On Laver's proof of Fraïssé's order type conjecture

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Contents

1 Introduction

This essay will be concerned with order types (isomorphism classes of linearly ordered sets). A scattered order type is one that doesn't contain an isomorphic copy of the rationals. A quasi-order (or QO) is a transitive and reflexive binary relation and a well quasi-order (or WQO) is a quasi-order in which there are no infinite antichains and no infinite descending sequences. A better quasi-order (or BQO) is a particularly nice kind of well quasi-order. We quasi-order the order types by embeddability.[1]

The aim of this essay is to prove a conjecture of Fraïssé [1] which states that the class of countable order types is well quasi-ordered. This conjecture was later extended to include scattered order types. These assertions follow from the main theorem of this essay: the class $\mathcal{M}$ of countable unions of scattered order types is better quasi-ordered.

My approach will be to follow sections 3 and 4 of Richard Laver's paper [6] (stating without proof Theorem 3.4). I have elaborated on Laver's proofs and substantially expanded some arguments to make them more accessible to the reader.

[1] Full definitions will be given in section 2.

I would like to thank the BQO reading group consisting of Thomas Forster, Oren Kolman, Lovkush Agarwal, Zach Norwood, Zachiri McKenzie, and David Matthai for general understanding of the subject. In particular, I would like to thank Zach Norwood for the summary of Laver's paper that he presented to the group, and Lovkush Agarwal for some helpful discussions.

Simpson [12] gives an alternative proof of Fraïssé's conjecture, by reformulating the theory in terms of the usual (product) topology on $[\omega]^\omega$ and using a theorem by Galvin and Prikry [2] on Borel partitions. He proves a slightly weaker version of the theorem than the one given here: the class of scattered order types and the class of countable order types are better quasi-ordered.

This result contributes to the aim of showing that many large classes of mathematical objects are well quasi-ordered under natural orderings. Other classes that have been considered are trees under tree embeddability (see Kruskal [5] for finite trees and Nash-Williams [8] for infinite trees) and transfinite Q-sequences where Q is a BQO (see Nash-Williams [9]). More recently, Robertson and Seymour [11] proved that finite graphs under the graph immersion relation are well quasi-ordered.

Overview

Section 2 contains definitions and some fundamental general properties of WQOs and BQOs are proved.

In section 3 I will define a class of trees $\mathcal{T}_Q$ whose vertices are labelled by elements of a QO Q. $\mathcal{T}_Q$ is quasi-ordered by embeddability. Theorem 3.4 states that Q BQO $\Rightarrow \mathcal{T}_Q$ BQO.

The next step is to define and characterise a useful class of order types $\eta_{\alpha\beta}$ which generalise the order type of the rationals (in fact, the rationals have order type $\eta_{\omega_1\omega_1}$). Theorem 4.4 gives some properties of the types $\eta_{\alpha\beta}$ and Theorem 4.5 shows that these properties suffice to describe these types up to equivalence.

We let $\mathcal{D}_{\alpha\beta}$ be the class of types that strictly embed into $\eta_{\alpha\beta}$. In Theorem

5.14 we will want to use this class for a proof by induction, so an inductive characterisation of $\mathcal{D}_{\alpha\beta}$ is given in Theorem 4.9.

Next, we define another class of types $\mathcal{H}(Q)$ and a quasi-ordered class Q^+ containing Q. We prove that Q BQO $\Rightarrow Q^+$ BQO. Theorem 5.10 shows that $\mathcal{T}_{Q^+}$ BQO $\Rightarrow \mathcal{H}(Q)$ BQO by assigning trees in $\mathcal{T}_{Q^+}$ to types in $\mathcal{H}(Q)$.

$Q^{\mathcal{M}}$ is the class of countable unions of scattered order types with points labelled by elements of Q. Theorem 5.14 essentially shows that $\mathcal{H}(Q)$ BQO $\Rightarrow Q^{\mathcal{M}}$ BQO.

The strongest result in the paper is Theorem 5.15 which states that Q BQO $\Rightarrow Q^{\mathcal{M}}$ BQO and is proved by combining all previously established implications. Finally, the fact that $\mathcal{M}$ is a BQO drops out as an easy corollary. We conclude the essay with an application of this result.

Some preliminary definitions

An *order type* (or just *type*) is an isomorphism class of linearly ordered sets. Letters ϕ, ψ will usually range over order types. Write $\mathrm{tp}(L)$ for the order type of the linearly ordered set L. Letters L, M, N will usually range over linearly ordered sets.

Ordinals are the order types of well-ordered sets (they will also be taken to be well-ordered sets). The class of ordinals is denoted by On. Ordinals are usually denoted by α, β, γ, δ, κ, λ, ξ. For limit ordinals λ, the *cofinality* of λ is the least ordinal α such that λ is the limit of an α-sequence of ordinals $< \lambda$ and is denoted by $\mathrm{cf}(\lambda)$. If $\mathrm{cf}(\lambda) = \lambda$, λ is said to be *regular*.

I will use the Axiom of Choice without further comment throughout the essay. In particular, cardinals are initial ordinals. The class of cardinals is denoted by $Card$. If κ is a cardinal, then κ^+ is the least cardinal greater than κ. Cardinals of the form κ^+ are *successor cardinals*. Cardinals that are not of this form are called *limit cardinals*. RC is the class of infinite regular cardinals.

If X is a set, then X^α is the set of sequences $\langle x_\beta \rangle_{\beta<\alpha}$ in X and $X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$. $\mathbb{N} = \{0, 1, ...\}$.

Definition 1.1. $\phi \leqslant \psi$ means that ϕ is *embeddable* in ψ, i.e. if $\mathrm{tp}(L) = \phi$

and $\mathrm{tp}(M) = \psi$ then there is a strictly increasing function from L into M.

Definition 1.2. η is the order type of $\mathbb{Q}$. An order type ϕ is *scattered* if $\eta \nleqslant \phi$. $\mathcal{S}$ is the class of all scattered order types.

Definition 1.3. $\mathcal{M}$ is the class of countable unions of scattered types, i.e. $\phi \in \mathcal{M} \Leftrightarrow \phi = \mathrm{tp}(L)$ where L can be written as $\bigcup_{i<\omega} L_i$ with each $\mathrm{tp}(L_i) \in \mathcal{S}$.

For example, $\eta = \mathrm{tp}(\mathbb{Q}) \in \mathcal{M}$ ($\mathbb{Q}$ is a countable union of singletons), and $On \subseteq \mathcal{M}$.

2 Basic WQO and BQO Theory

In this section I will introduce the notions of well quasi-orders and better quasi-orders, and I will establish some elementary properties that will turn out to be useful in the course of the essay.

The concept of WQO is not strong enough to prove our main theorem, so we will work with BQOs. This strengthens the hypothesis and makes the proof work. See [6] for further motivation of the definitions.

While it is often easier to prove theorems about WQOs using BQOs, the notion of WQO is more natural and easier to apply in concrete situations. A particularly striking example is the application of Kruskal's theorem on finite trees [5] to logic: H. Friedman derived special cases of the theorem that can be expressed but not proved in first-order arithmetic (see for example Smorynski [13]).

Definition 2.1. A *quasi-order* or *QO* is a transitive and reflexive binary relation on a set or class Q. This relation will be denoted by $\leqslant$ (possibly with subscripts). Quasi-orders are usually denoted by the letters Q and R.

If $q_1, q_2 \in Q$, write $q_1 \equiv q_2$ if $q_1 \leqslant q_2 \wedge q_2 \leqslant q_1$ (q_1 and q_2 are said to be *equivalent*) and write $q_1 < q_2$ if $q_1 \leqslant q_2 \wedge q_2 \nleqslant q_1$. Whenever a subset $Q' \subseteq Q$ is considered, Q' is assumed to be quasi-ordered by the restricted QO of Q.

Definition 2.2. If Q is a QO, then a sequence $\langle q_i \rangle_{i<\omega}$ in Q is said to be *bad* if $(\forall i < j) q_i \nleqslant q_j$. A QO Q is a *well quasi-order* or *WQO* if there are no bad sequences in Q.

Remark. This definition is equivalent to the definition given in the introduction (via Ramsey's Theorem).

Given QOs Q_1, Q_2, define QOs on spaces obtained from them:

1. $Q_1 \times Q_2$ is quasi-ordered by

$$\langle \alpha_1, \alpha_2 \rangle \leqslant \langle \beta_1, \beta_2 \rangle \Leftrightarrow (\alpha_1 \leqslant_{Q_1} \beta_1 \wedge \alpha_2 \leqslant_{Q_2} \beta_2).$$

2. If $Q_1 \cap Q_2 = \varnothing$ then $Q_1 \cup Q_2$ is quasi-ordered by

$$\alpha \leqslant \beta \Leftrightarrow ((\alpha, \beta \in Q_1 \wedge \alpha \leqslant_{Q_1} \beta) \vee (\alpha, \beta \in Q_2 \wedge \alpha \leqslant_{Q_2} \beta)).$$

If Q is a QO, order $\bigcup_{\alpha \in On} Q^\alpha$ by embeddability, i.e. $\langle q_\alpha \rangle_{\alpha<\gamma} \leqslant \langle r_\beta \rangle_{\beta<\delta} \Leftrightarrow$ there is a strictly increasing $f : \gamma \to \delta$ such that $q_\alpha \leqslant r_{f(\alpha)}$ for all $\alpha < \gamma$. In particular, this induces a quasi-order on $Q^{<\omega}$ (as a subset of $\bigcup_{\alpha \in On} Q^\alpha$).

A set $B \subseteq \mathbb{N}^{<\omega}$ of strictly increasing finite sequences is called a *block* if for any strictly increasing $\langle x_i \rangle_{i<\omega} \in \mathbb{N}^\omega$ there is some $n \in \mathbb{N}$ with $\langle x_i \rangle_{i<n} \in B$. If $t, u \in B$ we write $t \lhd u$ if there exist $m < n, x_1 < \ldots < x_n \in \mathbb{N}$ such that $t = \langle x_i \rangle_{i=1,\ldots,m}$ and $u = \langle x_i \rangle_{i=2,\ldots,n}$ (for example $(1,2,3) \lhd (2,3,4,5,6)$). Given a QO Q, a function from a block to Q is called an *array*. An array $a : B \to Q$ is *bad* if $t, u \in B \wedge t \lhd u \Rightarrow a(t) \nleqslant a(u)$.

Definition 2.3. A QO Q is a *better quasi-order* or *BQO* if there are no bad arrays in Q. In other words, there is no block B that admits a bad array $a : B \to Q$.

Remark. By taking $B = \mathbb{N} \subseteq \mathbb{N}^{<\omega}$ in the definition of BQOs, it follows that every BQO is a WQO. There are, however, WQOs that are not BQOs: see for example Rado's counterexample given in [10]. Hence being BQO is a strictly stronger property than being WQO.

This combinatorial definition of BQOs only plays a small role in this essay. Instead we use some nice basic properties of BQOs:

Theorem 2.4 (Nash-Williams [8])**.** *(i) Q BQO $\Rightarrow Q$ WQO.*

(ii) Q well-ordered $\Rightarrow Q$ BQO.

(iii) $Q = Q_1 \cup Q_2$ and Q_1, Q_2 BQO $\Rightarrow Q$ BQO.

(iv) Q_1, Q_2 BQO $\Rightarrow Q_1 \times Q_2$ BQO.

(v) Q BQO $\Rightarrow Q^{<\omega}$ BQO.

Definition 2.5. For Q a QO and $q \in Q$, define $Q_q = \{r \in Q : q \nleqslant r\}$.

Lemma 2.6 (Induction Principle for WQOs)**.** *If $\Phi(Q)$ is an assertion about a WQO Q such that $((\forall q \in Q)\Phi(Q_q)) \Rightarrow \Phi(Q)$ and $\Phi(\boldsymbol{0})$ holds (where $\boldsymbol{0}$ is the empty quasi-order), then $\Phi(Q)$ holds for all WQOs Q.*

Proof. Suppose the conditions above hold and there is a WQO Q such that $\Phi(Q)$ is false. Then $\Phi(Q_{q_1})$ is false for some $q_1 \in Q$. Write $Q^{(1)} = Q_{q_1}$. Continue inductively: If $\Phi(Q^{(n-1)})$ is false, then there is $q_n \in Q^{(n-1)}$ such that $\Phi(Q^{(n-1)}_{q_n})$ is false. Write $Q^{(n)} = Q^{(n-1)}_{q_n}$.

So we get a nested sequence of WQOs $Q^{(1)} \supseteq Q^{(2)} \supseteq \ldots$ and a sequence $\langle q_n \rangle_{n<\omega}$ in Q such that for $i < j$ we have $q_j \in Q^{(j-1)} \subseteq Q^{(i)} = Q^{(i-1)}_{q_i} = \{r \in Q^{(i-1)} : q_i \nleqslant r\}$, i.e. $q_i \nleqslant q_j$. So $\langle q_n \rangle_{n<\omega}$ is a bad sequence. Contradiction.

□

Lemma 2.7 (Homomorphism Property for BQOs)**.** *If Q is a BQO, $Q' \subseteq Q$, and there is an oder-preserving surjection $Q' \to R$ (any quasi-order R), then R is a BQO.*

Proof. Suppose there is a bad array $a : B \to R$. Define an array $a' : B \to Q'$ with $a'(x) =$ any element of $h^{-1}(\{a(x)\})$ (this is $\neq \varnothing$ by surjectivity). If a' is good, then $(\exists r, s \in B) r \lhd s \wedge a'(r) \leqslant a'(s)$. Then $a(r) = h(a'(r)) \leqslant h(a'(s)) = a(s)$, contradicting the badness of a. Hence a' is bad, contradicting Q BQO.

□

3 Trees

This section states a theorem concerning the better quasi-ordering of a class of trees with vertices labelled by elements of a BQO and defines all concepts required to understand the result. Some of the definitions will be used later in the essay.

Here a *tree* will be a set T, partially ordered by the relation $\leqslant_T$, such that $\{y \in T : y \leqslant_T x\}$ is well-ordered by $\leqslant_T$ for any $x \in T$. If there is a point $r \in T$ with $(\forall x \in T) r \leqslant_T x$ then T is said to be *rooted* and the *root* of T is $\rho(T) = r$. For $x \in T$, $S(x)$ is the set of immediate successors of x in T. If Q is any set or class, a Q-*tree* is a pair (T, l) where T is a tree and $l : T \to Q$ is a function (l labels the vertices of T). The *branch* of (T, l) with root node x, written $\mathrm{br}_{(T,l)}(x)$ (or just $\mathrm{br}(x)$), is the Q-tree obtained from T by restricting T to the vertices that are $\geqslant_T x$. $\mathbf{0}$ is the empty tree and, for $q \in Q$, $\mathbf{1}^q$ is the singleton tree labelled by q.

Definition 3.1. $\mathcal{T}$ is the class of all rooted trees with no path of length $> \omega$. $\mathcal{T}_Q$ is the class of Q-trees (T, l) with $T \in \mathcal{T}$.

Definition 3.2. If $q \in Q$ and $\mathcal{B} \subseteq \mathcal{T}_Q$, then $[q; \mathcal{B}]$ denotes the Q-tree $(T, l) \in \mathcal{T}_Q$ with $l(\rho(T)) = q$ and $\{\mathrm{br}(x) : x \in S(\rho(T))\} = \mathcal{B}$ (assume a convention where the trees in $\mathcal{B}$ are disjoint).

Definition 3.3. Define the following natural quasi-order on $\mathcal{T}_Q$: $(T_1, l_1) \leqslant_m (T_2, l_2)$ if there is a strictly increasing function $f : T_1 \to T_2$ such that $(\forall x \in T_1) l_1(x) \leqslant l_2(f(x))$.

Theorem 3.4 (Laver [6])**.** *Q BQO $\Rightarrow \mathcal{T}_Q$ BQO under $\leqslant_m$.*

Remark. The proof of this theorem given in [6] assumes familiarity with [8].

4 Characterisation of $\eta_{\alpha\beta}$

I will prove some preliminary lemmas before defining the types $\eta_{\alpha\beta}$ mentioned in the introduction. The main aim of this section is to prove some properties of these types and show that these properties define the $\eta_{\alpha\beta}$ up to equivalence.

The *sum* $\phi + \psi$ of two order types is $\mathrm{tp}(L)$ where L is the disjoint union of linear orders P_1, P_2 ($\mathrm{tp}(P_1) = \phi$, $\mathrm{tp}(P_2) = \psi$), linearly ordered by $x \leqslant_L y \Leftrightarrow (x, y \in P_1 \wedge x \leqslant_{P_1} y) \vee (x, y \in P_2 \wedge x \leqslant_{P_2} y) \vee (x \in P_1 \wedge y \in P_2)$.

If M is a linearly ordered set and, for each $x \in M$, ϕ_x is an order type, define the *ordered sum* $\sum_{x \in M} \phi_x$ to be $\mathrm{tp}(N)$, where N is obtained from M by replacing each point $x \in M$ by a linearly ordered set of order type ϕ_x and defining the new ordering on N appropriately.

The *product* $\phi \cdot \psi$ of two order types is $\sum_{x \in L} \phi$ where $\mathrm{tp}(L) = \psi$.

If ψ is an order type (or $\mathcal{R}$ is a collection of order types), then we will often express the fact that $\phi = \sum_{x \in L} \phi_x$, where $\mathrm{tp}(L) = \psi$ (or $\mathrm{tp}(L) \in \mathcal{R}$), by saying that ϕ is a ψ*-sum* (or an $\mathcal{R}$*-sum*) of the ϕ_x.

If $\phi = \mathrm{tp}(L)$ then the *converse* of ϕ is $\phi^* = \mathrm{tp}(L')$ where L' is a linear order that has the same underlying set as L and $x \leqslant_{L'} y \Leftrightarrow x \geqslant_L y$. If α is well-ordered, then α^* is said to be *conversely well-ordered*.

0 denotes the linear order with empty underlying set, **1** denotes the linear order with a one-element underlying set. **0** and **1** will also be used for the corresponding order types.

The following characterisation of $\mathcal{S}$ will allow us to perform induction over scattered types.

Theorem 4.1 (Hausdorff [3]). *Let $\mathcal{S}_0 = \{\mathbf{0}, \mathbf{1}\}$ and, for $\beta > 0$, let $\mathcal{S}_\beta = \{\phi : \phi$ is a well-ordered or conversely well-ordered sum of membersof $\bigcup_{\gamma<\beta} \mathcal{S}_\gamma\}$. Then $\mathcal{S} = \bigcup_{\alpha \in On} \mathcal{S}_\alpha$.*

Lemma 4.2. *(i) A scattered sum of scattered types is scattered.*

(ii) If $\kappa \in RC$ and $\kappa \leqslant \sum_{y \in M} \phi_y$ then either $\kappa \leqslant tp(M)$ or $\kappa \leqslant \phi_y$ for some $y \in M$.

(iii) If $\kappa \in RC$, $\lambda < \kappa$, $L = \bigcup_{\gamma<\lambda} L_\gamma$, and $\kappa \leqslant tp(L)$, then $\kappa \leqslant tp(L_\gamma)$ for some $\gamma < \lambda$.

(iv) If $\kappa \in RC$, $\phi \in \mathcal{S}$, and $\kappa \leqslant Card(\phi)$, then $\kappa \leqslant \phi$ or $\kappa^* \leqslant \phi$.

Proof. (i) Let $\mathrm{tp}(L)$ be scattered, and for each $x \in L$ let $\phi_x = \mathrm{tp}(L_x)$ be scattered. Suppose $f : \mathbb{Q} \to \bigcup_{x\in L} L_x$ gives an embedding $\eta \leqslant \sum_{x\in L} \phi_x$. For each $x \in L$ let $\mathbb{Q}_x = f^{-1}(L_x)$.

Then $\mathrm{tp}(\mathbb{Q}_x) \leqslant \phi_x \Rightarrow$ each $\mathbb{Q}_x$ is scattered and $\eta = \sum_{x\in L} \mathrm{tp}(\mathbb{Q}_x)$ is a scattered sum of scattered types. Now f order-preserving $\Rightarrow$ each $\mathbb{Q}_x$ is an interval in $\mathbb{Q}$, so $\mathrm{tp}(\mathbb{Q}_x)$ scattered $\Rightarrow \mathbb{Q}_x$ is a singleton.

Hence $\eta = \sum_{x\in L} \mathrm{tp}(\mathbb{Q}_x) \equiv \mathrm{tp}(L)$, contradicting $\mathrm{tp}(L)$ scattered.

(ii) Let $\mathrm{tp}(L_y) = \phi_y$ and let $f : \kappa \to \bigcup_{y\in M} L_y$ be an embedding of κ into $\sum_{y\in M} \phi_y$.

Note first that

$$\text{if } S \subseteq \kappa \text{ is any subset of cardinality } \kappa \text{ then } \kappa \leqslant S \tag{1}$$

by defining an embedding $g : \kappa \to S : \alpha \mapsto$ the αth element of S in the well-ordering on S induced by κ.

If $\kappa \leqslant M$ we're done.

Assume $\kappa \not\leqslant M$. Then $\mathrm{Card}(\{y \in M : \mathrm{im}(f) \cap L_y \neq \varnothing\}) < \kappa$, since if this cardinality was equal to κ then $\kappa \leqslant M$ by equation (1). Hence $\kappa = \mathrm{Card}(\bigcup_{y\in M}(\mathrm{im}(f) \cap L_y)) = \sum_{y\in M} \mathrm{Card}(\mathrm{im}(f) \cap L_y)$ is a sum of $< \kappa$ non-zero cardinals, so by regularity of κ, $(\exists y \in M)\mathrm{Card}(\mathrm{im}(f) \cap L_y) = \kappa$. So again by equation (1), $\kappa \leqslant L_y$.

(iii) Let $f : \kappa \to L$ be an embedding. Then by regularity of κ, $(\exists\gamma < \kappa)\mathrm{Card}(\mathrm{im}(f) \cap L_\gamma) = \kappa$. So $\kappa \leqslant L_\gamma$ as before.

(iv) The proof will be by induction on the hierarchy given in Theorem 4.1.

The result is trivial for $\mathcal{S}_0$.

Assume it holds for $\bigcup_{\gamma<\beta} \mathcal{S}_\gamma$ and suppose $\phi \in \mathcal{S}_\beta$. Then ϕ is an α-sum or an

α^*-sum of elements of $\bigcup_{\gamma<\beta} \mathcal{S}_\gamma$ for some $\alpha \in On$. Write $\phi = \sum_{x\in A} \phi_x$ where $\text{tp}(A) = \alpha$ and $(\forall x \in A)\phi_x \in \bigcup_{\gamma<\beta} \mathcal{S}_\gamma$ (the case $\text{tp}(A) = \alpha^*$ is similar). If $\kappa \leqslant \text{Card}(\phi_x)$ for some x the we're done by the induction hypothesis. So assume $\text{Card}(\phi_x) < \kappa$ for all x. Since $\kappa \leqslant \text{Card}(\phi)$ we get $\kappa \leqslant \text{Card}(A)$ by regularity of κ. $\text{Card}(A) = \alpha$, so $\kappa \leqslant \alpha \leqslant \phi$. □

Call a pair $\langle \alpha, \beta \rangle \in On \times On$ *admissible* if α and β are uncountable regular cardinals and $\max\{\alpha, \beta\}$ is a successor cardinal.

Given an admissible pair $\langle \alpha, \beta \rangle$ we define the order type $\eta_{\alpha\beta}$. To do this we first define an auxiliary type $\sigma_{\alpha\beta}$ as follows:

> If $\alpha = \gamma^+$ and $\beta = \delta^+$ are both successor cardinals, $\sigma_{\alpha\beta} = \gamma^* \cdot \delta$.
>
> If α is a limit cardinal, then $\alpha < \beta = \delta^+$ for some cardinal δ and we define $\sigma_{\alpha\beta} = \sum_{x\in M} \phi_x$ where $\text{tp}(M) = \delta$, $(\forall x \in M)\phi_x < \alpha^*$, and for each cardinal $\lambda < \alpha$ there is some $x \in M$ with $\phi_x \geqslant \lambda^*$ (this last condition is satisfiable because $\alpha \leqslant \delta$).
>
> If β is a limit cardinal, then $\sigma_{\alpha\beta} = (\sigma_{\beta\alpha})^*$.

Definition 4.3. $\eta_{\alpha\beta} = \text{tp}(L)$ where $L = \bigcup_{n<\omega} L_n$ and the sets $L_0 \subseteq L_1 \subseteq \ldots$ are chosen as follows:

(i) $\text{tp}(L_0) = \sigma_{\alpha\beta}$

(ii) L_{n+1} is obtained from L_n by inserting a set of type $\sigma_{\alpha\beta}$ into each empty interval of L_n.

The next theorem identifies the essential properties of the types $\eta_{\alpha\beta}$.

Theorem 4.4. *(i) $\eta_{\alpha\beta} \in \mathcal{M}$. (Recall definition 1.3)*

(ii) $\alpha^* \nleqslant \eta_{\alpha\beta}, \beta \nleqslant \eta_{\alpha\beta}$.

(iii) If $\alpha_0 < \alpha$ and $\beta_0 < \beta$ are ordinals then $\alpha_0^* \leqslant tp((x,y))$ and $\beta_0 \leqslant tp((x,y))$ for any interval $(x,y) \subseteq L$.

Proof. (i) Cardinals are scattered, so $\sigma_{\alpha\beta}$ is a scattered sum of scattered types. By Lemma 4.2(i), $\text{tp}(L_0) = \sigma_{\alpha\beta} \in \mathcal{S}$.

Given $\text{tp}(L_n) \in \mathcal{S}$ we get $\text{tp}(L_{n+1}) \in \mathcal{S}$ by writing $\text{tp}(L_{n+1}) = \sum_{x \in L_n} \phi_x$ where each ϕ_x is $\mathbf{1}$ or $\sigma_{\alpha\beta} + \mathbf{1}$ and applying Lemma 4.2(i). So $L = \bigcup_{n<\omega} L_n$ is a countable union of scattered types.

(ii) I will prove the result for β. The case α^* is similar.

First, note that $\beta \nleqslant \sigma_{\alpha\beta} = \text{tp}(L_0)$ by the definition of L_0. Suppose that, for some n, $\beta \nleqslant \text{tp}(L_n)$ but $\beta \leqslant \text{tp}(L_{n+1})$. Writing $\text{tp}(L_{n+1}) = \sum_{x \in L_n} \phi_x$ as in (i) we get $\beta \leqslant \text{tp}(L_n)$ or $\beta \leqslant \phi_x$ for some x by Lemma 4.2(ii) ($\omega < \beta \in RC$ because $\langle \alpha, \beta \rangle$ is admissible). Both of these cases are impossible. Hence, by induction, $\beta \nleqslant \text{tp}(L_n)$ for all n.

Hence $\omega < \beta \in RC$ and $L = \bigcup_{n<\omega} L_n$. If $\beta \leqslant \eta_{\alpha\beta} = \text{tp}(L)$ then Lemma 4.2(iii) $\Rightarrow \beta \leqslant \text{tp}(L_n)$ for some n, giving a contradiction.

(iii) I will prove the result for β_0. The proof for α_0^* is similar.

The proof uses induction on ordinals $\gamma < \beta$. Assume that the result holds for all ordinals $\delta < \gamma$. I will find an embedding $\gamma \leqslant (x, y)$. The L_n are nested, so $x, y \in L_m$ for some m.

$[x, y] \subseteq L_m$ *has an empty subinterval* (u, v)

> Recall that the only countable dense order type with no endpoints is $\text{tp}(\mathbb{Q})$ (Theorem 4.3(i) in [4]). If there is no empty subinterval, construct such a linear order inside $[x, y]$ by picking points in subintervals. Then $\text{tp}(\mathbb{Q}) \leqslant \text{tp}([x, y]) \leqslant \text{tp}(L_m)$, contradicting the fact that $\text{tp}(L_m) \in \mathcal{S}$ which was established in theorem 4.4(i).

Hence $(x, y) \subseteq L$ contains a copy of $\sigma_{\alpha\beta}$, so all cardinals $< \beta$ embed into $(x, y) \subseteq L$. We still need to extend the result to all ordinals $< \beta$.

$\text{cf}(\gamma) \leqslant \gamma < \beta$ is a regular cardinal, so $\text{cf}(\gamma) \leqslant (x, y)$. If $\text{cf}(\gamma) = \gamma$ then $\gamma \leqslant (x, y)$ and we're done. So assume $\text{cf}(\gamma) < \gamma$.

Let $f : \text{cf}(\gamma) \to (x, y)$ be an embedding. Write γ as a $\text{cf}(\gamma)$-sum of smaller ordinals γ_τ ($\tau < \text{cf}(\gamma)$). By induction it is possible to embed each γ_τ into the subinterval $(f(\tau), f(\tau + 1)) \subseteq (x, y)$. Combining these embeddings gives an embedding $\gamma \leqslant (x, y)$.

$\square$

The next theorem gives a converse to Theorem 4.4. It implies that the non-uniqueness of the construction of $\eta_{\alpha\beta}$ is unimportant in the sense that all types constructed in this way are equivalent.

The proof will make use of Dedekind cuts: A *Dedekind cut* (L^1, L^2) of a linear order L is a partition $L = L^1 \cup L^2$ such that $(\forall x \in L^1)(\forall y \in L^2) x <_L y$.

Theorem 4.5. *Suppose $\phi \neq \mathbf{0}, \mathbf{1}$, $\phi = tp(M)$, and there are ordinals α, β such that*

(i) $\phi \in \mathcal{M}$;

(ii) $\alpha^* \nleqslant \phi$, $\beta \nleqslant \phi$;

(iii) If (x, y) is an interval of M, then $tp((x, y)) \geqslant \alpha_0^*$ (for all $\alpha_0 < \alpha$) and $tp((x, y)) \geqslant \beta_0$ (for all $\beta_0 < \beta$).

Then $\langle \alpha, \beta \rangle$ is admissible and $\phi \equiv \eta_{\alpha\beta}$.

Proof. By (i) I can write $M = \bigcup_{n<\omega} M_n$ with each $\mathrm{tp}(M_n) \in \mathcal{S}$. To show that $\langle \alpha, \beta \rangle$ is admissible, I need to check the following four conditions:

α, β are limit ordinals

I will prove it for β.

Suppose $\beta = \gamma + 1$. Let $(x, y) \subseteq M$ be any interval. Then (iii) gives $\gamma \leqslant \mathrm{tp}((x, y))$, so $\beta \leqslant \mathrm{tp}((x, y]) \leqslant \phi$, contrary to (ii).

α, β are regular

I will prove it for β.

Suppose $\mathrm{cf}(\beta) < \beta$. Then (iii) can be used to embed β (expressed as a $\mathrm{cf}(\beta)$-sum of smaller ordinals) into ϕ in the same way as this was done in the proof of Theorem 4.4(iii), contradicting (ii).

α, β are uncountable cardinals

I will prove it for β.

$\phi \neq \mathbf{0}, \mathbf{1} \Rightarrow \text{Card}(M) \geqslant 2$. So let $x \neq y \in M$ and consider the interval $(x, y) \subseteq M$. $\text{Card}(M) \geqslant 2 \Rightarrow \beta \geqslant 2$ by (ii). Hence $\mathbf{1} < \beta$, so (iii)$\Rightarrow \mathbf{1} \leqslant (x, y)$. In particular, $(\exists a \in M) x < a < y$. Repeating this process with the subinterval (a, y) gives an embedding $\omega \leqslant \phi$. Now β is a regular limit ordinal, so β is a cardinal, and hence condition (ii) gives $\beta \geqslant \omega_1$.

max$\{\alpha, \beta\}$ *is a successor cardinal*

Assume that $\max\{\alpha, \beta\} = \beta$ (wlog) is a limit cardinal. By (iii), every cardinal $< \beta$ embeds into ϕ, so $\text{Card}(M) \geqslant \sup_{\kappa<\beta}\kappa = \beta$. β is regular and uncountable, so $\text{Card}(M_n) \geqslant \beta$ for some n (otherwise $\langle \text{Card}(M_n) \rangle_{n<\omega}$ is an unbounded ω-sequence in β). Since $\beta \in RC$ and $\text{tp}(M_n) \in \mathcal{S}$, Lemma 4.2(iv) gives $\text{tp}(M_n) \geqslant \beta$ or $\text{tp}(M_n) \geqslant \beta^* \geqslant \alpha^*$, contrary to (ii).

$f : N_1 \to N$ is said to satisfy the *Dedekind cut condition* (or *DCC*) if for any Dedekind cut (N_1^1, N_1^2) of N_1 there is an interval $(x, y) \subseteq N$ such that $z \in (x, y), u \in N_1^1, v \in N_1^2 \Rightarrow f(u) < z < f(v)$.

I will show that $\phi \equiv \eta_{\alpha\beta}$ in three steps:

(1) Let N, N_1 be linear orders. Assume tp$(N_1) \in \mathcal{S}$, tp(N_1) satisfies (ii), Card$(N) > 1$, and N satisfies (iii). Then there is an embedding $N_1 \to N$ satisfying DCC.

The proof will be by induction on the Hausdorff hierarchy of $\mathcal{S}$. So assume $N_1 \in \mathcal{S}_\xi$ and (1) holds for all types in $\bigcup_{\zeta<\xi} \mathcal{S}_\zeta$. By Theorem 4.1 $\text{tp}(N_1)$ may be written as a δ-sum or a δ^*-sum of types in $\bigcup_{\zeta<\xi} \mathcal{S}_\zeta$ for some ordinal δ. I will assume the former (the other case is similar). Write $\text{tp}(N_1) = \sum_{x \in D} \psi_x$, where $\text{tp}(D) = \delta$, $\psi_x \in \bigcup_{\zeta<\xi} \mathcal{S}_\zeta$, and $\psi_x = \text{tp}(P_x)$.

$\text{tp}(N_1)$ satisfies (ii), so $\delta \leqslant \text{tp}(N_1) \Rightarrow \delta < \beta$. N satisfies (iii), so there is an embedding $f : D \to N$, and f certainly satisfies DCC. For each $x \in D$, the interval $(f(x), f(x+1)) \subseteq N$ satisfies (iii) and has cardinality > 1, so by the induction hypothesis there is an embedding $f_x : P_x \to (f(x), f(x+1))$ satisfying DCC. Patching together the f_x

gives an embedding $N_1 \to N$ that satisfies DCC.

(2) Let N be a linear order. Card$(N) > 1 \wedge N$ satisfies (iii) $\Rightarrow \phi \leqslant tp(N)$

Recall: $\phi = \text{tp}(M), M = \bigcup_{n<\omega} M_n, M_n \in \mathcal{S}$.

Note that M_0 satisfies (ii) (since $M_0 \subseteq M$ and M satisfies (ii)), and $M_0 \in \mathcal{S}$. By (1) there is an embedding $f_0 : M_0 \to N$ satisfying DCC. Using the DCC this can be extended to an embedding $f_1 : M_0 \cup M_1 \to N$. Since every interval $(x, y) \subseteq N$ satisfies (iii) and has cardinality > 1, we can take f_1 to satisfy DCC.

Continue this process, obtaining a nested sequence of embeddings $\langle f_n \rangle_{n<\omega}$. Then $f = \bigcup_{n<\omega} f_n$ is an embedding $M \to N$, i.e. $\phi \leqslant \text{tp}(N)$.

(3) $\phi \equiv \eta_{\alpha\beta}$

$\eta_{\alpha\beta} = \text{tp}(L)$ satisfies (iii) and Card$(L) > 1$, so $\phi \leqslant \eta_{\alpha\beta}$ by (2).

$\eta_{\alpha\beta} = \text{tp}(L)$ satisfies (i)-(iii), $\eta_{\alpha\beta} \neq \mathbf{0}, \mathbf{1}$, and $\phi = \text{tp}(M)$ satisfies (iii) and Card$(M) > 1$, so $\eta_{\alpha\beta} \leqslant \phi$ again by (2).

□

Corollary 4.6. *Let $\langle \alpha, \beta \rangle$ be admissible. $\phi \leqslant \eta_{\alpha\beta} \Leftrightarrow \phi \in \mathcal{M}, \alpha^* \nleqslant \phi, \beta \nleqslant \phi$.*

Proof. ($\Rightarrow$) Clear from properties (i)-(iii) of $\eta_{\alpha\beta}$.

($\Leftarrow$) $\eta_{\alpha\beta} = \text{tp}(L)$, Card$(L) > 1$, L satisfies (iii) $\Rightarrow \phi \leqslant \eta_{\alpha\beta}$ by step (2) in the proof of Theorem 4.5 (Note that the assumption that ϕ satisfies property (iii) isn't used in the proof of step (2), so it can be applied to this situation).

□

We now define the class $\mathcal{D}_{\alpha\beta}$ and prove an alternative definition which will allow us to perform induction on $\mathcal{D}_{\alpha\beta}$.

Definition 4.7. $\mathcal{D}_{\alpha\beta} = \{\phi : \phi < \eta_{\alpha\beta}\}$.

Definition 4.8. If L is a linear order, a sequence $\langle x_\alpha \rangle_{\alpha<\lambda}$ is called cofinal if $(\forall y \in L)(\exists \alpha < \lambda) y \leqslant x_\alpha$. The sequence is called co-initial if $(\forall y \in L)(\exists \alpha < \lambda) x_\alpha \leqslant y$.

Theorem 4.9. *(i) A $\mathcal{D}_{\alpha\beta}$-sum of elements of $\mathcal{D}_{\alpha\beta}$ is in $\mathcal{D}_{\alpha\beta}$.*

(ii) $\mathcal{D}_{\alpha\beta} = \bigcup_{\gamma<max\{\alpha,\beta\}}(\mathcal{D}_{\alpha\beta})_\gamma$, where $(\mathcal{D}_{\alpha\beta})_0 = \{\mathbf{0}, \mathbf{1}\}$, and for $\delta > 0$, $\phi \in (\mathcal{D}_{\alpha\beta})_\delta \Leftrightarrow \phi$ is an α_0^*-sum, or a β_0-sum, or an $\eta_{\alpha_0\beta_0}$-sum (for some $\alpha_0 < \alpha, \beta_0 < \beta$) of members of $\bigcup_{\gamma<\delta}(\mathcal{D}_{\alpha\beta})_\gamma$.

Proof. (i) First note that $(\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$, i.e. $\eta_{\alpha\beta} \equiv \sum_{x\in L}\eta_{\alpha\beta}$ where $\mathrm{tp}(L) = \eta_{\alpha\beta}$:

> $\eta_{\alpha\beta} \leqslant (\eta_{\alpha\beta})^2$ is clear.
>
> To show $(\eta_{\alpha\beta})^2 \leqslant \eta_{\alpha\beta}$ I will show that $(\eta_{\alpha\beta})^2 \in \mathcal{M}$, $\alpha^* \nleqslant (\eta_{\alpha\beta})^2$, $\beta \nleqslant (\eta_{\alpha\beta})^2$, and use Corollary 4.6.
>
> Write $\eta_{\alpha\beta} = \mathrm{tp}(L)$ where $L = \bigcup_{n<\omega} L_n$ as in definition 4.3. Define $M_{m,n}$ by $\mathrm{tp}(M_{m,n}) = \sum_{x\in L_m}\mathrm{tp}(L_n)$. By Lemma 4.2(i), $\mathrm{tp}(M_{m,n}) \in \mathcal{S}$. Let $M = \bigcup_{m,n<\omega} M_{m,n}$. Then $\mathrm{tp}(M) = (\eta_{\alpha\beta})^2$ is a countable union of scattered types, i.e. $(\eta_{\alpha\beta})^2 \in \mathcal{M}$.
>
> Suppose $\beta \leqslant (\eta_{\alpha\beta})^2$. Then $\beta \leqslant \eta_{\alpha\beta}$ by Lemma 4.2(ii) (recall that $\langle\alpha,\beta\rangle$ admissible, so $\beta \in RC$), contradicting Theorem 4.4(ii).
>
> $\alpha^* \nleqslant (\eta_{\alpha\beta})^2$ is similar.

Let $\sum_{x\in M}\phi_x$ be a $\mathcal{D}_{\alpha\beta}$-sum of members of $\mathcal{D}_{\alpha\beta}$. Then $\sum_{x\in M}\phi_x \leqslant \eta_{\alpha\beta}$ by the above discussion.

Suppose $\eta_{\alpha\beta} \leqslant \sum_{x\in M}\phi_x$. If an interval of $\eta_{\alpha\beta}$ is embedded into one ϕ_y, then $\eta_{\alpha\beta} \leqslant \phi_y$ since $\eta_{\alpha\beta}$ embeds into each of its non-empty intervals. This contradicts $\phi_y \in \mathcal{D}_{\alpha\beta}$. So the points of L (where $\mathrm{tp}(L) = \eta_{\alpha\beta}$) are mapped to different ϕ_x's, giving $\eta_{\alpha\beta} \leqslant \mathrm{tp}(M)$ and hence contradicting $\mathrm{tp}(M) \in \mathcal{D}_{\alpha\beta}$. So $\sum_{x\in M}\phi_x < \eta_{\alpha\beta}$, as required.

(ii) Let $\mathcal{C}_{\alpha\beta} = \bigcup_{\gamma<\max\{\alpha,\beta\}}(\mathcal{D}_{\alpha\beta})_\gamma$.

(1) A $\mathcal{C}_{\alpha\beta}$-sum of members of $\mathcal{C}_{\alpha\beta}$ is in $\mathcal{C}_{\alpha\beta}$

> Let $\mu = \mathrm{tp}(M) \in \mathcal{C}_{\alpha\beta}$ and for each $y \in M$, let $\phi_y \in \mathcal{C}_{\alpha\beta}$. $\mu \in (\mathcal{D}_{\alpha\beta})_\gamma$ for some $\gamma < \max\{\alpha,\beta\}$. I will show by induction on γ that $\sum_{y\in M}\phi_y \in \mathcal{C}_{\alpha\beta}$.

So assume that (1) holds for all $\mu' \in \bigcup_{\xi<\gamma}(\mathcal{D}_{\alpha\beta})_\xi$. I will show the result in the case where μ is a β_0-sum, the cases α_0^*, $\eta_{\alpha_0\beta_0}$ being similar.

So write $\mu = \sum_{x\in B_0}\mu_x$, where $\mathrm{tp}(B_0) = \beta_0 < \beta$ and $\mu_x = \mathrm{tp}(M_x) \in \bigcup_{\xi<\gamma}(\mathcal{D}_{\alpha\beta})_\xi$. Then $\sum_{y\in M}\phi_y = \sum_{x\in B_0}\sum_{y\in M_x}\phi_y$. By the induction hypothesis, $\sum_{y\in M_x}\phi_y \in \mathcal{C}_{\alpha\beta}$ for each x. Call these sums χ_x. I want to show that $\sum_{x\in B_0}\chi_x \in \mathcal{C}_{\alpha\beta}$.

Each χ_x is in some $(\mathcal{D}_{\alpha\beta})_{\gamma_x}$ ($\gamma_x < \max\{\alpha,\beta\}$). Let $\delta = \sup_{x\in B_0}\gamma_x$. Then $\delta < \max\{\alpha,\beta\}$ because $\mathrm{Card}(B_0) < \beta$ and β is regular. Hence $(\forall x \in B_0)\chi_x \in (\mathcal{D}_{\alpha\beta})_\delta \Rightarrow \sum_{x\in B_0}\chi_x \in (\mathcal{D}_{\alpha\beta})_{\delta+1} \subseteq \mathcal{C}_{\alpha\beta}$.

(2) $\mathcal{C}_{\alpha\beta} \subseteq \mathcal{D}_{\alpha\beta}$

Let $\alpha_0 < \alpha, \beta_0 < \beta$.

$\eta_{\alpha_0\beta_0}$ satisfies the conditions of Theorem 4.5 and $\eta_{\alpha\beta} = \mathrm{tp}(L)$ satisfies the conditions of claim (2) in the proof of Theorem 4.5, so $\eta_{\alpha_0\beta_0} \leqslant \eta_{\alpha\beta}$. If $\eta_{\alpha\beta} \leqslant \eta_{\alpha_0\beta_0}$ then $\beta_0 \leqslant \eta_{\alpha\beta} \leqslant \eta_{\alpha_0\beta_0}$ (by Theorem 4.4(iii), since $\beta_0 < \beta$), contradicting Theorem 4.4(ii). So $\eta_{\alpha_0\beta_0} < \eta_{\alpha\beta}$.

It is clear that $\alpha_0^*, \beta_0 \leqslant \eta_{\alpha\beta}$. Suppose $\eta_{\alpha\beta} \leqslant \beta_0$. Then $\mathrm{tp}(\mathbb{Q}) = \eta_{\omega_1\omega_1} \leqslant \eta_{\alpha\beta} \leqslant \beta_0$ (contradiction). So $\beta_0 < \eta_{\alpha\beta}$ and similarly $\alpha_0^* < \eta_{\alpha\beta}$.

Hence $\alpha_0^*, \beta_0, \eta_{\alpha_0\beta_0} \in \mathcal{D}_{\alpha\beta}$.

Since $(\mathcal{D}_{\alpha\beta})_0 \subseteq \mathcal{D}_{\alpha\beta}$ we obtain $\mathcal{C}_{\alpha\beta} \subseteq \mathcal{D}_{\alpha\beta}$ by induction and part (i) of this theorem.

The rest of the proof will demonstrate equality. Suppose that there is some L with $\mathrm{tp}(L) \in \mathcal{D}_{\alpha\beta}\backslash\mathcal{C}_{\alpha\beta}$.

(3) $(\psi \in \mathcal{C}_{\alpha\beta} \wedge \phi \leqslant \psi) \Rightarrow \phi \in \mathcal{C}_{\alpha\beta}$

Proof by induction: $\psi \in (\mathcal{D}_{\alpha\beta})_\gamma$ for some γ. Assume the claim holds for $\bigcup_{\xi<\gamma}(\mathcal{D}_{\alpha\beta})_\xi$ and let f be an embedding of ϕ into ψ.

Write $\psi = \sum_{x\in B_0}\psi_x$ where $\mathrm{tp}(B_0) = \beta_0 < \beta$, $\psi_x = \mathrm{tp}(P_x)$, and $(\forall x \in B_0)\psi_x \in \bigcup_{\xi<\gamma}(\mathcal{D}_{\alpha\beta})_\xi$ (the cases where ψ is an α_0^*-sum or an $\eta_{\alpha_0\beta_0}$-sum are similar).

Then $\phi = \sum_{x\in B_0}\mathrm{tp}(\mathrm{im}(f)\cap P_x)$ and $\mathrm{tp}(\mathrm{im}(f)\cap P_x) \leqslant \psi_x \in \bigcup_{\xi<\gamma}(\mathcal{D}_{\alpha\beta})_\xi \Rightarrow \mathrm{tp}(\mathrm{im}(f)\cap P_x) \in \mathcal{C}_{\alpha\beta}$ by the induction hypothesis. Hence ϕ is a $\mathcal{C}_{\alpha\beta}$-sum of members of $\mathcal{C}_{\alpha\beta}$, i.e. $\phi \in \mathcal{C}_{\alpha\beta}$ by (1).

Define a binary relation $\sim$ on L by setting $x \sim y$ if $\mathrm{tp}((x, y)) \in \mathcal{C}_{\alpha\beta}$ and set $x \sim x$ and $x \sim y \Rightarrow y \sim x$.

(4) $\sim$ is an equivalence relation

> I only need to check transitivity. Assume $x \sim y, y \sim z$ and $x <_L y <_L z$. Then $\mathrm{tp}((x, y)), \mathrm{tp}((y, z)) \in \mathcal{C}_{\alpha\beta} \Rightarrow \mathrm{tp}((x, z)) = \mathrm{tp}((x, y)) + \mathbf{1} + \mathrm{tp}((y, z)) \in \mathcal{C}_{\alpha\beta}$, so $x \sim z$.

(5) $\sim$ partitions L into intervals

> Suppose $x \sim z$ and $x <_L y <_L z$. Then $\mathrm{tp}((x, y)) \leqslant \mathrm{tp}((x, z)) \in \mathcal{C}_{\alpha\beta} \Rightarrow \mathrm{tp}((x, y)) \in \mathcal{C}_{\alpha\beta}$ by (3). So $x \sim y \sim z$.

Let $X \subseteq L$ be an equivalence class. Construct a co-initial α_0^*-sequence and a cofinal β_0-sequence (some α_0, β_0). Now $\mathrm{tp}(L) \in \mathcal{D}_{\alpha\beta} \Rightarrow \mathrm{tp}(L) < \eta_{\alpha\beta} \Rightarrow \alpha^*, \beta \nleqslant \mathrm{tp}(L)$, so $\alpha_0 < \alpha, \beta_0 < \beta$. Use these sequences to write $\mathrm{tp}(X)$ as an $(\alpha_0^* + \beta_0)$-sum of members of $\mathcal{C}_{\alpha\beta}$ (subintervals of X have type in $\mathcal{C}_{\alpha\beta}$ by definition of $\sim$). Since $\alpha_0^* + \beta_0 \in \mathcal{C}_{\alpha\beta}$, $\mathrm{tp}(X) \in \mathcal{C}_{\alpha\beta}$.

Let $L' \subseteq L$ be a set containing one representative from each equivalence class. Suppose some interval $(u, v) \subseteq L'$ has $\mathrm{tp}((u, v)) \in \mathcal{C}_{\alpha\beta}$. Write $[x]$ for the equivalence class of $x \in L$. If (u, v) is considered as an interval of L, then $(u, v) \subseteq \bigcup_{u \leqslant_{L'} x \leqslant_{L'} v} [x]$, i.e. $\mathrm{tp}((u, v)) \leqslant \sum_{u \leqslant_{L'} x \leqslant_{L'} v} \mathrm{tp}([x])$. Then the previous paragraph shows that this sum is a $\mathcal{C}_{\alpha\beta}$-sum of types in $\mathcal{C}_{\alpha\beta}$, so $\mathrm{tp}((u, v)) \leqslant$ some member of $\mathcal{C}_{\alpha\beta}$. By (3), $\mathrm{tp}((u, v)) \in \mathcal{C}_{\alpha\beta}$, contradicting $u \nsim v$. Hence all intervals in L' have type in $\mathcal{D}_{\alpha\beta} \backslash \mathcal{C}_{\alpha\beta}$.

(6) There is an interval $(x_0, y_0) \subseteq L'$ that doesn't contain a copy of α_0^* or of β_0 (for some $\alpha_0 < \alpha$ or $\beta_0 < \beta$)

> $L' \neq \mathbf{0}, \mathbf{1}$ and conditions (i) and (ii) of Theorem 4.5 are satisfied by L', α, β:
>
> (i): $\mathrm{tp}(L') \leqslant \mathrm{tp}(L) < \eta_{\alpha\beta} \in \mathcal{M} \Rightarrow \mathrm{tp}(L') \in \mathcal{M}$
>
> (ii): $\alpha^*, \beta \nleqslant \eta_{\alpha\beta} \Rightarrow \alpha^*, \beta \nleqslant \mathrm{tp}(L) \Rightarrow \alpha^*, \beta \nleqslant \mathrm{tp}(L')$

Since $\mathrm{tp}(L') \leqslant \mathrm{tp}(L) < \eta_{\alpha\beta}$, condition (iii) must fail. This is precisely the statement above.

Take (x_0, y_0) as above and assume $\beta_0 \nleqslant (x_0, y_0)$ (the other case is similar). Assume wlog that we have chosen (x_0, y_0) in such a way that the corresponding ordinal β_0 is minimal in the sense that if $\beta_1 < \beta_0$ then every interval of L' contains a copy of β_1. Let α_0 $(< \alpha)$ be the smallest ordinal such that α_0^* doesn't embed into some subinterval $(x_1, y_1) \subseteq (x_0, y_0) \subseteq L'$. Then (x_1, y_1), α_0, β_0 satisfy the conditions of Theorem 4.5, so $\mathrm{tp}((x_1, y_1)) \equiv \eta_{\alpha_0\beta_0}$ and $\langle\alpha_0, \beta_0\rangle$ is admissible. Now $\langle\alpha_0, \beta_0\rangle < \langle\alpha, \beta\rangle$, so $\mathrm{tp}((x_1, y_1)) \equiv \eta_{\alpha_0\beta_0} \in \mathcal{C}_{\alpha\beta}$, contradicting the fact that all intervals of L' have type in $\mathcal{D}_{\alpha\beta} \backslash \mathcal{C}_{\alpha\beta}$.

□

5 The Main Theorem

In this section I will define the classes $\mathcal{H}(Q)$ and Q^+ mentioned in the introduction, and I will use them to prove the chain

$$Q \text{ BQO } \Rightarrow Q^+ \text{ BQO } \Rightarrow \mathcal{T}_{Q^+} \text{ BQO } \Rightarrow \mathcal{H}(Q) \text{ BQO } \Rightarrow Q^{\mathcal{M}} \text{ BQO}.$$

The main theorem of this essay asserts that $\mathcal{M}$ is BQO – this is an easy corollary of $Q^{\mathcal{M}}$ being BQO.

A *Q-linear ordering* is a pair (L, l), where L is a linear order and $l : L \to Q$ is a function (the labelling function). Two Q-linear orders (L, l), (M, m) are *isomorphic* if there is an order isomorphism $f : L \to M$ satisfying $(\forall x \in L) m(f(x)) = l(x)$. A *$Q$-type* is the isomorphism type of a Q-linear ordering. In the following, Q-types are usually denoted by capital greek letters. $\mathrm{tp}(L, l)$ is the isomorphism type of the Q-linear order (L, l). The *base* of a Q-type $\Phi = \mathrm{tp}(L, l)$ is $\mathrm{bs}(\Phi) = L$. Sums and products of Q-types are defined as for order types. $\mathbf{0}$ will denote the Q-type with base $\mathbf{0}$, $\mathbf{1}_q$ will denote the Q-type with base $\mathbf{1}$ labelled by the point $q \in Q$.

Definition 5.1. Quasi-order the class of Q-types as follows: $\Phi \leqslant \Psi$ if $\Phi = \mathrm{tp}(L, l)$, $\Psi = \mathrm{tp}(M, m)$ and there is a strictly increasing function $f : L \to M$ satisfying $(\forall x \in L) l(x) \leqslant m(f(x))$.

Let Q^ϕ (or $Q^{\leqslant\phi}$, or $Q^{\equiv\phi}$) be the collection of all Q-types with base ϕ (or $\leqslant \phi$, or $\equiv \phi$). If $\mathcal{R}$ is a collection of order types, $Q^{\mathcal{R}} = \bigcup_{\phi\in\mathcal{R}} Q^\phi$.

We now define $(\mathcal{U},\kappa)$-unbounded sums and $(\mathcal{R},\alpha,\beta)$-shuffles and prove some preliminary results. These concepts will be used to define the class or types $\mathcal{H}(Q)$.

Definition 5.2. If Φ is a Q-type, $\mathcal{U}$ is a set of Q-types, and κ is an infinite cardinal, then Φ is called a *$(\mathcal{U},\kappa)$-unbounded sum* if $\Phi = \sum_{x\in K}\Phi_x$ where $\mathrm{tp}(K) = \kappa$, $\mathcal{U} = \{\Phi_x : x \in K\}$, and

$$(\forall x \in K)(\exists Y \subseteq K)(\mathrm{Card}(Y) = \kappa \wedge (y \in Y \Rightarrow \Phi_x \leqslant \Phi_y)).$$

If $\mathrm{tp}(K) = \kappa^*$ instead of κ in the above definition, then Φ is called a *$(\mathcal{U},\kappa^*)$-unbounded sum.*

Lemma 5.3. *Suppose $\delta \in RC$, $\kappa \leqslant \delta$, Φ is a $(\mathcal{U},\kappa)$-unbounded sum, Ψ is a $(\mathcal{V},\delta)$-unbounded sum (or Φ is a $(\mathcal{U},\kappa^*)$-unbounded sum and Ψ is a $(\mathcal{V},\delta^*)$-unbounded sum) and $(\forall\Theta \in \mathcal{U})(\exists\chi \in \mathcal{V})\Theta \leqslant \chi$. Then $\Phi \leqslant \Psi$.*

Proof. Write $\Phi = \sum_{x\in K}\Phi_x$, $\Psi = \sum_{x\in D}\Psi_y$ as in definition 5.2. Define an embedding h of Φ into Ψ by induction on initial segments of κ. Suppose h has been defined on $\sum_{x<x_0}\Phi_x$ and embeds that segment into $\sum_{y<y_0}\Psi_y$. By assumption, $(\exists y_1 \in D)\Phi_{x_0} \leqslant \Psi_{y_1}$. Ψ is a $(\mathcal{V},\delta)$-unbounded sum, so $(\exists y_2 \geqslant_D y_1)\Psi_{y_1} \leqslant \Psi_{y_2}$ (since the cardinality of an initial segment of δ is $< \delta$). So extend h to embed Φ_{x_0} into Ψ_{y_2}.

For $\gamma < \kappa$, a γ-limit of initial segments of δ is an initial segment of δ, since $\kappa \leqslant \delta$ and δ is regular. Hence this construction of the embedding h defines it on each Φ_x, showing that $\Phi \leqslant \Psi$.

The argument for $(\mathcal{U},\kappa^*)$ and $(\mathcal{V},\delta^*)$ is similar.

$\square$

Definition 5.4. Let $\mathcal{R}$ be a set of Q-types and $\Psi \in \mathcal{R}^\phi$ (some ϕ), where $\mathcal{R}$ is quasi-ordered as in definition 5.1: $\Psi = \mathrm{tp}(L,l)$ where $\mathrm{tp}(L) = \phi$ and $l : L \to \mathcal{R}$ labels elements of L with Q-types. Then we define

$$\overline{\Psi} = \sum_{x\in L} l(x)$$

Definition 5.5. (i) If Q is any QO and Φ is a Q-type, then Φ is called (Q, α, β)-*universal* if $\Phi \in Q^{\equiv \eta_{\alpha\beta}}$ and $\Psi \in Q^{\leqslant \eta_{\alpha\beta}} \Rightarrow \Psi \leqslant \Phi$.

(ii) We say Φ is an $(\mathcal{R}, \alpha, \beta)$-*shuffle* if $\mathcal{R} \subseteq Q^{\mathcal{M}}$ and $\Phi = \overline{\Phi'}$ for some $(\mathcal{R}, \alpha, \beta)$-universal Φ'.

Remark. Recall that $Q^{\mathcal{M}}$ is the class of countable unions of scattered types, with points labelled by elements of Q.

Lemma 5.6. *If Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle, Ψ a $(\mathcal{V}, \gamma, \delta)$-shuffle, $\langle \alpha, \beta \rangle \leqslant \langle \gamma, \delta \rangle$, and $(\forall \Theta \in \mathcal{U})(\exists \chi \in \mathcal{V})\Theta \leqslant \chi$, then $\Phi \leqslant \Psi$.*

Proof. Ψ is a $(\mathcal{V}, \gamma, \delta)$-shuffle: $\Psi = \overline{\Psi'}$ where Ψ' is $(\mathcal{V}, \gamma, \delta)$-universal, i.e. $\Psi' \in \mathcal{V}^{\equiv \eta_{\gamma\delta}}$ and $\chi \in \mathcal{V}^{\leqslant \eta_{\gamma\delta}} \Rightarrow \chi \leqslant \Psi'$. Assume that $\Psi' = \mathrm{tp}(M, m)$, where $\mathrm{tp}(M) \equiv \eta_{\gamma\delta}$ and $m : M \to \mathcal{V}$. Then $\Psi = \overline{\Psi'} = \sum_{y \in M} m(y)$.

Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle: $\Phi = \overline{\Phi'}$, where $\Phi' \in \mathcal{U}^{\equiv \eta_{\alpha\beta}}$. Assume that $\Phi' = \mathrm{tp}(L, l)$, where $\mathrm{tp}(L) \equiv \eta_{\alpha\beta}$ and $l : L \to \mathcal{U}$. Then $\Phi = \overline{\Phi'} = \sum_{x \in L} l(x)$.

Define $l' : L \to \mathcal{V}$ by letting $l'(x)$ be some $\chi \in \mathcal{V}$ such that $l(x) \leqslant \chi$ (as guaranteed by the assumptions of the lemma). Since $\eta_{\alpha\beta} \leqslant \eta_{\gamma\delta}$ it follows that $\Phi' \leqslant \Phi'' \in \mathcal{V}^{\leqslant \eta_{\gamma\delta}}$ where $\Phi'' = \mathrm{tp}(L, l')$. By $(\mathcal{V}, \gamma, \delta)$-universality of Ψ' we get $\Phi' \leqslant \Phi'' \leqslant \Psi'$, i.e. there is a strictly increasing $f : L \to M$ such that $(\forall x \in L) l(x) \leqslant m(f(x))$. So $\sum_{x \in L} l(x) \leqslant \sum_{x \in L} m(f(x)) \leqslant \sum_{y \in M} m(y)$, i.e. $\Phi \leqslant \Psi$.

$\square$

Definition 5.7. Define $\mathcal{H}(Q)$ to be $\bigcup_{\alpha \in On} \mathcal{H}_\alpha(Q)$ where

(i) $\mathcal{H}_0(Q) = \{\mathbf{0}\} \cup \{\mathbf{1}_q : q \in Q\}$ and

(ii) for $\alpha > 0$, $\Phi \in \mathcal{H}_\alpha(Q) \Leftrightarrow \Phi$ is a $(\mathcal{U}, \kappa)$-unbounded sum, or a $(\mathcal{U}, \kappa^*)$-unbounded sum, or a $(\mathcal{U}, \alpha, \beta)$-shuffle for some $\mathcal{U} \subseteq \bigcup_{\beta < \alpha} \mathcal{H}_\beta(Q)$ (and some $\kappa \in RC$, some admissible $\langle \alpha, \beta \rangle$).

Definition 5.8. If Q is a quasi-ordered set or class, construct a new QO Q^+ from it as follows: For each $\kappa \in RC$, add elements a_κ and b_κ, and for each admissible pair $\langle \alpha, \beta \rangle$, add a point $c_{\alpha\beta}$ to Q. Quasi-order Q^+ as the disjoint union of sets $Q \cup \{a_\kappa : \kappa \in RC\} \cup \{b_\kappa : \kappa \in RC\} \cup \{c_{\alpha\beta} : \langle \alpha, \beta \rangle \text{ admissible}\}$, where

$$a_\kappa \leqslant a_\lambda \Leftrightarrow b_\kappa \leqslant b_\lambda \Leftrightarrow \kappa \leqslant \lambda \text{ and } c_{\alpha\beta} \leqslant c_{\gamma\delta} \Leftrightarrow \langle \alpha, \beta \rangle \leqslant \langle \gamma, \delta \rangle.$$

Definition 5.9. Define a class function $T : \mathcal{H}(Q) \to \mathcal{T}_{Q^+}$ by induction on the 'levels' of $\mathcal{H}(Q)$ as follows:

(i) $T(\mathbf{0})$ is the empty Q^+-tree, and $T(\mathbf{1}_q) = \mathbf{1}^q$.

(ii) Assume $T(\Psi)$ has been defined for all $\Psi \in \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$ and $\Phi \in \mathcal{H}_\alpha(Q) \backslash \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$. Then there is $\mathcal{U} \subseteq \bigcup_{\beta<\alpha} \mathcal{H}_\beta(Q)$ such that one of the following three conditions holds.

> (1) Φ is a $(\mathcal{U}, \lambda)$-unbounded sum for some $\lambda \in RC$. Let $T(\Phi) = [a_\lambda; \{T(\Theta) : \Theta \in \mathcal{U}\}]$;
>
> (2) Φ is a $(\mathcal{U}, \lambda^*)$-unbounded sum for some $\lambda \in RC$. Let $T(\Phi) = [b_\lambda; \{T(\Theta) : \Theta \in \mathcal{U}\}]$;
>
> (3) Φ is a $(\mathcal{U}, \gamma, \delta)$-shuffle for some admissible $\langle \gamma, \delta \rangle$. Let $T(\Phi) = [c_{\gamma\delta}; \{T(\Theta) : \Theta \in \mathcal{U}\}]$.

Remark. Recall that trees in $\mathcal{T}_{Q^+}$ are rooted and have no path of length $> \omega$. The first condition is immediate for trees $T(\Phi)$ ($\Phi \in \mathcal{H}(Q)$) from the definition of T, the second condition holds by induction. Hence T really is a function into $\mathcal{T}_{Q^+}$.

The next theorem reduces the problem of showing that $\mathcal{H}(Q)$ is BQO to that of showing that $\mathcal{T}_{Q^+}$ is BQO.

Theorem 5.10. *If $\Phi \in \mathcal{H}_\alpha(Q)$ and $\Psi \in \mathcal{H}_\beta(Q)$, then $T(\Phi) \leqslant_m T(\Psi) \Rightarrow \Phi \leqslant \Psi$.*

Proof. The proof is by induction on $\langle \alpha, \beta \rangle$. Assume the result for all $\langle \alpha_0, \beta_0 \rangle < \langle \alpha, \beta \rangle$. Let $T(\Phi) = (T_1, l_1)$, $T(\Psi) = (T_2, l_2)$, and let $f : T_1 \to T_2$ be an embedding of $T(\Phi)$ into $T(\Psi)$.

Suppose $f(\rho(T_1)) \neq \rho(T_2)$. Then $T(\Phi) \leqslant_m \mathrm{br}_{(T_2,l_2)}(x)$ for some $x \in S(\rho(T_2))$, $\mathrm{br}_{(T_2,l_2)}(x) = T(\chi)$ for some $\chi \in \bigcup_{\gamma<\beta} \mathcal{H}_\gamma(Q)$, and $\chi \leqslant \Psi$. By the induction hypothesis, $\Phi \leqslant \chi \leqslant \Psi$ and we're done. So assume $f(\rho(T_1)) = \rho(T_2)$. There are 4 cases:

(1) $T(\Phi)$ is empty or $l_1(\rho(T_1)) = q$ for some $q \in Q$

> The theorem is clear in the first case.

In the second case we must have $l_2(\rho(T_2)) = r$ for some $r \geqslant q$ (recall that the trees are labelled by Q^+). Hence $\Phi = \mathbf{1}_q \leqslant \mathbf{1}_r = \Psi$.

(2) $l_1(\rho(T_1)) = a_\delta$ *for some* $\delta \in RC$

We must have $l_2(\rho(T_2)) = a_\kappa$ for some $\kappa \geqslant \delta$, $\kappa \in RC$.

$$\begin{aligned} l_1(\rho(T_1)) = a_\delta \quad &\Rightarrow \quad \Phi \text{ is a } (\mathcal{U}, \delta)\text{-unbounded sum of members of } \bigcup_{\gamma<\alpha} \mathcal{H}_\gamma(Q) \\ &\Rightarrow \quad T(\Phi) = [a_\delta; \{T(\Theta) : \Theta \in \mathcal{U}\}] \\ &\Rightarrow \quad \mathcal{U} = \{\Theta : T(\Theta) = \mathrm{br}_{(T_1,l_1)}(x) \text{ for some } x \in S(\rho(T_1))\} \end{aligned}$$

Similarly, Ψ is a $(\mathcal{V}, \kappa)$-unbounded sum where $\mathcal{V} = \{\chi : T(\chi) = \mathrm{br}_{(T_2,l_2)}(x)$ for some $x \in S(\rho(T_2))\}$.

The function f above gives an embedding of any $\mathrm{br}_{(T_1,l_1)}(x)$ $(x \in S(\rho(T_1)))$ into some $\mathrm{br}_{(T_2,l_2)}(y)$ $(y \in S(\rho(T_2)))$, i.e. $(\forall \Theta \in \mathcal{U})(\exists \chi \in \mathcal{V}) T(\Theta) \leqslant T(\chi)$. By induction hypothesis, $(\forall \Theta \in \mathcal{U})(\exists \chi \in \mathcal{V}) \Theta \leqslant \chi$. The conditions of Lemma 5.3 are satisfied, so $\Phi \leqslant \Psi$.

(3) $l_1(\rho(T_1)) = b_\delta$ *for some* $\delta \in RC$

Similar to case (2).

(4) $l_1(\rho(T_1)) = c_{\alpha\beta}$ *for some admissible* $\langle \alpha, \beta \rangle$

We must have $l_2(\rho(T_2)) = c_{\gamma\delta}$ with $\langle \alpha, \beta \rangle \leqslant \langle \gamma, \delta \rangle$. Also, Φ is a $(\mathcal{U}, \alpha, \beta)$-shuffle and Ψ is a $(\mathcal{V}, \gamma, \delta)$-shuffle, where $\mathcal{U}$ and $\mathcal{V}$ are as in case (2). As above, we get $(\forall \Theta \in \mathcal{U})(\exists \chi \in \mathcal{V}) \Theta \leqslant \chi$ from the induction hypothesis.

Hence the conditions of Lemma 5.6 are satisfied and it follows that $\Phi \leqslant \Psi$, as required.

□

Corollary 5.11. *Q BQO $\Rightarrow \mathcal{H}(Q)$ BQO*

Proof. Theorem 2.4(ii) $\Rightarrow \{a_\kappa : \kappa \in RC\}, \{b_\kappa : \kappa \in RC\}$ BQO.

Theorem 2.4(ii),(iv) $\Rightarrow \{c_{\alpha\beta} : \langle\alpha,\beta\rangle$ admissible$\}$ BQO.

Hence Theorem 2.4(iii) $\Rightarrow Q^+$ is BQO. By Theorem 3.4, $\mathcal{T}_{Q^+}$ is BQO under $\leqslant_m$. If $a : B \to \mathcal{H}(Q)$ is an array, then the composition of a with T is an array in $\mathcal{T}_{Q^+}$ which can't be bad since $\mathcal{T}_{Q^+}$ is BQO. So by Theorem 5.10, a can't be bad. Hence $\mathcal{H}(Q)$ is BQO.

□

Theorem 5.12. *Suppose Q is WQO and $\Phi \in Q^{\leqslant\eta_{\alpha\beta}}$. Then Φ is a $\mathcal{D}_{\alpha\beta}$-sum of types $\mathbf{1}_q$ ($q \in Q$) and of (R,α_0,β_0)-universal types ($R \subseteq Q$, $\langle\alpha_0,\beta_0\rangle \leqslant \langle\alpha,\beta\rangle$).*

Proof. The theorem is trivial for $Q = \mathbf{0}$. Let Q be a WQO. By the induction principle (Lemma 2.6), it suffices to assume that the result is true for $Q_q = \{r \in Q : q \nleqslant r\}$ for each $q \in Q$ and deduce it for Q.

$\Phi \in Q^{\leqslant\eta_{\alpha\beta}} \Rightarrow \Phi = \mathrm{tp}(L,l)$ for some $l : L \to Q$ and $\mathrm{tp}(L) \leqslant \eta_{\alpha\beta}$.

Define a binary relation $\sim$ on L by setting $y \sim z$ if $y < z$ and every subinterval of (y,z) is a $\mathcal{D}_{\alpha\beta}$-sum of types $\mathbf{1}_q$ ($q \in Q$) and (R,α_0,β_0)-universal types ($R \subseteq Q$, $\langle\alpha_0,\beta_0\rangle \leqslant \langle\alpha,\beta\rangle$) and by setting $x \sim x$ and $x \sim y \Rightarrow y \sim x$.

$\sim$ is an equivalence relation that partitions L into intervals

> $\sim$ is transitive: Suppose $x < y < z$ and $x \sim y \sim z$. Then every subinterval of (x,z) is either a subinterval of (x,y) or a subinterval of (y,z) or of the form $(u,y) + \mathbf{1} + (y,v)$. All of these have the right form, so $x \sim z$. Hence $\sim$ is an equivalence relation.
>
> If $x \sim z$ and $x < y < z$ then $x \sim y \sim z$ by definition of $\sim$. Hence the equivalence classes are intervals.

Write $[x]$ for the equivalence class of $x \in L$.

Choose a co-initial γ^*-sequence and a cofinal δ-sequence in $[x]$ such that all points of the former are below all points of the latter in the ordering of L. $\mathrm{tp}(L) \leqslant \eta_{\alpha\beta}$, so by Theorem 4.4(ii), $\gamma < \alpha$ and $\delta < \beta$. Using these sequences, write $[x]$ as a $(\gamma^* + \delta)$-sum of types $\mathbf{1}_q + (u,v)$ (some $q \in Q$, $(u,v) \subseteq [x]$). $\gamma^* + \delta < \eta_{\alpha\beta}$, so $\gamma^* + \delta \in \mathcal{D}_{\alpha\beta}$. Since $u \sim v$ for each subinterval of $[x]$, $[x]$ is thus written as a $\mathcal{D}_{\alpha\beta}$-sum of types $\mathbf{1}_q$ and (R,α_0,β_0)-universal types (using Theorem 4.9(i)).

If L consists of one equivalence class, we're done.

So suppose $(\exists x, y \in L) x \nsim y$. I will show that Φ itself is a (Q, α, β)-universal type, thereby completing the proof.

Write $[L]$ for the set $\{[x] : x \in L\}$ linearly ordered by $[x] \leqslant_{[L]} [y] \Leftrightarrow x \leqslant_L y$. Let $([x], [y])$ be an interval in $[L]$, $[x] \neq [y]$.

(1) tp$(([x],[y])) \equiv \eta_{\alpha\beta}$

We know that $\mathrm{tp}(([x],[y])) \leqslant \mathrm{tp}([L]) \leqslant \mathrm{tp}(L) \leqslant \eta_{\alpha\beta}$.

Suppose $\mathrm{tp}(([x],[y])) < \eta_{\alpha\beta}$, i.e. $\mathrm{tp}(([x],[y])) \in \mathcal{D}_{\alpha\beta}$. Then (x, y) is a $\mathcal{D}_{\alpha\beta}$-sum of $\mathcal{D}_{\alpha\beta}$-sums of $\mathbf{1}_q$'s and $(\mathcal{R}, \alpha_0, \beta_0)$-universal types, so by Theorem 4.9(i), $x \sim y$ (contradiction).

(2) $(\forall q \in Q)(\exists z \in L)[z] \in ([x],[y]) \wedge l(z) \geqslant q$

Otherwise there is $q \in Q$ such that

$$\{l(z) : [z] \in ([x],[y])\} \subseteq \{p \in Q : q \nleqslant p\} = Q_q.$$

We are assuming the theorem for Q_q, so $\{z : [z] \in ([x],[y])\}$ is a subset of one equivalence class (all of Q_q), contradicting (1).

To show that Φ is (Q, α, β)-universal, note first that $\Phi \in Q^{\equiv\eta_{\alpha\beta}}$ because $\Phi \in Q^{\leqslant\eta_{\alpha\beta}}$ (by assumption) and $\mathrm{bs}(\Phi) \geqslant \mathrm{tp}([L]) \geqslant \mathrm{tp}(([x],[y])) \equiv \eta_{\alpha\beta}$ by (1).

Now assume $\mathrm{tp}(M, m) \in Q^{\leqslant\eta_{\alpha\beta}}$. I want $\mathrm{tp}(M, m) \leqslant \Phi$. Recall that $\mathrm{tp}([L]) \equiv \eta_{\alpha\beta}$ and $(\eta_{\alpha\beta})^2 \equiv \eta_{\alpha\beta}$. By considering $\eta_{\alpha\beta}$ copies of $[L]$ it follows that there is an embedding $f : M \to [L]$ such that for each $x \in M$ there is an interval $([y],[z]) = [L]_x$ of $[L]$ such that x is the only element of M mapping into $[L]_x$ and $x \neq y \Rightarrow [L]_x \cap [L]_y = \varnothing$.

By (2), f gives rise to an embedding $f' : M \to L$ such that $(\forall y \in M) m(y) \leqslant l(f'(y))$. So $\mathrm{tp}(M, m) \leqslant \Phi$ and Φ is (Q, α, β)-universal, as required.

$\square$

Lemma 5.13. *$\chi \in \mathcal{H}_\gamma(\mathcal{H}(Q)) \Rightarrow \overline{\chi} \in \mathcal{H}(Q)$ (on the left-hand side, $\mathcal{H}(Q)$ is considered to be a QO, quasi-ordered by embeddability).*

Proof. The proof will be by induction on γ. The result is clear for $\gamma = 0$, since $\mathbf{1}_\Phi \in \mathcal{H}_0(\mathcal{H}(Q)) \Rightarrow \overline{\mathbf{1}_\Phi} = \Phi \in \mathcal{H}(Q)$. Let $\chi \in \mathcal{H}_\gamma(\mathcal{H}(Q))$ and assume the result for $\beta < \gamma$. There are three cases:

(1) χ is a $(\mathcal{U}, \kappa)$-unbounded sum ($\mathcal{U} \subseteq \bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q))$ and $\kappa \in RC$)

Suppose $\mathcal{U} = \{\chi_i : i \in I\}$, $\mathrm{tp}(I) = \kappa$, $\mathrm{bs}(\chi_i) = X_i$, and let $\overline{\mathcal{U}} = \{\overline{\chi_i} : i \in I\}$. First I will show that

$$\chi = \sum_{i \in I} \chi_i \Rightarrow \overline{\chi} = \sum_{i \in I} \overline{\chi_i} \tag{2}$$

$\chi = \mathrm{tp}(L, l)$ for some $l : L \to \mathcal{H}(Q)$ and $\mathrm{tp}(L) = \sum_{i \in I} X_i$. Hence $\overline{\chi} = \sum_{x \in L} l(x) = \sum_{i \in I} \sum_{x \in X_i} l(x) = \sum_{i \in I} \overline{\chi_i}$, as required.

Hence $\overline{\chi}$ is a $(\overline{\mathcal{U}}, \kappa)$-sum.

Furthermore, since $\chi_1 \leqslant \chi_2 \Rightarrow \overline{\chi_1} \leqslant \overline{\chi_2}$, $\overline{\chi}$ is a $(\overline{\mathcal{U}}, \kappa)$-unbounded sum.

As all χ_i are in $\bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q))$ it follows by the induction hypothesis that the $\overline{\chi_i}$ are in $\mathcal{H}(Q)$, so by equation 2, $\overline{\chi} = \sum_{i \in I} \overline{\chi_i} \in \mathcal{H}(Q)$.

(2) χ is a $(\mathcal{U}, \kappa^*)$-unbounded sum ($\mathcal{U} \subseteq \bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q))$ and $\kappa \in RC$)

This case is similar to case (1).

(3) χ is a $(\mathcal{U}, \delta, \lambda)$-shuffle ($\mathcal{U} \subseteq \bigcup_{\beta<\gamma} \mathcal{H}_\beta(\mathcal{H}(Q))$ and $\langle \delta, \lambda \rangle$ is admissible)

I will show that $\overline{\chi}$ is a $(\overline{\mathcal{U}}, \delta, \lambda)$-shuffle. As in case (1), it follows that $\overline{\chi} \in \mathcal{H}(Q)$.

χ is a $(\mathcal{U}, \delta, \lambda)$-shuffle, so I may assume $\chi = \overline{\chi'}$ where

(a) $\chi' \in \mathcal{U}^{\equiv \eta_{\delta\lambda}}$

(b) $\Theta \in \mathcal{U}^{\leqslant \eta_{\delta\lambda}} \Rightarrow \Theta \leqslant \chi'$.

Write Φ for $\overline{\chi}$. I need to show that Φ is a $(\overline{\mathcal{U}}, \delta, \lambda)$-shuffle, i.e. $\Phi = \overline{\Phi'}$ where (a') $\Phi' \in \overline{\mathcal{U}}^{\equiv \eta_{\delta\lambda}}$

(b') $\Theta \in \overline{\mathcal{U}}^{\leqslant \eta_{\delta\lambda}} \Rightarrow \Theta \leqslant \Phi'$.

By (a) we have $\chi' = \mathrm{tp}(L, l')$ for some $l' : L \to \mathcal{U}$ and $\mathrm{tp}(L) \equiv \eta_{\delta\lambda}$. Define a new labelling $l : L \to \overline{\mathcal{U}} : x \mapsto \overline{l'(x)}$. Then $\chi = \overline{\chi'} = \sum_{x \in L} l'(x)$, so by equation 2 above, $\Phi = \overline{\chi} = \sum_{x \in L} \overline{l'(x)} = \sum_{x \in L} l(x)$. Define $\Phi' = \sum_{x \in L} \mathbf{1}_{l(x)} \in \overline{\mathcal{U}}^{\equiv \eta_{\delta\lambda}}$. Then $\overline{\Phi'} = \sum_{x \in L} l(x) = \Phi$, so we have (a').

Suppose $\Theta \in \overline{\mathcal{U}}^{\leqslant \eta_{\delta\lambda}}$, i.e. $\Theta = \mathrm{tp}(M, m)$ for some $m : M \to \overline{\mathcal{U}}$ and $\mathrm{tp}(M) \leqslant \eta_{\delta\lambda}$. Define a new labelling $m' : M \to \mathcal{U}$ that sends $x \in M$ to some $m(x) \in \mathcal{U}$ such that $\overline{m(x)} = m'(x)$. Define $\Theta' = \sum_{x \in M} \mathbf{1}_{m'(x)} \in \mathcal{U}^{\leqslant \eta_{\delta\lambda}}$. Then (b)$\Rightarrow \Theta' \leqslant \chi'$, i.e. there is an embedding $f : M \to L$ such that $(\forall x \in M) m'(x) \leqslant l'(f(x))$. Hence $m(x) = \overline{m'(x)} \leqslant \overline{l'(f(x))} = l(f(x))$, i.e. f gives an embedding $\Theta = \sum_{x \in M} \mathbf{1}_{m(x)} \to \sum_{x \in L} \mathbf{1}_{l(x)} = \Phi'$. This is (b').

□

Theorem 5.14 collects many of the previously proved results in one sentence. This will then be used to prove Theorem 5.15, which is the strongest result in this essay.

Theorem 5.14. *If Q is a BQO and $\Phi \in Q^{\leqslant \eta_{\alpha\beta}}$ then Φ is a finite sum of members of $\mathcal{H}(Q)$.*

Proof. The proof is by induction on admissible $\langle \alpha, \beta \rangle$. So assume the theorem holds for all admissible $\langle \alpha_0, \beta_0 \rangle < \langle \alpha, \beta \rangle$.

I will first prove the theorem for $\Phi \in Q^{< \eta_{\alpha\beta}} = Q^{\mathcal{D}_{\alpha\beta}}$ by induction on the hierarchy for $\mathcal{D}_{\alpha\beta}$ given in Theorem 4.9(ii). The result is trivial for $\Phi \in Q^{(\mathcal{D}_{\alpha\beta})_0}$ (as $\mathrm{bs}(\Phi)$ is finite). So assume the result for $\delta < \gamma$ and let $\Phi \in Q^{(\mathcal{D}_{\alpha\beta})_\gamma}$. There are three cases to be considered:

(1) bs(Φ) is a β_0-sum ($\beta_0 < \beta$)

Suppose the theorem fails for Φ. Then it fails for some $\Theta \in Q^{(\mathcal{D}_{\alpha\beta})_\gamma}$, where Θ is a λ-sum of types in $Q^{\bigcup_{\delta<\gamma} (\mathcal{D}_{\alpha\beta})_\delta}$ with λ minimal (in particular, $\lambda \leqslant \beta_0$). In other words, Θ is not a finite sum of members of $\mathcal{H}(Q)$, but $\Theta = \sum_{x \in L} \Theta_x$ where $\mathrm{tp}(L) = \lambda$ and each Θ_x is a finite sum of members of $\mathcal{H}(Q)$ by induction hypothesis.

Clearly, λ must be an infinite limit ordinal. I will show that λ is regular. Hence λ is a cardinal and $\lambda \in RC$. To show that λ is regular, write Θ as a cf(λ)-sum $\sum_{y \in M} \Theta^y$ (tp$(M) = $ cf(λ)), where each Θ^y is a $< \lambda$-sum of types Θ_x $(x \in L)$. Each Θ^y is a finite sum of members of $\mathcal{H}(Q)$ by minimality of λ. Again by minimality of λ, cf$(\lambda) = \lambda$, as required.

Now Θ is a λ-sum of finite sums of members of $\mathcal{H}(Q)$, so Θ can be written as $\sum_{x \in L} \Theta'_x$ where each $\Theta'_x \in \mathcal{H}(Q)$ and tp$(L) = \lambda$. Claims:

(1.1) $(\exists x_0 \in L)(\forall y, z \in L)(x_0 \leqslant_L y \leqslant_L z \Rightarrow (\exists u \in L)(z \leqslant_L u \wedge \Theta'_y \leqslant \Theta'_u))$

Suppose this is false, i.e.

$(\forall x_0 \in L)(\exists y, z \in L)(x_0 \leqslant_L y \leqslant_L z \wedge (\forall u \in L)(z \leqslant_L u \Rightarrow \Theta'_y \nleqslant \Theta'_u))$.

This means that for arbitrarily large $y \in L$, $(\exists z \in L)(y \leqslant_L z \wedge (\forall u \in L)(z \leqslant_L u \Rightarrow \Theta'_y \nleqslant \Theta'_u))$. Hence it is possible to choose an increasing ω-sequence $\langle y_n \rangle_{n<\omega}$ in L such that $m < n \Rightarrow \Theta'_{y_m} \nleqslant \Theta'_{y_n}$, contradicting the fact that $\mathcal{H}(Q)$ is BQO (by Corollary 5.11, since Q is BQO) and hence WQO (by Theorem 2.4(i)).

Fix the point x_0 given in (1.1).

(1.2) $\sum_{x \geqslant x_0} \Theta'_x$ *is a* $(\{\Theta'_x : x \geqslant x_0\}, \lambda)$*-unbounded sum*

$T = \{x \in L : x \geqslant x_0\}$ is a terminal segment of the regular cardinal λ, so tp$(T) = \lambda$. I still need to show unboundedness:

$$(\forall x \geqslant x_0)(\exists Y \subseteq L)(\text{Card}(Y) = \lambda \wedge (y \in Y \Rightarrow \Theta'_x \leqslant \Theta'_y)).$$

By (1.1) there is an unbounded (in λ) set Y that satisfies $y \in Y \Rightarrow \Theta'_x \leqslant \Theta'_y$. λ is regular, so Card$(Y) = \lambda$ and we are done.

Now each $\Theta'_x \in \mathcal{H}(Q)$ by definition, so $\sum_{x \geqslant x_0} \Theta'_x \in \mathcal{H}(Q)$. Note that $I = \{x \in L : x < x_0\}$ is an initial segment of λ, so $I < \lambda$. By minimality of λ, $\sum_{x < x_0} \Theta'_x$ is a finite sum of members of $\mathcal{H}(Q)$. hence $\sum_{x \in L} \Theta'_x$ is a finite sum of types in $\mathcal{H}(Q)$, contradicting the definition of λ and hence giving case (1).

(2) bs(Φ) is an α_0^*-sum $(\alpha_0 < \alpha)$

This case is similar to (1).

(3) bs(Φ) is an $\eta_{\alpha_0\beta_0}$-sum ($\langle\alpha_0,\beta_0\rangle < \langle\alpha,\beta\rangle$)

By the induction hypothesis on $\mathcal{D}_{\alpha\beta}$, Φ is an $\eta_{\alpha_0\beta_0}$-sum of finite sums of members of $\mathcal{H}(Q)$. So Φ is a ϕ-sum of types in $\mathcal{H}(Q)$, where $\phi \equiv \eta_{\alpha_0\beta_0}$ because $\omega\cdot\eta_{\alpha_0\beta_0} \leqslant (\eta_{\alpha_0\beta_0})^2 \equiv \eta_{\alpha_0\beta_0}$. Hence Φ can be written as $\sum_{x\in F}\Phi_x$, where $\mathrm{tp}(F) = \phi$ and $\Phi_x \in \mathcal{H}(Q)$. Define $\chi = \sum_{x\in F}\mathbf{1}_{\Phi_x} \in \mathcal{H}(Q)^{\equiv\eta_{\alpha_0\beta_0}}$ and note that $\overline{\chi} = \sum_{x\in F}\Phi_x = \Phi$.

$\mathcal{H}(Q)$ is BQO by Corollary 5.11 and we are assuming the theorem for all admissible $\langle\alpha_0,\beta_0\rangle < \langle\alpha,\beta\rangle$ and arbitrary BQOs. Replacing Q by $\mathcal{H}(Q)$ in the statement of the theorem and recalling that

$$\chi \in \mathcal{H}(Q)^{\equiv\eta_{\alpha_0\beta_0}} \subseteq \mathcal{H}(Q)^{\leqslant\eta_{\alpha_0\beta_0}} \tag{3}$$

we get that χ is a finite sum of members of $\mathcal{H}(\mathcal{H}(Q))$, i.e. $\chi = \sum_{i<n}\chi_i$ for some $n < \omega$ and $\chi_i \in \mathcal{H}(\mathcal{H}(Q))$. By Lemma 5.13, $\Phi = \overline{\chi} = \sum_{i<n}\overline{\chi_i}$ is a finite sum of members of $\mathcal{H}(Q)$, as required.

These three cases complete the theorem in the case that $\Phi \in Q^{<\eta_{\alpha\beta}}$.

Now consider the general case where $\Phi \in Q^{\leqslant\eta_{\alpha\beta}}$. By Theorem 5.12, Φ is a ϕ-sum of types $\mathbf{1}_q$ ($q \in Q$) and (R,γ,δ)-universal types ($R \subseteq Q$, $\langle\gamma,\delta\rangle \leqslant \langle\alpha,\beta\rangle$), and $\phi \in \mathcal{D}_{\alpha\beta}$. I will show that (R,γ,δ)-universal types are in $\mathcal{H}(Q)$. Hence Φ is a ϕ-sum of members of $\mathcal{H}(Q)$.

Let χ be an (R,γ,δ)-universal type ($R \subseteq Q$, $\langle\gamma,\delta\rangle \leqslant \langle\alpha,\beta\rangle$). I will show that $\chi \in \mathcal{H}_1(Q) \subseteq \mathcal{H}(Q)$ by proving that χ is an $(\mathcal{R},\gamma,\delta)$-shuffle for some $\mathcal{R} \subseteq \mathcal{H}_0(Q)$. In fact, $\mathcal{R} = R^{\mathbf{1}} = \{\mathbf{1}_r : r \in R\}$ will do.

χ is (R,γ,δ)-universal: $\chi \in R^{\equiv\eta_{\gamma\delta}}$ and $\Psi \in R^{\leqslant\eta_{\gamma\delta}} \Rightarrow \Psi \leqslant \chi$. So we can write $\chi = \sum_{x\in E}\mathbf{1}_{l(x)}$ where $\mathrm{tp}(E) \equiv \eta_{\gamma\delta}$ and $l : E \to R$ is the labelling. Define a new labelling $l' : E \to \mathcal{R} : x \mapsto \mathbf{1}_{l(x)}$ and let $\chi' = \sum_{x\in E}\mathbf{1}_{l'(x)}$.

Since $\mathbf{1}_a \leqslant \mathbf{1}_b \Leftrightarrow a \leqslant b$ we have $\Psi \in \mathcal{R}^{\leqslant\eta_{\gamma\delta}} \Rightarrow \Psi \leqslant \chi'$ by the corresponding property of χ. Hence χ' is $(\mathcal{R},\gamma,\delta)$-universal. Note that $\mathcal{R} = R^{\mathbf{1}} \subseteq Q^{\mathcal{M}}$ and $\overline{\chi'} = \sum_{x\in E} l'(x) = \sum_{x\in E}\mathbf{1}_{l(x)} = \chi$, so χ is indeed an $(\mathcal{R},\gamma,\delta)$-shuffle.

As in case (3) above, there is some $\chi \in \mathcal{H}(Q)^{\phi}$ such that $\overline{\chi} = \Phi$. As $\mathcal{H}(Q)$ is BQO and $\mathrm{bs}(\chi) = \phi \in \mathcal{D}_{\alpha\beta}$, the first part of this proof shows that χ is a finite sum of members of $\mathcal{H}(\mathcal{H}(Q))$. As in case (3), Lemma 5.13 gives that $\Phi = \overline{\chi}$ is a finite sum of types in $\mathcal{H}(Q)$, as we wanted.

□

Remark. The following theorem uses Theorem 5.14, and for this use it would suffice to prove Theorem 5.14 only for the case $\Phi \in Q^{<\eta_{\alpha\beta}}$. However, the last part of the proof is still necessary to make the induction step at equation (3) go through.

Theorem 5.15. *Q BQO $\Rightarrow Q^{\mathcal{M}}$ BQO.*

Proof. If $\Phi \in Q^{\mathcal{M}}$ then Corollary 4.6 $\Rightarrow \mathrm{bs}(\Phi) \leqslant \eta_{\alpha\beta}$ for some α, β. Define a homomorphism $f : (\mathcal{H}(Q))^{<\omega} \to Q^{\mathcal{M}} : \langle \Phi_i \rangle_{i<n} \mapsto \sum_{i<n} \Phi_i$. Theorem 5.14 $\Rightarrow f$ is surjective. Now Corollary 5.11 $\Rightarrow \mathcal{H}(Q)$ BQO, so Theorem 2.4(v) $\Rightarrow (\mathcal{H}(Q))^{<\omega}$ BQO. Hence Lemma 2.7 (the homomorphism property) $\Rightarrow Q^{\mathcal{M}}$ BQO.

□

By letting Q be a singleton BQO we get the main theorem:

Corollary 5.16. *$\mathcal{M}$ is BQO.*

An immediate consequence of Theorem 5.15 is the main result of Nash-Williams' paper [9], which states that if Q is BQO, then so is the class of transfinite Q-sequences $\bigcup_{\alpha \in On} Q^{\alpha}$:

$$Q \text{ BQO } \Rightarrow Q^{\mathcal{M}} \text{ BQO } \Rightarrow Q^{On} = \bigcup_{\alpha \in On} Q^{\alpha} \text{ BQO (since } On \subseteq \mathcal{M}).$$

The fact that On is an important subclass of $\mathcal{M}$ gives rise to questions concerning the generalisation of properties of the ordinals to members of $\mathcal{M}$. In [7], Richard Laver investigates the generalisation of combinatorial results about the ordinals and gives some sample applications to partition relations and decompositions.

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