular successors* of x — is Laver's $K(x)$.

Put $f(y) = j(w)$, an element of $\mathrm{br}(z)$. By $(*)$ and the fact that j is an embedding of Q-trees, we have

$$l_1(y) \leq_Q l_2(w) \leq_Q l_2(j(w)) = l_2(f(y)).$$

That is, f preserves the Q-labelling at y.

The difficulty now is that $A_1(y)$ must be sufficiently large so that (i) holds, and $A_2(y)$ must be large enough for (ii) to hold. To help balance $A_1(y)$ and $A_2(y)$, we define

$$V := \left\{ w' \in \mathrm{S}(w) : \mathrm{br}(w') \in \mathrm{sba}(w) \text{ and } j(w') \in \mathrm{br}(x) \text{ for some } x \in \mathrm{rs}(j(w)) \right\}.$$

(V is the set of 'strict' successors of w that j sends above 'regular' successors of $j(w)$.) Now set

$$A_1(y) := \{x \in \mathrm{S}(f(y)) : x \leq_{T_2} j(t) \text{ for some } t \in V\}; \quad A_2(y) := \mathrm{rs}(f(y)) \smallsetminus A_1(y).$$

First we show (i), that $\mathrm{sba}(y) \leq \mathrm{sba}(f(y)) \cup \{\mathrm{br}(x) : x \in A_1(y)\}$. Recall $(*)$, which implies that $\mathrm{sba}(y) \leq_1 \mathrm{sba}(w)$. Use an injection witnessing that $\mathrm{sba}(y) \leq_1 \mathrm{sba}(w)$ to map branches $\mathrm{br}(a) \in \mathrm{sba}(y)$ to branches in $\mathrm{sba}(w)$. Then apply j to get distinct (since j is an embedding) branches of the tree $\mathrm{br}(f(y))$. Because j is an embedding, distinct branches obtained in this way sit above distinct successors b of $f(y)$. This argument shows that the map sending a branch $\mathrm{br}(a) \in \mathrm{sba}(y)$ to $\mathrm{br}(b) \in \mathrm{S}(f(y))$ is injective. We defined V to arrange that the resulting $\mathrm{br}(b)$ belongs either to $\mathrm{sba}(f(y))$ or to $\{\mathrm{br}(x) : x \in A_1(y)\}$, so this establishes (i).

Now we want to prove (ii), that $|\mathrm{rs}(y)| \leq |A_2(y)|$. The embedding j sends distinct members of $\mathrm{rs}(w)$ above distinct members of $\mathrm{rs}(j(w)) = \mathrm{rs}(f(y))$. (Note that an embedding $(T, l) \hookrightarrow (T, l)$ must send 'regular' nodes — x such that $\mathrm{br}(x)$ is regular — to regular nodes, though a node x with $\mathrm{br}(x)$ strict may be sent to a regular node.) That is, there is an injection $\mathrm{rs}(w) \hookrightarrow \mathrm{rs}(f(y))$ whose image is contained in $A_2(y)$ by the definitions of A_1 and V. Therefore $|\mathrm{rs}(y)| \leq |\mathrm{rs}(w)| \leq |A_2(y)|$. $\blacksquare_{\text{Claim}}$

This completes the proof of the proposition, as we have constructed inductively an embedding $(T_1, l_1) \hookrightarrow (T_2, l_2)$. $\blacksquare$

The main complication in proving Proposition 3.2.3 is the asymmetry between 'regular' nodes and 'strict' nodes. As noted in the proof's penultimate paragraph, an embedding must send regular nodes to regular nodes, though it needn't send strict nodes to strict nodes. Thus it's important to keep track of the partition $A_1 \cup A_2$ during the construction, because this ensures that we have above $f(v)$ enough regular nodes to map the regular successors of v into.

Proposition 3.2.3 shows that we can transform a bad $\mathcal{F}_Q$-array f into a bad $\mathscr{P}(\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q)$-array; now our effective results from §2.4 will allow us to whittle this new array down to a bad $\mathcal{F}_Q$-array that is $<^* f$. Our partial ranking $<'$ on $\mathcal{F}_Q$ will be the relation 'is a strict branch of', which is well-founded by the definition of *descensionally finite.*

The following result is essentially a Q-labelled version of Lemma 7 in [5]. Our approach is similar in flavour to Laver's discussion on page 91 of [6], though we provide a slightly cleaner argument by using Simpson's topological approach rather than Nash-Williams' original combinatorial definition of bqo. (Laver's paper [6] predates Simpson's paper [18].)

Proposition 3.2.4. If $f\colon [A]^\omega \to Q$ is a bad $\mathcal{F}_Q$-array, then there is a bad $\mathcal{F}_Q$-array g such that $g <^* f$ (wrt $<'$, the relation 'is a strict branch of').

Proof. By Proposition 3.2.3 the composite $\Theta \circ f$ is a bad $\mathscr{P}(\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q)$-array, also with domain $[A]^\omega$. By Corollary 2.4.9 (our effective version of 'Q bqo $\to (\mathscr{P}Q, \leq_m)$ bqo') there is a bad $(\mathscr{P}\mathcal{F}_Q \times \textsc{card} \times Q)$-array f_1 with domain $[A_1]^\omega \subseteq [A]^\omega$ such that

$$f_1(X) \in \Theta(f(X)) \text{ for every } X \in [A_1]^\omega.$$

Since a product of bqos is bqo (Proposition 2.4.5), $\textsc{card} \times Q$ is bqo. Apply Proposition 2.4.5 (our effective version of 'binary product of bqos is bqo') to f_1 to obtain a bad $(\mathscr{P}\mathcal{F}_Q, \leq_1)$-array f_2 such that $\operatorname{dom} f_2 = [A_2]^\omega \subseteq [A_1]^\omega$ and

$$f_2(X) \text{ is the first coordinate of } f_1(X) \text{ for every } X \in [A_2]^\omega.$$

Now apply Corollary 2.4.8 (our effective version of 'Q bqo $\to (\mathscr{P}Q, \leq_1)$ bqo') to f_2 to obtain a bad $\mathcal{F}_Q$-array g such that $\operatorname{dom} g = [B]^\omega \subseteq [A_2]^\omega$ and

$$g(X) \in f_2(X) \text{ for every } X \in [B]^\omega.$$

For every $X \in [B]^\omega$ we have $g(X) \in f_2(X)$, and $f_2(X)$ is the first coordinate of a member of $\Theta(f(X))$. Unraveling this and the definition of Θ, we see that $g(X)$ is a strict branch of $f(X)$ for every $X \in [B]^\omega$. So $g <^* f$. ∎

That was the final ingredient in our proof that $\mathcal{T}_Q$ is bqo.

Theorem 3.2.5. $\mathcal{T}_Q$ is bqo.

Proof. Suppose f is a bad $\mathcal{F}_Q$-array. By the Minimal Bad Array Lemma 2.3.3, we may assume f is minimal bad. By Proposition 3.2.4, this is impossible. Therefore $\mathcal{F}_Q$ is bqo. By Corollary 3.2.2, $\mathcal{F}_Q = \mathcal{T}_Q$, so $\mathcal{T}_Q$ is bqo. ∎

Conclusion

By offering an ample supply of bqo-preserving operations, this essay has hopefully provided its reader with an appreciation for the efficacy both of the Minimal Bad Array Lemma and of Simpson's approach to bqo theory. The interested reader will find several other examples of theorems — some very hard — like those we have presented here. One recent example introduces an exciting connection between bqo theory and forcing axioms: in [10] Carlos Martinez-Ranero proves that, under the Proper Forcing Axiom (PFA) the class of Aronszajn lines is bqo.

To date the most imposing result in wqo theory is the Robertson–Seymour Theorem, that the class of graphs under the graph minor relation is wqo. The frightening length of

Robertson & Seymour's proof makes one wonder whether there is a more digestible proof of a result like Q bqo $\to \mathscr{G}_Q$ bqo, where $\mathscr{G}_Q$ is the class of 'Q-labelled' graphs under a suitable Q-minor relation.

One avenue of research that bqo theorists have not yet pursued (as far as I know) is a pleasing characterisation of the operations that preserve bqoness. Such a characterisation would add another layer of richness to an already-beautiful theory.

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