PROOF BY COMPLETE INDUCTION

Exercise 3.4 in the notes on the *Foundations of Computer Science* course (2000 Edition) by Larry Paulson presents a recurrence which may be expressed as:

$$
\begin{aligned}
&T(1) = 1\\
&T(2) = 1\\
&T(3) = 1\\
&T(n) = T(\lceil n/4 \rceil) + T(\lfloor 3n/4 \rfloor) + n \qquad \text{for } n \geqslant 4
\end{aligned} \tag{1}
$$

The problem is to prove that the recurrence is $O(n \log n)$.

Preliminary Analysis

Given the recurrence as expressed above one can set up the following table:

n	$\lceil n/4 \rceil$	$\lfloor 3n/4 \rfloor$	$T(n)$	$n \log_2 n$
1	1	0	1	0.0
2	1	1	1	2.0
3	1	2	1	4.8
4	1	3	6	8.0
5	2	3	7	11.6
6	2	4	13	15.5
7	2	5	15	19.7
8	2	6	22	24.0
9	3	6	23	28.5
10	3	7	26	33.2
11	3	8	34	38.1
12	3	9	36	43.0
13	4	9	42	48.1
14	4	10	46	53.3
15	4	11	55	58.6
16	4	12	58	64.0

From the table, it seems that for values of $n > 1$ it is reasonable to conjecture that $T(n)$ is $O(n \log n)$ and this conjecture will be proved by the method of complete induction.

Lemmas

In the proof that follows, three lemmas are assumed:

$$
\begin{array}{rll}
\text{I} & \lceil k/4 \rceil < 3k/4 & \text{for integer } k \geqslant 2\\
\text{II} & \lfloor 3k/4 \rfloor \leqslant 3k/4 & \text{for integer } k \geqslant 0\\
\text{III} & \lceil k/4 \rceil + \lfloor 3k/4 \rfloor = k & \text{for integer } k \geqslant 0
\end{array}
$$

The proof of these lemmas is left as an exercise for the reader.

Preliminary Observation

Inspection of the table suggests too little leeway between $T(n)$ and $n\log n$ for comfort. Indeed, when $n = 64$ it is easy to show that $T(n) = 389$ and $n\log n = 384$. Accordingly, an attempt to prove that $T(n) = O(n\log n)$ by demonstrating that $T(n) \leqslant n\log n$ for $n \geqslant 2$ will fail. Fortunately it *can* be shown that $T(n) \leqslant 4\,n\log n$ for $n \geqslant 2\ldots$.

Proof by the Method of Complete Induction

The proposition is that $T(n) \leqslant 4\,n\log n$ for $n \geqslant 2$.

Take as the induction hypothesis that, for any $k > 2$, one may assume that for all i such that $2 \leqslant i < k$ that $T(i) \leqslant 4\,i\log i$.

Now consider $T(k)$ itself. If $1 < k < 5$ it can be seen by inspection of the table that $T(k) \leqslant 4\,k\log k$ holds. Now consider $T(k)$ for $k \geqslant 5$:

$$
\begin{aligned}
T(k) &= T(\lceil k/4\rceil) + T(\lfloor 3k/4\rfloor) + k && \text{from the recurrence (1)}\\
&\leqslant 4\lceil k/4\rceil\log\lceil k/4\rceil + 4\lfloor 3k/4\rfloor\log\lfloor 3k/4\rfloor + k && \text{by hypothesis but see note below}\\
&\leqslant 4\lceil k/4\rceil\log(3k/4) + 4\lfloor 3k/4\rfloor\log(3k/4) + k && \text{by Lemmas I and II}\\
&= 4\,k\log(3k/4) + k && \text{by Lemma III}\\
&= 4\,k\big(\log(3k/4) + \tfrac{1}{4}\big)\\
&= 4\,k\big(\log(3k/4) + \tfrac{1}{4}\log 2\big)\\
&= 4\,k\big(\log(3k/4) + \log 2^{\frac{1}{4}}\big)\\
&= 4\,k\big(\log(2^{\frac{1}{4}}.3k/4)\big)\\
&\leqslant 4\,k\log k && \text{given that } 2^{\frac{1}{4}}.3/4 < 1
\end{aligned}
$$

Accordingly, $T(k) \leqslant 4\,k\log k$ and since this is the induction hypothesis with i replaced by k the proof is complete.

Note on the use of the Induction Hypothesis

For a given k, the range of values of i for which the induction hypothesis applies is $2 \leqslant i < k$. It is assumed in the proof that, by the induction hypothesis, $T(\lceil k/4\rceil) \leqslant 4\lceil k/4\rceil\log\lceil k/4\rceil$ and $T(\lfloor 3k/4\rfloor) \leqslant 4\lfloor 3k/4\rfloor\log\lfloor 3k/4\rfloor$. For these assumptions to be valid, the induction hypothesis requires $2 \leqslant \lceil k/4\rceil < k$ and $2 \leqslant \lfloor 3k/4\rfloor < k$.

Demanding $k \geqslant 5$, ensures that $\lceil k/4\rceil \geqslant 2$ and that $\lfloor 3k/4\rfloor \geqslant 2$. Moreover this demand ensures that $\lceil k/4\rceil < k$ and that $\lfloor 3k/4\rfloor < k$. Accordingly, meeting this demand ensures the validity of the induction step in the above proof.

Note that if $k < 5$ the condition $\lceil k/4\rceil \geqslant 2$ fails (see table). Accordingly, the main part of the proof requires preliminary confirmation that the proposition holds for $2 \leqslant n < 5$. This is achieved by inspection of the table.

F.H. King
October 2009