2015 Paper 6 Question 4
Scott numerals in λ-calculus

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Part (c)

Never had to know about Scott numerals until one of my 1B pairs had a look at this old tripos question. I had heard about it before ... i think the idea is that you get a nice implementation of `pred`....

Scott numeral $\mathtt{Sc}_0$ is $\mathtt{K}$; thereafter Scott numeral $\mathtt{Sc}_{n+1}$ is $\mathtt{K}$ of `enumerate-at-Sc`$_n$: in symbols $\lambda x.\lambda f.f\mathtt{Sc}_n$.

They ask you to show that $\mathtt{Sc}_0 MN \twoheadrightarrow_\beta M$ and $\mathtt{Sc}_{n+1}$ MN$\twoheadrightarrow_\beta M\ \mathtt{Sc}_n$.

Seems to me that this is from the Department Of The Bleedin' Obvious [and they are offering only two marks] but let's Do As We Are Asked.

$$\mathtt{Sc}_0\ M\ N \text{ is } \mathtt{K}\ M\ N \twoheadrightarrow_\beta M$$

so *that's* OK.

and

$$\underline{\lambda x.\lambda f.f\mathtt{Sc}_n\ M}\ N$$

and β-reduce the underlined bit first:

$$\lambda f.f\mathtt{Sc}_n\ N$$

β-reduce again to get

$$N\ \mathtt{Sc}_n$$

as desired.

Now they want `pred`. What do we have to do to $\mathtt{Sc}_{n+1}$ to get $\mathtt{Sc}_n$?

Starting with $\lambda x.\lambda f.f\mathtt{Sc}_n$ we apply it to $\mathtt{I}$ to get $\lambda f.f\mathtt{Sc}_n$ and then apply that to $\mathtt{I}$ to get $\mathtt{Sc}_n$. So `pred` must be $\lambda n.n\ \mathtt{I}\ \mathtt{I}$.

Part (i)

(*We seek a Scott-definable term for the successor function*)

One lambda-term that does the trick is

$$\lambda y.\lambda x.\lambda f.fy$$

Part (ii)

(*We seek a Scott-definable term for the predecessor function*)

These β-reductions are fairly straightforward if you don't get flustered.

$$\mathbb{N}_0\, M\, N = (\lambda x.\lambda f.x)\, N \to (\lambda f.M)\, N \to M$$

and

$$\mathbb{N}_{n+1}\, M\, N = (\lambda x.\lambda f.f\,\mathbb{N}_n)\, N \to (\lambda f.f\,\mathbb{N}_n)\, N \to N\,\mathbb{N}_n \qquad (*)$$

Part (iii)

(*We seek a Scott-definable term for the* `plus` *function*)

The obvious S to try is the S we obtained in Part (c)(i). We are obviously going to have to do an induction. The thing to try to prove is ... fix a natural number m and prove

$$(\forall n \in \mathbb{N})(P_m \mathbb{N}_n \twoheadrightarrow \mathbb{N}_{m+n}) \tag{1}$$

To prove 1 we use the following fact from part (b) (not proved here)

$$P_m \twoheadrightarrow (\lambda f.\lambda y.y\,\mathbb{N}_m(\lambda z, S\,(f z)))P_m \to \lambda y.y\,\mathbb{N}_m(\lambda z.S\,(P_m z))$$

which gives

$$P_m \mathbb{N}_n \mapsto \mathbb{N}_n \mathbb{N}_m(\lambda z.S\,(P_m z)) \tag{2}$$

Now we can prove the induction.

Base case, $n = 0$

$$\begin{aligned} P_m\,\mathbb{N}_0 &\twoheadrightarrow \mathbb{N}_0\mathbb{N}_m(\lambda z.S\,(P_m z)) && \text{by 2} \\ &\twoheadrightarrow \mathbb{N}_m \end{aligned}$$

Induction Step:

$$\begin{aligned} P_m\,\mathbb{N}_{n+1} &\twoheadrightarrow \mathbb{N}_{n+1}\mathbb{N}_m(\lambda z.S\,(P_m z)) && \text{by 2} \\ &\twoheadrightarrow \lambda z.S\,(P_m z))\mathbb{N}_n && \text{by (*) from part (c)(i)} \\ &\to S\,(P_m \mathbb{N}_n) \\ &\twoheadrightarrow S\,(\mathbb{N}_{m+n}) && \text{by induction hypothesis} \\ &\twoheadrightarrow \mathbb{N}_{m+n+1} && \text{by (c)(i)} \end{aligned}$$