Fränkel-Mostowski Models: Notes for a Reading Group

Thomas Forster

October 8, 2012

1 Let's get started

'ZFU' is my name for ZF with extensionality weakened to allow *urelemente*. Hence the 'U'. It's sometimes also called 'ZFA', but not by me. 'ZFA' is sometimes used to denote ZF with the Forti-Honsell Antifoundation axiom (usually misattributed to Peter Aczel but never mind); 'ZFU' is preferable here co's it has no other use. (The name 'FM' set theory is sometimes seen in this setting. However, since it suggests both 'base theory, point of departure for FM models' and 'theory of a [particular] FM model'—and I have seen both uses—its employment can only cause confusion and should be eschewed. There are, after all, many—wildly different—FM models!)

Now, what are these *urelemente*? We can use either Quine atoms ($x = \{x\}$) or empty atoms; it makes no difference to the result, and they are equally easy to add. (i) Both flavours of atoms can be got rid of, by the simple device of considering the substructure of the model-in-hand that consists of elements that do not have any atom in their transitive closure. And (ii) a model with atoms of one flavour can be straightforwardly transformed to a model with atoms of the other flavour by Rieger-Bernays permutation models[1]. However the Quine atoms treatment is perhaps *ever-so-slightly* preferable because it admits additional Rieger-Bernays constructions which can remove Quine atoms and thereby give us models that lack Quine atoms but otherwise resemble the input model very closely. I don't think we are going to need any of those constructions here, but their existence and availability amount to a small plus for the Quine-atoms way of doing things and thereby explain my preference for Quine atoms.

In what follows i am going to be a bit naughty in using 'ZFU' to denote the version of ZF with Quine atoms instead of *urelemente* (which is usually what is meant when people talk about ZF-with-atoms) but i don't think anyone is going to be seriously misled. I shall follow Felgner [2] in using the expression

[1]The difference between Quine atoms and empty atoms matters only if we have an axiom of complementation, for then all Quine atoms have distinct complements whereas all empty atoms have the same complement. And of course ZFU has no axiom of complementation. I owe thanks to Randall Holmes for pointing out to me that one can't just add ordered pairs to (or delete them from) the membership relation of the model

'FMS' ('S' for 'Specker') to refer to FM-methods-with-Quine-atoms—at least in settings where it is clear that what we are doing relies on the constructions being FMS not merely FM. I shall *not* be following Felgner in calling Quine atoms 'reflexive sets': that word 'reflexive' has far too much work to do already. Forti-Honsell call them 'autosingletons' which is self-explanatory, but 'Quine atom' is shorter and is in general use.

1.1 Definitions

We assume the reader is familiar with Rieger-Bernays methods. They almost certainly know at least one proof using R-B (even if they don't know that it is a R-B construction!) namely the proof of independence of foundation from the other axioms of ZF. You know, the proof that considers the transposition $(\emptyset, \{\emptyset\})$. The fact that the model that results from that permutation satifies the other axioms of ZF does not rely on any features of that permutation. It works whatever permutation is used. That particular construction adds a Quine atom. But by judicious choice of permutations one can control in some detail what appears in the resulting model and obtain a wider variety of outcomes. There is a detailed theory of R-B constructions but we will not need very much of it.

Definition 1

1. *$Symm(X)$ is the full symmetric group on X.*
2. *$Stab_G(x)$ is the set of those elements of G that stabilise x. We assume that the action of G is understood. If G is also understood we will omit the subscript.*
3. *We write '(x, y)' for the permutation that transposes x and y and fixes everything else. The only other use for '(' and')' is punctuation.*
4. *A $\mathcal{P}$-**embedding** i from $\mathfrak{A}$ into $\mathfrak{B}$ is an injection $i : \mathfrak{A} \to \mathfrak{B}$ for which the power set operation is absolute. "No new members or subsets of old sets." If i is the identity we say $\mathfrak{B}$ is a $\mathcal{P}$-**extension** of $\mathfrak{A}$.*

2 A Basis Lemma

If we use Quine atoms in our FM construction we have violated foundation, so we cannot use $\in$-induction. However, if the collection of Quine atoms satisfies a simple condition, we can recover $\in$-induction.

Definition 2
Let $\langle X, R\rangle$ be a binary structure.
*An R-**bottomless set** is a subset $Y \subseteq X$ s.t. $(\forall x \in Y)(\exists y \in Y)(yRx)$.*
*We say $B \subseteq X$ is a **basis** if it meets every R-bottomless set.*

If R is clear from context we shall merely say "bottomless". (Of course for us R is going to be $\in$.)

We can now prove a version of the $\in$-recursion theorem.

THEOREM 3 *Let $\langle X, R\rangle$ be a binary structure and*

g an arbitrary (total)[2] *function $X \times V \to V$;*
f a (total) function $B \to V$;
$B \subseteq X$ a basis for the R-bottomless sets of $\langle X, R\rangle$ (subsets of X),

Then

There is a unique total function $f^* : X \to V$ satisfying

(i) $f^* \restriction B = f$;
(ii) $(\forall x \in (X \setminus B))(f(x) = g(x, f``\{y : R(y, x)\}))$.

Proof: The idea is very simple. We obtain our best candidate for f^* by closing the graph of f under the operation that adds ordered pairs according to the clause that says that $f(x) =: g(x, f``\{y : R(y, x)\})$.

Now suppose (ii) fails, so the subset Y of X on which f^* is not uniquely defined is nonempty. This set Y is R-bottomless because $f(x) =: g(x, f``\{y : R(y, x)\})$. So it meets B, since B is a basis. But then f^* is defined on at least some of Y.

■

This modification of the recursion theorem looks trivial, but there are cases where it is useful, such as the one before us, where we add Quine atoms to a model of ZFC in a well controlled way so that every $\in$-bottomless class contains a Quine atom. But then any function defined on the Quine atoms can be extended to the whole universe.

[What with this version of the recursion theorem being essential for the Quine-atom route to FM models—as in [20]—i would imagine that it is to be found there, but i confess i have never read it! I actually worked this out for myself!]

Another application (which is so cute that i can't bear to omit it even tho' it's no use to us here) is the factoid that the late lamented Jon Barwise used to call "The Solution Lemma". (See [1] p ???). It is a consequence of the Forti-Honsell antifoundation axiom that every system of equations in the style

$$x_1 = \{\emptyset, x_2, x_3\};\ x_2 = \{\{\emptyset\}, x_3\};\ x_3 = \{\emptyset, x_1\}$$

has a unique solution.

Let's show how to add such bad sets to a model of ZF by Rieger-Bernays permutations. Start with three sets x_1, x_2 and x_3—it won't much matter what they are, but

[2] Here V is the universe, so that when we say "$g : X \times V \to V$" we mean only that we are not putting any constraints on what the values of g (or its second inputs) are to be.

let's take them to be von Neumann reals, or something large and remote like that—and consider the product of the three transpositions: $(x_1, \{\emptyset, x_2, x_3\})$, $(x_2, \{\{\emptyset\}, x_3\})$ and $(x_3, \{\emptyset, x_1\})$. In the resulting Rieger-Bernays permutation model x_1, x_2, x_3 form a solution to the system of equations.

What is the feature of interest here? There are these things we're inventing, namely the xs, and we are declaring them in terms of each other and some *wellfounded* sets. In this new model the bad sets x_1, x_2, x_3 form a basis. This basis B has the nice property that $TC(B) \setminus B$ is wellfounded: the bad sets are declared in terms of each other and wellfounded sets only.

A final thought about bases before we set to work. If the Quine atoms in a model $\mathfrak{M}$ of ZFU form a basis for the illfounded sets then there is a permutation model $\mathfrak{M}^\pi$ of $\mathfrak{M}$ from which the Quine atoms have vanished—and which of course satisfies the same stratified sentences as $\mathfrak{M}$.

3 Construction of a Simple FMS model

Whenever we have a model of ZFU we can consider a submodel (in fact an "inner model" in the jargon of Set Theorists) consisting of those elements that are fixed under a certain action of a group of permutations of the atoms of that model. There are several degrees of freedom here, so it is a good idea to start with a simple illustration.

We will exhibit the simplest possible FMS construction (which i think is actually the original construction of Fraenkel [3]) in which the axiom of choice for countable sets of pairs fails. This is just to get us started: more complicated constructions will follow.

We start with a model $\mathfrak{M}$ of ZF(C) + foundation, and use Rieger-Bernays methods to obtain a permutation model with a countable set A of Quine atoms. The permutation σ we use to achieve this is

$$\sigma = \prod_{n \in \omega} (n, \{n\})$$

the product of all transpositions $(n, \{n\})$ for $n \in \omega$. (Here ω is the von Neumann ω—with $\emptyset$ removed just to be on the safe side.) In $\mathfrak{M}^\sigma$ the old ω has become a set of atoms, which we will call A.

Lemma 4 *A is a basis for the illfounded sets of $\mathfrak{M}^\sigma$.*

Proof: Suppose X is a bottomless set in the sense of $\mathfrak{M}^\sigma$. That is to say

$$(\forall x \in X)(\exists y \in X)(y \in x)^\sigma.$$

this is

$$(\forall x \in \sigma(X))(\exists y \in \sigma(X))(y \in \sigma(x)).$$

Reletter $\sigma(X)$ as X to obtain:

$$(\forall x \in X)(\exists y \in X)(y \in \sigma(x)) \qquad \text{(bot)}$$

If everything in X is fixed by σ this would become $(\forall x \in X)(\exists y \in X)(y \in x)$ which would mean that X was bottomless in $\mathfrak{M}$. But $\mathfrak{M} \models$ foundation, so there can be no such x. So at least one thing in X is moved. So X contains either a natural number n (in which case X contains an atom, since natural numbers in $\mathfrak{M}$ become atoms in $\mathfrak{M}^\sigma$), or contains a singleton $\{n\}$ for some natural number n. Sticking this singleton in for 'x' in expression (bot) above gives

$$(\exists y \in X)(y \in n)$$

Now we exploit that fact that our natural numbers are von Neumann naturals, so that the witness y which lives inside x is a natural number, so X contains a natural number after all. ■

[We probably didn't need the countable set on which σ acts to be any particular set, but making it the von Neumann ω make life easier. Also i think that there is a $\mathcal{P}$-embedding from $\mathfrak{M}$ into $\mathfrak{M}^\sigma$. Not that it matters.]

We are now going to forget that our model with the countably many atoms arose as a permutation model with a permutation σ, so that we can recycle the letter 'σ'! The cardinality of the set of atoms won't matter much either. All that will matter is that the atoms form a basis for the illfounded sets, so that we can use the modified version of $\in$-recursion from theorem 3

Definition 5 *Any permutation σ of A lifts to an $\in$-automorphism of the universe (and this automorphism, too, will be written σ) by means of the recursion $\sigma(x) =: \sigma``x$.*

This is of course justified by theorem 3, the modified recursion theorem.

[Of course one can instead think of this—as Jamie Gabbay does—as the group Symm(A) acting on the universe as a group of $\in$-automorphisms. Quite which of these two approaches is more convenient is a debatable point. Perhaps the reading group will come down heavily on one side or the other.]

Definition 6 *A set s of atoms* **supports** *a set x of $\mathfrak{M}^\sigma$ if every element of Symm($A \setminus s$) fixes x. If x is supported by a finite set of atoms we say x* **is of finite support**. *We will say that s is <u>the</u> support of x if s is $\subseteq$-minimal among the sets that support x.*

If x is of finite support then its support is in fact defined, as we will now show. We will show that if x is supported by s_1 and by s_2 both finite, then it is supported by $s_1 \cap s_2$, from which it will follow that if x is a set of finite support then there is a unique $\subseteq$-minimal set s that supports x.

The situation we consider is quite elementary and general, and doesn't necessarily have anything to do with FMS models at all.

Lemma 7 *Let A be an infinite set, and $a_1 \ldots a_k$ and $b_1 \ldots b_j$ two disjoint finite subsets of it. Then any permutation τ of A can be expressed as the composition of three permutations of A, the first one fixing all the as, the second one fixing all the bs and the third one fixing all the as (we can do it the other way round—"bab"—if you prefer).*

Proof:

The argument to τ might be an a, or a b or an ordinary ("O"), and so might the values. This gives nine cases

τ sends an a to an a	Fix it; move it to the a; fix it.
τ sends an a to an b	Fix it; move it to an O; move the O to the b.
τ sends an a to an O	Fix it; move it to the O; fix it.
τ sends a b to an a	Move it to an O; move the O to the a; fix it.
τ sends a b to a b	Move it to an O; fix it; move the O to the b.
τ sends a b to an O	Move it to the O; fix it; fix it.
τ sends an O to an a	Fix it; move it to the a; fix it.
τ sends an O to a b	Move it to the b; fix it; fix it.
τ sends an O to an O	Move it to the O; fix it; fix it.

The second, fourth and fifth lines compel us to use Os as stepping stones. But there are only finitely many stepping stones needed. The ninth line is a correct way to procede in sending an O to an O in all but finitely many cases. ■

So we have proved that if x has two (finite) supports s_1 and s_2 then it is supported by $s_1 \cap s_2$. Any permutation π that fixes everything in $s_1 \cap s_2$ is a product—as above—of three permutations that fix x; two of them permutations that fix everything in s_1 (and therefore fix x) and a third permutation that fixes everything in s_2 (and therefore fixes x). So π fixes x. So every permutation fixing everything in $s_1 \cap s_2$ fixes x.

Let us extend the use of 'stab(x)' to denote also that subgroup of Symm(A) consisting of those permutations of A [that induce $\in$-automorphisms] that fix x.

Suppose x has support $A' \subset A$. That is to say: every permutation of $A \setminus A'$ fixes x. Can there be permutations fixing x that also move some things in A'? Suppose stab(x) contained a permutation σ that swapped b and c, both in A'. Now we can compose σ with anything in Symm($A \setminus A'$) to obtain something that fixes x, and $\sigma^{-1} \restriction (A \setminus A')$ is a permutation that fixes everything in A'. So $\sigma \cdot (\sigma^{-1} \restriction (A \setminus A'))$ fixes x, and we can compose it with *any* element of Symm($A \setminus A'$) to obtain something that fixes x. But that means that any permutation of A that swaps b and c but fixes everything else in A' will fix x. So stab(x) is the product of symm($A \setminus A'$) with a group G of permutations of A'. Is G is a product of symmetric groups?

Every set x gives rise to an equivalence relation on atoms. Say $a \sim_x b$ if (a, b) fixes x. I think it will turn out that x is of finite support iff $\sim_x$ has a cofinite equivalence class. (If it has a cofinite equivalence class it can have only one, and all the others will be finite). Clearly whenever a and b belong to the same equivalence class then the transposition (a, b) belongs to stab(x).

Randall supplies the example of a set $x = \{\{a, b\}, \{c, d\}\}$. $\sim_x$ has five equivalence classes: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ and $A \setminus \{a, b, c, d\}$ but stab(x) is bigger than the product of the symmetric group on the $\sim_x$-equivalence classes, co's it contains the permutation $(a,b)(c,d)$. So G might not be a product of symmetric groups ... it might contain more.

It would be nice if the class of sets of finite support gave us a model of something sensible, but extensionality fails: if X is of finite support then $\mathcal{P}(X)$ and the set $\{Y \subseteq X : Y \text{ is of finite support}\}$ are both of finite support and have the same members with finite support. We have to consider the class of elements hereditarily of finite support. Let's call it HF. This time we *do* get a model of ZF.

DEFINITION 8 *HF is the inner model of sets hereditarily of finite support*

LEMMA 9 *The class of sets of finite support is closed under all the definable operations that the universe is closed under.*

Proof:
The idea is so simple that I shall wave my arms over it. Suppose $\phi(x, \vec{y})$ is a formula that says that $x = f(\vec{y})$. If σ is a permutation [inducing an automorphism] that fixes all the $\vec{y}$, then it must also fix x—because it is an automorphism. Thus if $x = f(\vec{y})$ we must have $\mathrm{supp}(x) \subseteq (\mathrm{supp}(y_1) \cup \ldots \mathrm{supp}(y_n))$.
■

COROLLARY 10 *HF is a model of all the axioms of ZF except choice: AC for countable sets of pairs fails.*

Proof: Lemma 9 takes care of the axioms of empty set, pairing, sumset and power set. To verify the axiom scheme of replacement we have to check that the image of a set hereditarily of finite support in a definable function (with parameters among the sets hereditarily of finite support and all its internal variables restricted to sets hereditarily of finite support) is hereditarily of finite support too. The operation of translating a set under a definable function (with parameters among the sets hereditarily of finite support and all its internal variables restricted to sets hereditarily of finite support) is definable and will (by lemma 9) take sets of finite support to sets of finite support.

So if X is in HF and f a definable operation as above, $f``X$ is of finite support. And since we are interpreting this in $HF$, all members of $f``X$ are in HF, so $f``X$ is in HF too, as desired.

To verify the axiom of infinity we reason as follows. Every wellfounded set x is fixed under all automorphisms, and is therefore of finite support. Since all members of any wellfounded set are wellfounded they will all be of finite support as well, so every wellfounded set is hereditarily of finite support. So HF will contain all wellfounded sets that were present in the model we started with. In particular it will contain the von Neumann ω.

It remains only to show that AC_2 fails in HF. Consider the set of (unordered) pairs of atoms. This set is in HF. (It has empty support). However no selection function for it can be in HF. Suppose f is a selection function and $\{a, b\}$ an arbitrary pair of atoms. f must pick one element from $\{a, b\}$, so $\text{supp}(f)$ meets every pair of atoms and is therefore infinite.

■

Observe that in HF the collection A of atoms is a set, but—altho' it is manifestly countable seen from outside (and in the sense of the original model)—it is not countable in the sense of HF, and indeed it is not wellorderable (since there is no choice function on the set of its two-membered subsets). In fact (tho' we shall not prove this) we cannot even find a wellfounded set to which it is equinumerous! Thus HF also falsifies *Coret's axiom B*: "Every set is the same size as a wellfounded set".

4 Generalising the Construction

Now let's stand back and have a look at what features we have used.

Definition 11 *Definition of General FM Model*

An FM model arises from:
(i) A set U of atoms;
(ii) A group G of permutations of U;
(iii) A normal filter $\mathcal{F}$ over G.

A normal filter over a group G is a collection of subgroups of G closed under $\cap$, superset and conjugation.

We extend permutations in G to $\in$-automorphisms of our model of ZFU as above (definition 5). We then say that a **stable** *set is one whose stabiliser is in $\mathcal{F}$. The FM model reulting from this choice of U, G and $\mathcal{F}$ is the collection of hereditarily stable sets.*

In the model with which we started U is the (countable) set A of all atoms, G is the full symmetric group $\text{Symm}(A)$ and $\mathcal{F}$ is the filter generated by the set $\{\text{stab}_{\text{Symm}(A)}(X) : (X \subseteq A) \wedge |A \setminus X| < \aleph_0\}$ of setwise stabilisers in $\text{Symm}(A)$ of cofinite subsets of A.

Typically G will be the full symmetric group $\text{Symm}(U)$, but it doesn't have to be, and if it isn't we can arrange for lemma 7 to fail. [Thanks to Randall for this illustration.] Consider a model in which we start with countably many atoms, partitioned into pairs ("socks"). We consider only permutations that respect the pairs. Then consider the sets hereditarily of finite support as usual. Now let x be $\{\{a, b\}, \{c, d\}\}$, where $\{a, b\}$ and $\{c, d\}$ are two pairs of socks. Then the set $\{a, c\}$ supports x (beco's among the permutations-that-preserve-pairs, all those that fix both a and c also fix x) and so does the set $\{b, d\}$ (beco's among the permutations-that-preserve-pairs, all those that fix both b and d also fix x).

Note that for lemma 9 to hold we need $\mathcal{F}$ to be closed under finite intersections and superset.

Why do we need it closed under conjugation?

5 Embeddings between FM Models

Usually an FM model is a one-off construction, done to order to solve a particular independence question, and there is not normally any reason to consider relations between FM models, no need to consider more than one model at a time. However more complicated situations can arise in which we are juggling several FM models and we want to know about embedding relations between them. As is our habit, we will start with a simple illustration.

If we fix U and G, but rerun the construction with an $\mathcal{F}' \supset \mathcal{F}$ we obtain more sets, and the first FM model is a proper substructure of the second. (For example, $\mathcal{F}'$ could be the filter generated by $\{\mathrm{stab}_{\mathrm{Symm}(A)}(X) : X \in \mathcal{U}\}$ for $\mathcal{U}$ some nonprincipal ultrafilter on A) then we have in the resulting FM model all the sets we had in our original model plus some more.) What can one say about the inclusion embedding?

Or again, let A be a set of atoms, and $B \subset A$ with $|B| < |A|$, both infinite. We obtain two FM models from this, $\mathfrak{M}_B$ and $\mathfrak{M}_A$: $\mathfrak{M}_B$ is the collection of sets hereditarily of finite support ("HF") with support a subset of B, and $\mathfrak{M}_A$ similarly consists of those things hereditarily of finite support with support a subset of A. Observe that $\mathfrak{M}_B$ is not a substructure of $\mathfrak{M}_A$, co's there are sets of atoms in $\mathfrak{M}_B$ that are not in $\mathfrak{M}_A$ (for example the cofinite subsets of B).

It's pretty clear that there should be an injection $i : \mathfrak{M}_B \hookrightarrow \mathfrak{M}_A$. It sends atoms to themselves, and if $b \subset B$ is a set of atoms in $\mathfrak{M}_B$ we want $i(b)$ to be b if b is finite and $b \cup (A \setminus B)$ otherwise. So far so good. The challenge now is to show that i can be extended naturally to something defined on the whole of $\mathfrak{M}_B$, and that the result is an elementary embedding.

Observe that, for $x \in \mathfrak{M}_B$, $\mathrm{supp}(x) = \mathrm{supp}(i(x))$, at least when x is a set of atoms. Is this always going to be true? What is to stop us declaring $i(x) =: i``x$? As Randall says, consider the set $[B]^2$ of pairs of atoms from B. It's a set of $\mathfrak{M}_B$ but not of $\mathfrak{M}_A$. Is i a $\mathcal{P}$-embedding?

From here on things become a bit sketchy!

5.1 John Truss writes

I'm sorry if I ignored (or seemed to ignore) your message. Looking it up, there was just one, that's to say whether if you take an infinite subset of the set of atoms, and take the basic Fraenkel models (I think that's what they are, before Mostowski got involved) one's an elementary substructure of the other. Well, that must be true! But if you say it's tricky to prove, then I'd take your word for it. At any rate, they're isomorphic (which isn't what you asked). I looked at that sort of thing in the old APAL paper entitled 'The structure of amorphous sets', [21]. Basically, given two FM models which 'ought' to be the same, you force using finite maps between subsets of the sets of atoms (in the general case preserving some structure on them) and pass to the generic extension; then prove that the original models are suitable submodels of the generic extensions, or something like that...

[tf interjects: of course you can play an Ehrenfeucht-Fraïssé game to show that they are elementarily equivalent! mind you, that isn't enough for what we want.]

...Another related thing which should be true, but perhaps you're not interested in, is this: if M and N are Mostowski ordered models formed by starting with the sets of atoms as the rationals and the reals respectively, then M is an elementary substructure of N. I think I worked out that they are elementarily equivalent, but presumably the stronger statement should be true.

I'll see if there's anything I can send you. Some of it is from the pre-digital age, so might have to be scanned first.

All the best, John

5.2 Randall on Elementary Embeddings

Randall sez (and i'm going to edit this)

Notations are of two kinds. (x_m, n), where $m \leq n$, represents the function from n-term injective finite sequences of atoms which returns the mth term of the sequence: $(x_m, n)[(a_1, \ldots, a_n)]$ is defined as a_m.

(A, n), where A is a set of notations such that for every $(a, m) \in A$, $m \geq n$, represents a more complex function taking n-element sequences of atoms as input:

$(A, n)[(a_1, \ldots, a_n)][\mathbf{v}]$ is defined as $\{(\mathbf{a}[\mathbf{w}] : \mathbf{a} \in A \wedge \mathbf{v} \subseteq \mathbf{w}\}$.

All notations are defined in this way.

We prove by induction on the structure of notations that $\pi(n[\mathbf{v}] = n(\pi \circ \mathbf{v})$ and $n[\mathbf{v}]$ has the range of $\mathbf{v}$ as a support for any notation n. This is obvious for the atomic notations: $\pi((x_m, n)[(a_1, \ldots, a_n)]) = \pi(x_m) = (x_m, n)[\pi(a_1), \ldots, \pi(a_n)]$. a support of $(x_m, n)[(a_1, \ldots, a_n)] = a_m$ is $\{a_m\} \subseteq \{a_1, \ldots, a_n\}$, and of course a finite superset of a support is a support.

Now suppose for a set notation A that every element of A satisfies these conditions. $\pi((A, n)[(a_1, \ldots, a_n)][\mathbf{v}]) = \{\pi((\mathbf{a}[\mathbf{w}]) : \mathbf{a} \in A \wedge \mathbf{v} \subseteq \mathbf{w}\} = \{(\mathbf{a}[\pi \circ \mathbf{w}] : \mathbf{a} \in A \wedge \mathbf{v} \subseteq \mathbf{w}\} = \{(\mathbf{a}[\pi \circ \mathbf{w}] : \mathbf{a} \in A \wedge \pi \circ \mathbf{v} \subseteq \pi \circ \mathbf{w}\} = \pi((A, n)[(a_1, \ldots, a_n)][\pi \circ \mathbf{v}])$. Further, if π fixes each element of $\mathbf{v} = (a_1, \ldots, a_n)$, then for each element $a[a_1, \ldots, a_n, b_1, \ldots, b_m]$ of $(A, n)[(a_1, \ldots, a_n)][\mathbf{v}]$, its image under π is $a[\pi(a_1), \ldots, \pi(a_n), \pi(b_1), \ldots, \pi(b_m)] = a[a_1, \ldots, a_n, \pi(b_1), \ldots, \pi(b_m)]$ which also belongs to$(A, n)[(a_1, \ldots, a_n)][\mathbf{v}]$ and its inverse image under π also belongs to $(A, n)[(a_1, \ldots, a_n)][\mathbf{v}]$ for the same reason, so A is fixed under π as desired.

This means that every set thus denoted belongs to the FM model.

Now we show that every set in the FM model is thus denoted (lies in the range of one of the functions represented by the notations). We show this by induction on rank.

Clearly every atom is in the range of a notation function (specifically $(x_1, 1)$). The empty set is in the range of $(\emptyset, 0)$, Suppose that each set of rank $\beta < \alpha$ is in the range of some notation function. Let A be a set of rank α. Each $a \in A$ can be written in the form $n_a[\mathbf{w}]$ by inductive hypothesis. Let S be a support of A. By changing indices of component notations x_i, we can arrange for each $\mathbf{w}$ to have the same initial segment $\mathbf{v} = (a_1, \ldots, a_n)$ whose range is S. Now observe that for any notation n_a that we have used, since $n_a[\mathbf{w}] = n_a[(a_1, \ldots, a_n.b_1, \ldots, b_m) = a$ is in this set, the value of any notation $n_{a^*}[\mathbf{w}] = n_a[(a_1, \ldots, a_n.b_1^*, \ldots, b_m^*)$ is also included in the set, because it is the result of the action of a permutation fixing S on an element of the set A (it is important here that the argument lists do not contain repeated elements). Thus (after the indicated modifications) we see that $(\{n_a a \in A\}, |S|)(\mathbf{v}) = A$.

Now we show that a model of this kind with atom set $\mathbf{A}$ is elementarily equivalent to its submodel with atom set $\mathbf{B}$ an infinite subset of $\mathbf{A}$. The elementary embedding sends each element of the model with atoms $\mathbf{B}$ written in the form $A[\mathbf{v}]$ to the element of the model with atoms $\mathbf{A}$ represented by exactly the same notation.

We need the following fact. $x = y \leftrightarrow \pi(x) = \pi(y)$ and $x \in y$ iff $\pi(x) \in \pi(y)$ for the particular permutations π induced by actions on atoms that we work with. Thus any atomic sentence $M[(a_1, \ldots, a_n)]\, R\, N[(b_1, \ldots, b_n)]$ has the same truth value as $M[(\pi(a_1), \ldots, \pi(a_n))]\, R\, N[(\pi(b_1), \ldots, \pi(b_n))]$ (R being membership or equality). Here we have used what we showed above about actions of permutations on our notation functions. And this implies that the truth value of such a sentence in any model depends only on the identities of M and N and the truth values of sentences $a_i = b_j$.

For any x in the smaller model (with atoms B) we associate an x^* in the larger model (with atoms A) written using the same notation. For any x in the smaller model, let x^* be the object with the same notation in the larger model.

We need to establish that this is well-defined: for any notations m and n, and vectors taken from $\mathcal{B}$, $m[\mathbf{v}] = n[\mathbf{w}]$ in the smaller model iff $m[\mathbf{v}] = n[\mathbf{w}]$ in the larger model.

If one of the notations m and n is a projection map x_m, then the other must also be a projection map, as its value must also be an atom, and clearly $m[\mathbf{v}] = n[\mathbf{w}] = a$ in either model implies $m[\mathbf{v}] = n[\mathbf{w}] = a$ in the other model in this case.

Suppose that $m[\mathbf{v}] = n[\mathbf{w}]$ in the smaller model. Let x be an element of $m[\mathbf{v}]$. It must be of the form $m'[\mathbf{v}']$ where $m' \in \pi_1(m)$ and $\mathbf{v}'$ extends $\mathbf{v}$. It must also be of the form $n'[\mathbf{w}']$ where $n' \in \pi_1(n)$ and $\mathbf{w}'$ extends $\mathbf{w}$. Since we have $m'[\mathbf{v}'] = n'[\mathbf{w}']$ for the smaller model, we must also have $m'[\mathbf{v}'] = n'[\mathbf{w}']$ for the larger model (induction on complexity of notations). Now any element of the set denoted by $m[\mathbf{v}]$ in the larger model must be of the same form $m'[\mathbf{v}']$ described above. Change atoms in $\mathcal{A} \setminus \mathcal{B}$ which occur in $m'[\mathbf{v}']$ into atoms in $\mathcal{B}$ using a permutation σ of A which fixes the atoms in $\mathcal{B}$ which appear in $m'[\mathbf{v}']$. The resulting $m'[\mathbf{v}'']$ will be equal to a $n'[\mathbf{v}'']$ whose referent in the smaller model is equal to $m'[\mathbf{v}'']$. We can then apply σ^{-1} to send $m'[\mathbf{v}'']$ back to $m'[\mathbf{v}']$ and $n''[\mathbf{w}'']$ to $n'[\mathbf{w}']$, which is the same object (of course) but also clearly an element of $n[\mathbf{w}]$ in the larger model. The argument is symmetrical, so the two sets are the same.

Now suppose that $m[\mathbf{v}] = n[\mathbf{w}]$ in the larger model. An element of $m[\mathbf{v}]$ in the smaller model has notation $m'[\mathbf{v}']$. This refers to an element of the larger model as well, and must be equal there to an $n'[\mathbf{w}']$. Because the set $m'[\mathbf{v}']$ has support included in $\mathcal{B}$ in the larger model, we can apply a permutation to get a notation $n'[\mathbf{w}'']$ in wich no atoms not in $\mathcal{B}$ without changing its value. So we have established $m'[\mathbf{v}'] = n'[\mathbf{w}'']$ in the larger model and so also in the smaller model by induction on complexity. But clearly $n'[\mathbf{w}''] \in n[\mathbf{w}]$ in the smaller model, and by symmetry of the situation we see that the two sets are the same.

We also need $x \in y \leftrightarrow x^* \in y^*$. This follows fairly easily. The truth of $m(\mathbf{x}) \in n(\mathbf{y})$ in either model depends on the equality of $m(\mathbf{x})$ with some term $n'(\mathbf{y}')$ (and in the larger model considerations of support allow you to assume that additional atoms in $\mathbf{y}'$ are in $\mathcal{B}$); then this equality fact can be transferred from one model to the other by the previous discussion.

Now that we have handled atomic statements, we indicate how the inductive argument for statements with quantifiers will go.

Suppose that $(\exists y.P(x^*, y))$ is true in the larger model, where x is in the smaller model. Find y witnessing this. Apply a permutation to send all arguments in a notation representing y to elements of $\mathcal{B}$ (which will not affect the truth value of the statement, as shown above). Then $y = z^*$ for some z, so we have $P(x^*, z^*)$, so we have $P(x, z)$ in the smaller model, so we have $(\exists x.P(x, z))$ in the smaller model.

Suppose that $(\exists y.P(x, y))$ is true in the smaller model. Then $P(x, y)$ is true in the smaller model, $P(x^*, y^*)$ is true in the larger model, so $(\exists y.P(x^*, y))$ is true in the larger model.

An argument by induction on the structure of formulas here would also need to include an extension of the assertion that applying permutations to parameters in a statement will not affect its truth value. This isn't doubtful (we showed it for atomic statements above), just annoying.

G'day, it's me again. Every element x of $\mathfrak{M}_B$ is a union of $\mathrm{Symm}(B\setminus \mathrm{supp}(x))$-orbits. So, once we know what to send orbits to, we have a definition of i by $\in$-recursion. The obvious thing to do is to send a $\mathrm{Symm}(B\setminus \mathrm{supp}(x))$-orbit $\mathcal{O}$ to is the $\mathrm{Symm}(A\setminus \mathrm{supp}(i(x)))$-orbit of $i(x)$. Or do we mean the $\mathrm{Symm}(A\setminus \mathrm{supp}(x))$-orbit of $i(x)$? It turns out that it won't make any difference beco's—according to Randall—$\mathrm{supp}(i(x)) = \mathrm{supp}(x)$ anyway. (Which is what one would expect)

For a start, we'd have to check that it doesn't make any difference to the answer if we pick a different representative of $\mathcal{O}$.

But i have a worry of higher priority. Every element x of $\mathfrak{M}_B$ is a union of $\mathrm{Symm}(B\setminus \mathrm{supp}(x))$-orbits. But—more to the point—x is also a union of $\mathrm{Stab}(x)$-orbits. . . and it is presumably this group we need rather than $\mathrm{Symm}(B\setminus \mathrm{supp}(x))$.

5.3 A message from Jamie Gabbay

Dear Randall and Thomas,

I'm writing following conversations with both of you to describe some maths that has not yet been put together (and perhaps cannot be put together), but I see its shape in other papers. I'll be as brief and clear as I can, and I welcome input.

It turns out that FM sets have a capture-avoiding substitution action. That means that given elements x and u and an atom a you can form $x[a \to u]$ which is in some sensible sense "x with a substituted for u".

(This substitution is capture-avoiding; e.g. Atoms$[a \to u]$ =Atoms. So this is not the obvious replacement of a with u in x.)

Call such an abstract substitution action a *σ-action*. So the universe of FM sets supports a σ-action. This is documented in two papers (both technical): [8] and [9]

It also turns out that if X has σ-action then (after the fashion of a Stone Duality construction) also $\mathcal{P}(\mathcal{P}(X))$ has a corresponding σ-action. And further, $\mathcal{P}(\mathcal{P}(X))$ is a model of first-order logic. Also, $\mathcal{P}(X)$ has an inverted dual to the σ-action, which I all an *amgis-action*.

The sense in which I mean "model of first-order logic" is unorthodox. $\mathcal{P}(\mathcal{P}(X))$ is a Boolean algebra, but using *nominal* algebra we can define a generalisation of Boolean algebra that is to predicate logic, as propositional logic is to Boolean algebra, like so:

- Algebra, Boolean algebra, powerset
- Nominal algebra, FOLeq algebra, nominal powerset

To use some jargon, if X is a σ-algebra then $\mathcal{P}(\mathcal{P}(X))$ is a FOLeq algebra (first-order logic with equality).

This is documented in [10] and also in an unpublished draft under consideration for a journal (semocn), which I attach (please do not distribute it).

What semocn adds over stodnf is the observation that a FOLeq algebra allows an *absolute* denotation of first-order logic. By "absolute" I mean there is no valuation: given a predicate ϕ its sets denotation $[[\phi]]$ is a set, and we do not need a valuation mapping free variables of ϕ to elements. Intuitively, free variables of phi are interpreted as themselves, as FM atoms.

Now for some speculation:

- We can we extend this from first-order logic with equality to the language of set theory, i.e. we can add $\in$. So for the specific case that X has a σ-action and a notion of set inclusion (e.g. take X to be some "big" FM set), then $\mathcal{P}(\mathcal{P}(X))$ is itself a model (or nearly so, depending on the size of X) of FM set theory. This is like a Stone Duality result, but for set theory. Weird.
- The resemblance with stratification is unclear. The "$\mathcal{P}(\mathcal{P}(X))$" construction is stratification-flavoured, in that elements of X and $\mathcal{P}(\mathcal{P}(X))$ are "positively oriented" and have a σ-action, but elements of $\mathcal{P}(X)$ are negatively oriented, and have this *amgis-action. * This means that x and $\{x\}$ are different, because if $\{x\}$ is positive then x is negative; x might be the same but in one case it is required to have a σ-action and in the other an amgis-action. For me, this is reminiscent of a stratification condition.

- I am also reminded of Thomas's work on the semantic characterisation of stratification—you use permutations of the universe to detect how far "down" a predicate looks into a set. I wonder if the σ-action is doing a similar job: a predicate ϕ can't "see" too far into x when $\phi(x')$ if and only if $\phi(x)$ for every x', a, and u such that $x'[a \to u] = x$.

- I lied about $\mathcal{P}(\mathcal{P}(X))$ being a FOLeq algebra. It's actually a restriction to a subset of $\mathcal{P}(\mathcal{P}(X))$ that forms the FOLeq algebra; it's a pretty big subset, but the difference is "noticeable" (to see the precise conditions, see Definition 3.16 of the attached paper). So I had hoped to build a model of set theory by iterated restricted powersets—analogously to how we build a model of FM sets by iterated finitely-supported powersets—but that may not be possible. Intuitively, this reflects that it may be possible to take a set representing a predicate and remove a single element from it "by hand" to obtain a set that does not represent a predicate. However, perhaps for sets that are "large enough", this cannot be done, because our language of set theory can't identify individual elements any more. I just don't know, and I'd welcome ideas.

I'm not sure how this all fits together, and fits in with your work, but it seems to me that there is something going on here.

Let me sum this up as follows:

- FM sets has a σ-action. [he means: every model of ZFU has a σ-action]

- There is a notion of FOLeq algebra which generalises Boolean algebra to first-order logic with equality.

- This notion can be extended further to include a notion of $\in$, set membership, though I haven't written that up.

- That basically means that given any model of set theory X (or just a rather big set) we get another model out of (nearly) $\mathcal{P}(\mathcal{P}(X))$, though $x = y$ and $x \in y$ are not interpreted as literal equality and set membership any more, but as elements of $\mathcal{P}(\mathcal{P}(X))$. Is that interesting?

- The "double powerset" construction has a "σ-amgis-σ" structure that reminds me of stratification.

- I wonder if this can be usefully iterated, as the $\mathcal{P}$ construction can be iterated. Note: it's not difficult to build an X with both a sigma *and* an amgis-action.

- I wonder if the σ-action can be linked to Thomas's work on semantically characterising stratification.

- Very concretely, I wonder if it might be interesting to just take stodfo and extend it with $\in$ to obtain a topological analogue of set theory.

I welcome comments.

5.4 Correspondence Gabbay-Forster

tf writes

Good. Now i have to understand what this capture-avoiding substitution is. You say it isn't the obvious thing..

Let's start with an FM model. Specifically with the model in which every set has finite support. I'm in with a chance if you do that.

Jamie writes

We're thinking in terms of FM sets. So we want some definition of substitution on the sets universe $x[a \mapsto u]$.

Substitution should interact consistently with the FM notion of support. In particular this means that if $a\#x$

> ...which means that a is not in the support of x.? [interjects tf]

then $x[a \mapsto u] = x$.

It is a fact that $supp(\mathbb{A}) = \varnothing$. So $a\#\mathbb{A}$. Therefore, whatever we take $x[a \mapsto u]$ to be, it should satisfy $\mathbb{A}[a \mapsto u] = \mathbb{A}$.

The 'obvious' definition of substitution includes that for a set X, $X[a \mapsto u] = \{x'[a \mapsto u] : x' \in X\}$.

tf writes

Yep, it does, indeed, as you say. So far i'm with you.

Presumably, when x is an atom, $x[a \mapsto b]$ is x unless $x = a$ in which case it is b. So the problem is that the function $[a \mapsto b]$ doesn't just permute the atoms. Am i right?

Jamie writes

This would imply that $\mathbb{A}[a \mapsto b] = \mathbb{A}\setminus\{a\}$. We do not want this behaviour, so we cannot take the 'obvious' definition above.

$\mathbb{A}[a \mapsto b] = \mathbb{A}\setminus\{a\}$....?

tf writes

Does this mean the same as

$\mathbb{A}[a \mapsto b] = \{x[a \mapsto b] : x \in \mathbb{A}\}$.

Jamie writes

Dear Thomas,

OK, so the challenge is to define a capture-avoiding substitution on FM sets. The literature contains two approaches:

1. stusun “A study of substitution”. [8]
2. stodfo “Stone duality for first-order logic”. [10]

I’ll sketch both.

For $A \subseteq \mathbb{A}$ define $\mathit{fix}(A) = \{\pi : \forall a{\in}A.\pi(a) = a\}$ (as standard).

[So $\mathit{fix}(A)$ is the pointwise stabiliser of A]

The idea of stusun is this:

* It is a fact that every FM set X (not an atom) can be expressed as a union of orbits $x\circlearrowleft_A$ where $x \in X$ and $A \subseteq \mathit{supp}(X)$, and

[so this will be enuff, beco’s the operation $[a \mapsto b]$ commutes with set union in the sense that the result of substituting a for b in X is the union of the results of substituting a for b in all the PLANES (yipee!) included in X.]

Yes. Exactly!

No! I’m glad you brought this up. $[a \mapsto b]$ does *not* commute with set union [in a different sense].

Consider that $a[a \mapsto b] = b$ and (it is a fact that) $(\mathbb{A}\backslash\{a\})[a \mapsto b] = \mathbb{A}\backslash\{b\}$ and also $\mathbb{A}[a \mapsto b] = \mathbb{A}$ (since $a\#\mathbb{A}$). You can do the sets calculations yourself, writing $\mathbb{A}$ as $\{a\} \cup (\mathbb{A}\backslash\{a\})$.

The substitution of [8] (there are actually two in [8], but they both display this behaviour) does not commute with $\cup$ or $\cap$ in general.

Here is what I know:

* The second substitution action in stusun commutes with Δ (exclusive or), but again not with $\cup$ or $\cap$ in general.

* Both substitution actions from stusun commute with nominal abstraction $[a]x$; the notion of atoms-abstraction from my PhD thesis which in the notation I have given you previous is defined by $[a]x = (a, x)\circlearrowleft_{supp(x)\backslash\{a\}}$. See Definition 3.8 of fountl. * frenrs identifies a subclass of FM sets which I call *positive* sets, for which substitution is better-behaved. The class of positive sets is not closed under negation. * stodfo defines a substitution action that commutes with $\cap$, $\cup$, and complements $\backslash$. However, this is a completely different substitution action and is not defined orbit-wise. The substitution action of stodfo does not commute with nominal abstraction.

One of the puzzles of my current research is why there are these two (or three) substitution actions on FM sets, one pair of which commutes with ‘orbit’ or ‘datatype’ structure $[a]x$, and the other commutes with ‘logical’ structure $\cup$, $\cap$, and $\backslash$.

$x\circlearrowleft_A = \{\pi{\cdot}x : \pi \in \mathit{fix}(A)\}$. In words, $x\circlearrowleft_A$ is the set of $\pi{\cdot}x$ such that π fixes A pointwise.

[he means that $x\circlearrowleft_A$ is the orbit of x under the action of the pointwise stabiliser of A]

In stusun, I call the set of (x, A) pairs necessary to generate X, the * planes* of X.

[he means the planes are the orbits into which you decompose x..?]

(The definitions are 2.16 and 3.10 of [8]. However, I would suggest you might prefer the presentation—which I wrote some years later—in [9], definitions 3.1 and 3.3.)

$\mathbb{A}$ has one plane (up to α-conversion of planes): $a\circlearrowleft_{\varnothing}$;
$\mathbb{A} \setminus \{a\}$ has one plane (up to α-conversion): $b\circlearrowleft_{a}$;
$\{a\}$ has one plane: $a\circlearrowleft_{a}$.

Now defining substitution is kind of simple if we can just define it on the planes.

So we define $\in$-inductively:

$a[a \mapsto u] = u$ etc

and

$(x\circlearrowleft_A)[a \mapsto u] = x[a \mapsto u]\circlearrowleft_{A\setminus(a \cup supp(u))}$

as usual.

(The above is a lie; the actual definition is more complex, but not for very interesting reasons. See 3.14 of [8] if you dare. I would write it differently today. It doesn't matter for now, though.)

Let's try this with $\mathbb{A}$ to get the idea. $\mathbb{A} = b\circlearrowleft_{\varnothing}$ so $\mathbb{A}[a \mapsto u] = (b[a \mapsto u])\circlearrowleft_{\varnothing} = b\circlearrowleft_{\varnothing} = \mathbb{A}$.

Perfect!

The definition in [8] has the following good property: *if* x consists of nominal abstract syntax (syntax-with-binding) as constructed in my PhD thesis, *then* $x[a \mapsto u]$ is equal to what x with a substituted for u should be. So in this sense, [8] is a generalisation of my PhD from syntax to ... all of FM sets. Amazing.

stodfo takes a completely different approach. We introduce an auxiliary amgis-action $[u \hookleftarrow a]$ and set $X[a \mapsto u] = \{x : x[u \hookleftarrow a] \in X\}$ and $X[u \hookleftarrow a] = \{x : x[a \mapsto u] \in X\}$.

This introduces the rather odd (in the context of FM sets) 'polarity' of sets; given x, it can be considered *positively* and we calculate $x[a \mapsto u]$, or *negatively* and we calculate $x[u \hookleftarrow a]$.

You might guess that the definition above will not give us $a\#X$ implies $X[a \mapsto u] = X$. You would be right—it doesn't!

However, if we restrict to the X that *do* have this property, and construct the set of such X and call it $pow_\sigma(X)$, then it turns out that such X are closed under the substitution action so $pow_\sigma(X)$ does have a capture-avoiding substitution action.

The feature of [10] is that by virtue of being a powerset, $pow_\sigma(X)$ also has the structure of a Boolean algebra—but there's more, because we have substitution too, and so by taking intersections we can interpret universal quantification. In fact, $pow_\sigma(X)$ is a model not just of Boolean algebra but of first-order logic. Also amazing.

How are you with this so far?

Jamie

p.s. FYI there's more to being capture-avoiding than $a\#x \ \rightarrow \ x = x[a \mapsto u]$.

The three “capture-avoidance” properties are:

(1) $a\#x \;\rightarrow\; x[a\mapsto u] = x$;
(2) $b\#x \;\rightarrow\; ((b,a)\cdot x)[b\mapsto u] = x[a\mapsto u]$;
(3) $a\#v \;\rightarrow\; x[a\mapsto u][b\mapsto v] = x[b\mapsto v][a\mapsto u[b\mapsto v]]$.

Here (b,a) is the *swapping* permutation taking b to a, a to b, and c to c.

(3) is often called *the substitution lemma* when applied to syntax.

In [8] I generate a substitution action that satisfies (1) and (2) but only satisfies (3) for “a lot” of sets. I don’t *think* any of this matters for the embedding you want, because the embedding is a far simpler definition than the substitution of [8].

So if all you want to do is embed a copy of $x\circlearrowleft_A$ in $\mathfrak{M}_S$ as $x\circlearrowleft_A$ in $\mathfrak{M}_T$ then you can probably ignore (1), (2), and (3) above, and just concentrate on the theory of planes of FM sets.

Another message from Jamie

Dear Thomas,

So suppose for simplicity $B\subseteq A$. Consider two models of ZFA (a.k.a. ZFU), $\mathfrak{M}_B$ and $\mathfrak{M}_A$, as you described.

I need to change notation. I will write S and T where you wrote B and A. A and B will henceforth range over *finite* sets of atoms.

We want to inject $\mathfrak{M}_S$ into $\mathfrak{M}_T$.

Atoms map to themselves, as you observed.

Consider some set $X\in\mathfrak{M}_S$. By Theorem 3.12 of [8] we can write

$$X = \bigcup\{x\circlearrowleft_A \mid x\circlearrowleft_A \propto X\}$$

Here we use the following macro: $\propto$ and this notation is from [9], not [8], but means “$x\circlearrowleft_A$ is a plane in X”, which is written “$(x,A)\in \mathit{plane}(X)$” in [8]. It’s the same definition.

A **plane** in X a subset $x\circlearrowleft_A\subseteq X$ such that A is a minimal subset of $\mathit{supp}(X)$ such that $x\circlearrowleft_A\subseteq X$. See Definition 3.10 of [8].

[The point is that these As are finite, and that $x\circlearrowleft_A$ (for fixed x) gets bigger as A gets smaller.]

Planes can overlap, and planes-in-X do not all have to have the same A. Consider for instance

$$X = (a,b)\circlearrowleft_{\{a\}} \cup (a,b)\circlearrowleft_{\{b\}}.$$

It is easy to see that this consists of two planes which overlap and have different A.

$$(a,b)\circlearrowleft_{\{a\}} = \{(a,b),(a,c),(a,d),(a,e),...\}$$
$$(a,b)\circlearrowleft_{\{b\}} = \{(a,b),(c,b),(d,b),(e,b),...\}$$

The relationship between $\mathit{supp}(X)$ and the As in its planes is given by Theorem 3.13 of [8].

So we have identified the planes $x\circlearrowleft_A \propto X$. If we can map sets of the form $x\circlearrowleft_A$ from $\mathfrak{M}_B$ to $\mathfrak{M}_A$ then we are done, because we can extend this "planewise" to general X.

So we have essentially reduced the problem to mapping $x\circlearrowleft_A$ from $\mathfrak{M}_S$ to $\mathfrak{M}_T$.

Now we note that $x\circlearrowleft_A$ is actually dependent on S and T. Because:

In $\mathfrak{M}_S$ the set $x\circlearrowleft_A$ is equal to

$$\{\pi(x) : \pi \in \mathit{fix}(A \cup (\mathbb{A} \setminus S))\}.$$

In $\mathfrak{M}_T$ the set $x\circlearrowleft_A$ is equal to

$$\{\pi(x) : \pi \in \mathit{fix}(A \cup (\mathbb{A} \setminus T))\}.$$

[But i'm still not *entirely* happy. Then the argument to the '*fix*' is an infinite set. Is that what you mean?]

Yes, kind of. You build $\mathfrak{M}_S$ and $\mathfrak{M}_T$ using atoms from S and T, so obviously the notion of "atoms-orbit" $x\circlearrowleft_A$ is relativised to the "atoms available", which is S and T respectively.

Viewed externally, from the point of view of some larger set of "all possible atoms", which may be infinitely larger than S and T, this does indeed lead to an argument to *fix*—externally—that is infinite.

This is just an artefact of the way you set up your models.

So we map $x\circlearrowleft_A$ (in $\mathfrak{M}_S$) to $x\circlearrowleft_A$ (in $\mathfrak{M}_T$), and we are done.

[There may be merit to be gained by thinking of these two sets of permutations as $\{\pi(x) : \pi \in \mathit{fix}(S \Rightarrow A)\}$ and $\{\pi(x) : \pi \in \mathit{fix}(T \Rightarrow A)\}$?]

Of course this needs to be checked, but I would expect it to work.

Jamie

p.s. I have looked at such problems in the past.

* Section 9.5 of [6] performs (I think) essentially the construction above, for the case where $T = S \cup \{c\}$ where c is an atom and $c \notin S$ (actually something a little stronger, identifying the exact image of $\mathfrak{M}_S$ in $\mathfrak{M}_T$). The notation and language in which the result is expressed are completely different, though.

* A paper in the JSL "Finite and infinite support in nominal algebra and nominal logic" `gabbay.org.uk/papers.html#finisn` addresses a related question, of moving between universes with finite support and universes with infinite support. I don't suggest that this illuminates the discussion above of $\mathfrak{M}_S$ and $\mathfrak{M}_T$ directly, but I mention it in case the information is useful later.

References

[1] Jon Barwise and Laurence Moss, "Vicious Circles: On the Mathematics of Non-Wellfounded Phenomena" Center for the Study of Language and Information Publications 1996.

[2] Felgner, U. "Models of ZF set theory" Springer 1971

[3] A.A. Fränkel “Der Begriff “definit” und die Unabhügingkeit des Auswahlaxioms”. Sitzungsberichte d. Preussischen Akad. d. Wiss., Phys. Math. Klasse **21** (1922) pp 253–257

[4] A.A. Fränkel “Über eine abgeschwächte Fassung des Auswahlaxioms” JSL **2**(1937) pp 1–27.

[5] A.A. Fränkel Review of [18] JSL **4** (1939) pp 30–31.

[6] Jamie Gabbay “Foundations of nominal techniques”, `gabbay.org.uk/papers.html#fountl`

[7] Jamie Gabbay “Nominal terms and nominal logics: from foundations to metamathematics” `gabbay.org.uk/papers.html#nomtnl`

[8] Jamie Gabbay `www.gabbay.org.uk/papers.html#stusun`

[9] Jamie Gabbay “Freshness and name-restriction in nominal sets” `www.gabbay.org.uk/papers.html#frenrs`

[10] Jamie Gabbay `www.gabbay.org.uk/papers.html#stodfo`

[11] Jamie Gabbay “Finite and infinite support in nominal algebra and nominal logic” in JSL [. . .] `gabbay.org.uk/papers.html#finisn`

[12] Gauntt. R. J., “The undefinablity of cardinality” Lecture Notes prepared in Connection with the Summer Institute on Axiomatic Set Theory held at UCLA 1967 AMS. section IV M.

[13] Halbeisen, L. Combinatorial Set Theory. With a Gentle Introduction to Forcing, Springer Verlag, 2012. Chapter 7.

[14] Holmes on brackets math.boisestate.edu/∼holmes/holmes/newbracket.ps

[15] Jech, T. J., The Axiom of Choice, 1973. reissued by Dover

[16] A. Lévy “The Fraenkel-Mostowski method for Independence proofs in Set Theory” in The Skolem memorial volume on the theory of models North Holland (1965) pp 221–228.

[17] A Lindenbaum and A. Mostowski “Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen; C.R. Soc. Sci Lettr. Varsovie, Classe III, **31** (1938) pp 27–32.

[18] A. Mostowski “Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip”. Fund Math **32** (1939) pp 201–252.

[19] Steven O’Sullivan Part III essay 2012

[20] E. Specker “Zur Axiomatik der Mengenlehre (Fundierungs- und Auswahlaxiom)” Zeitschr. f. math. Logik. u. Grundlagen d. Math. **3** (1957) pp 173–210.

[21] J.K.Truss, “The structure of amorphous sets”, Annals of Pure and Applied Logic, 73 **1995**, 191–233.

[22] J.K.Truss,