COMPUTER SCIENCE TRIPOS PART 1A
2015 Paper 2 Question 9

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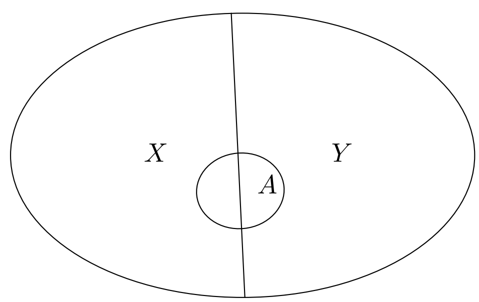
(a) This is bookwork.

(b) For 6 marks, this is a cushy number!

Do not succumb to the temptation to say that the LHS is of size $2^{|X|+|Y|}$ and the RHS is $2^{|X|} \cdot 2^{|Y|}$ and that you learnt in your crèche that these two numbers are the same. They are, but you are being called upon to *prove* this fact not *appeal* to it. So let's prove it.

Never be afraid to draw a good picture. You are told that $X \cap Y = \emptyset$ so you can draw the following, which depicts X and Y as disjoint. (Think briefly: what would the picture look like if X and Y were *not* disjoint?)



A is a subset of $X \cup Y$. I can uniquely identify $A \subseteq X \cup Y$ once I know $A \cap X$ and $A \cap Y$. Now $A \cap X$ lives inside $\mathcal{P}(X)$ and $A \cap Y$ lives inside $\mathcal{P}(Y)$. That is, I know A once I know the ordered pair $\langle A \cap X, A \cap Y \rangle$. But this is as much as to say that I have bijection between $\mathcal{P}(X \cup Y)$ and $\mathcal{P}(X) \times \mathcal{P}(Y)$. Notice that if X and Y are *not* disjoint then there will be more than one way of thinking of an $A \subseteq (X \cup Y)$ as a union–of–a–subset-of-X–with–a–subset-of-Y, so we wouldn't get a *bi*jection. We'd get a *sur*jection $\mathcal{P}(X) \times \mathcal{P}(Y) \twoheadrightarrow \mathcal{P}(X \cup Y)$. In those circumstances there is no *in*jection $\mathcal{P}(X \cup Y) \hookrightarrow \mathcal{P}(X) \times \mathcal{P}(Y)$ staring us in the face unless we have more information about X and Y.

(c) (i)

The way to prove that two sets are identical is to show that they have the same members. x is a member of $F(L_1 \cup L_2)$ iff

$$(\exists a \in \Sigma)(\exists y \in L_1 \cup L_2)(x = aya).$$

We will process this with a chain of biconditionals.

$$(\exists a \in \Sigma)[(\exists y \in L_1)(x = aya) \vee (\exists y \in L_2)(x = aya)]$$

We can now import the '$\exists a \in \Sigma$' past the '$\vee$' (with duplication) to get

$$(\exists a \in \Sigma)(\exists y \in L_1)(x = aya) \vee (\exists a \in \Sigma)(\exists y \in L_2)(x = aya)$$

and this becomes

$$x \in F(L_1) \vee x \in F(L_2)$$

(c) (ii)

Clearly if w is a palindrome over $L \subseteq \Sigma^*$ and $a \in \Sigma$ then awa is likewise a palindrome. Not much work for 2 marks. It's your lucky day.

(c) (iii)

Read the definitions slowly and `DON'T PANIC`. (A) you prove by induction on n. (B) ditto.