Equality of Church Numerals in the Lambda Calculus

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First we define lambda calculus notions of **true** and **false** as follows:

$$T \stackrel{\text{def}}{=} \lambda x.\lambda y.x$$
$$F \stackrel{\text{def}}{=} \lambda x.\lambda y.y$$

The Church numerals are defined for $n \in \mathbb{N}$ by the following induction:

$$C_n \stackrel{\text{def}}{=} \begin{cases} \lambda f.\lambda x.\ x & n = 0 \\ \lambda f.\lambda x.\ f\ (\ [C_{n-1}\ f]\ x\ ) & n > 0 \end{cases}$$

I provide a function which takes two Church numerals and returns T if they represent the same number, and F if they do not. The method is to instantiate the numerals with different fs (A and B below), and set the x of one to be the lambda term consisting of a string of applications of B, terminating in X (the x of the other numeral, also defined below). We choose A and B such that they cancel each other in the middle of the term, leaving either a string of applications of A terminating in X, or a string of applications of B terminating in X. By using a second argument to B we allow it to behave one way when called by A, and another when testing for X, and in the latter case it can return F and exit. X can also take advantage of this argument to return T when tested, but a special term when called by A that causes the stack of As to collapse, and return F when tested. Thus the test function Z can distinguish between a stack of applications of A terminating in an X, an X alone, and a stack of applications of B terminating in an X. We use a fixed point helper L to aid the collapse of the stack of applications of A.

$$A \stackrel{\text{def}}{=} \lambda x.x\ T$$
$$B \stackrel{\text{def}}{=} \lambda x.\lambda y.\ (\ [y\ x]\ F\ )$$
$$L \stackrel{\text{def}}{=} \lambda l.\lambda x.\ (\ [x\ (l\ l)]\ F\ )$$
$$X \stackrel{\text{def}}{=} \lambda x.\ (\ [x\ (L\ L)]\ T)$$
$$Z \stackrel{\text{def}}{=} \lambda x.\ (\ [x\ L]\ F\ )$$

We can now define the test for equality of Church numerals as follows:

$$\lambda a.\lambda b.\ Z\ (\ [a\ A]\ [(b\ B)\ X]\ )$$