2015 Paper 6 Question 4

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It's part (c) that people have difficulty with.

Part (i)

One lambda-term that does the trick is

$$\lambda y.\lambda x.\lambda f.fy$$

Part (ii)

These β-reductions are fairly straightforward if you don't get flustered.

$$\mathbb{N}_0\,M\,N = (\lambda x.\lambda f.x)\,N \to (\lambda f.M)\,N \to M$$

and

$$\mathbb{N}_{n+1}\,M\,N = (\lambda x.\lambda f.f\,\mathbb{N}_n)\,N \to (\lambda f.f\,\mathbb{N}_n)\,N \to N\,\mathbb{N}_n \qquad (*)$$

Part (iii)

The obvious S to try is the S we obtained in Part (c)(i). We are obviously going to have to an induction. The thing to try to prove is ... fix a natural number m and prove

$$(\forall n \in \mathbb{N})(P_m\mathbb{N}_n \twoheadrightarrow \mathbb{N}_{m+n}) \qquad (1)$$

To prove 1 we use the following fact from part (b) (not proved here)

$$P_m \twoheadrightarrow (\lambda f.\lambda y.y\,\mathbb{N}_m(\lambda z, S(fz)))P_m \to \lambda y.y\,\mathbb{N}_m(\lambda z.S(P_m z))$$

which gives

$$P_m\mathbb{N}_n \mapsto \mathbb{N}_n\mathbb{N}_m(\lambda z.S(P_m z)) \qquad (2)$$

Now we can prove the induction.

Base case, $n = 0$

$$
\begin{aligned}
P_m\,\mathbb{N}_0 &\twoheadrightarrow \mathbb{N}_0\mathbb{N}_m(\lambda z.S(P_m z)) && \text{by 2}\\
&\twoheadrightarrow \mathbb{N}_m
\end{aligned}
$$

Induction Step:

$$
\begin{aligned}
P_m\,\mathbb{N}_{n+1} &\twoheadrightarrow \mathbb{N}_{n+1}\mathbb{N}_m(\lambda z.S(P_m z)) && \text{by 2}\\
&\twoheadrightarrow \lambda z.S(P_m z))\mathbb{N}_n && \text{by (*) from part (c)(i)}\\
&\rightarrow\ S(P_m\mathbb{N}_n)\\
&\twoheadrightarrow S(\mathbb{N}_{m+n}) && \text{by induction hypothesis}\\
&\twoheadrightarrow \mathbb{N}_{m+n+1} && \text{by (c)(i)}
\end{aligned}
$$