The truth-functional arrow

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We will use standard notation for the connectives of propositional logic: '$\vee$', '$\wedge$' for 'or' and 'and'. We will also write $\bigwedge_{i\in I} p_i$ and suchlike for indexed conjunctions (and disjunctions). We write '$p \to q$' for the connective that will be equivalent to '$\neg(p \wedge \neg q)$' or to '$\neg p \vee q$'. $\to$ is the **material conditional**. A conditional[1] is a binary connective that is an attempt to formalise a relation of implication. The two components glued together by the connective are the **antecedent** (from which one infers something) and the **consequent** (which is the something that one infers). The material conditional is the simplest one: $p \to q$ evaluates to `true` unless p evaluates to `true` and q evaluates to `false`.

Lots of students dislike the material conditional as an account of implication. The usual cause of this unease is that in some cases a material conditional $p \to q$ evaluates to `true` for what seem to them to be spurious and thoroughly unsatisfactory reasons: namely, that p is false or that q is true. How can q follow from p merely because q happens to be true? The meaning of p might have no bearing on q whatever! This unease shows that we think we are attempting to formalise a relation between *intensions* rather than a relation between *extensions*. $\wedge$ and $\vee$, too, are relations between intensions but they also make sense applied to extensions. Now if p implies q, what does this tell us about what p and q evaluate to? Well, at the very least, it tells us that p cannot evaluate to `true` when q evaluates to `false`. This rule "from p and $p \to q$ infer q" is called **modus ponens**. q is the **conclusion**, p is the **minor premiss** and $p \to q$ is the **major premiss**. Thus we can expect the *extension* corresponding to a conditional to satisfy *modus ponens* at the very least.

How many extensions are there that satisfy *modus ponens*? For a connective C to satisfy *modus ponens* it suffices that in both rows of the truth table for C where p is true, if pCq is true then this is also a row in which q is true.

p	C	q
1	?	1
0	?	1
1	0	0
0	?	0

[1]This word 'conditional' is overloaded as well. Often a formula whose principal ('top level') connective is a conditional will be said to be a conditional.

We cannot make $p\ C\ q$ true in the third row, because that would cause C to disobey *modus ponens*, but it doesn't matter what we put in the centre column in the three other rows. This leaves eight possibilities:

(i) $\lambda pq.q$, (ii) $\lambda pq.(p \longleftrightarrow q)$, (ii) $\lambda pq.\neg p$, (iv) $\lambda pq.(\neg p \vee q)$, (v) $\lambda pq.\mathtt{false}$, (vi) $\lambda pq.p \wedge q$, (vii) $\lambda pq.\neg p \wedge q$, (viii) $\lambda pq.\neg p \wedge \neg q$.

These correspond to the eight rules of inference

$$(1): \frac{p \quad q}{q} \qquad (2): \frac{p \quad p \longleftrightarrow q}{q} \qquad (3): \frac{p \quad \neg p}{q} \qquad (4): \frac{p \quad p \to q}{q}$$

$$(5): \frac{p \quad \perp}{q} \qquad (6): \frac{p \quad p \wedge q}{q} \qquad (7): \frac{p \quad \neg p \wedge q}{q} \qquad (8): \frac{p \quad \neg p \wedge \neg q}{q}$$

obtained from the rule of *modus ponens* by replacing '$p \to q$' by each of the eight extensional binary connectives that satisfy the rule. (1) will never tell us anything we didn't know before; we can never use (5) because its major premiss is never true; (6) is a poor substitute for the rule of $\wedge$-elimination; (3), (7) and (8) we will never be able to use if our premisses are consistent.

(2), (4) and (6) are the only sensible rules left. They all obey the constraint of being truth-preserving. (2) is not what we are after because it is symmetrical in p and q whereas "if p then q" is not. The advantage of (4) is that you can use it whenever you can use (2) or (6). So it's of more use!

We had better check that this policy of evaluating $p \to q$ to `true` unless there is a very good reason not to does not get us into trouble. Fortunately, in cases where the conditional is evaluated to `true` *merely* for spurious reasons, then no harm can be done by accepting that evaluation. For consider: if it is evaluated to `true` *merely* because p evaluates to `false`, then we are never going to be able to invoke it (as a major premiss at least), and if it is evaluated to `true` *merely* because q evaluates to `true`, then if we invoke it as a major premiss, the only thing we can conclude—namely q—is something we knew all along.

This last paragraph is not intended to be a *justification* of our policy of using only the material conditional: it is merely intended to make it look less unnatural than it otherwise might. The astute reader who spotted that nothing was said there about conditionals as *minor* premisses should not complain. They may wish to ponder the reason for this omission.

Reasonable people might expect that what one has to do next is solve the problem of what the correct notion of conditional is for intensions. This is a very hard problem, since it involves thinking about the internal structure of intensions and nobody really has a clue about that. (This is connected to the fact that we do not really have robust criteria of identity for intensions, as mentioned earlier.) It has spawned a vast and inconclusive literature. Fortunately it turns out that we can duck it all, and resolve just to use the material conditional all the time.