COMPUTER SCIENCE TRIPOS 2007:2:5

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July 10, 2015

(a)

Don't think (at this stage) of what the action actually *is*: think of it as a black box. (This is a plot point for part (d)). $x \sim y$ if x can be sent to y by the action(!) of a permutation. (I know they say *bijection* not *permutation* but, in the only realistic cases you are going to have to think about, all the bijections of A that are of any interest will in fact be permutations). This relation is symmetric beco's the inverse of a permutation is a permutation; it's transitive beco's the composition of two permutations is a permutation; finally it's reflexive beco's of $\mathbb{1}_A$.

(b)

These equivalence classes are called *orbits*. This is nice: the orbit of a planet is all the places the planet can get sent to; the orbit of an element of X is the set of things it can be moved to by the action of $Bij(A)$.

When you look at 'e_x' and its definition, don't panic. e_x is going to be a function from $Bij(A)$ to $[x]_\sim$. Take some time out to think along the following lines .... If i have an $x \in X$ in my mind, and i want to define a function $Bij(A) \to [x]_\sim$—using x—what function can i dream up? Think about it for a bit, and you will come up with the definition the examiners have supplied. Do this **before** attempting to understand the definition.

Here, with e_x, we are thinking of $x * \sigma$ not as the result of doing σ to x but as the result of doing x to σ. The result is the same, you're just thinking of it differently.

What has to happen for e_x to be surjective? It has to be that, for every $y \in [x]_\sim$, there is a $\sigma \in Bij(A)$ that sends x to y. But $[x]_\sim$ is precisely the set of things that x can be sent to in this way! The fact that e_x is surjective can be expressed as $|Bij(A)| \geq^* |[x]_\sim|$ and $\geq^*$ is the same as $\geq$ when all numbers concerned are finite, so we infer $n! \geq |[x]_\sim|$.

(c)

(i) Clearly $n!$.

(ii) Reality check (always useful in cases like this!) the answer had better be a whole number. There are $|X|$ things in X and they are divided amongst equivalence classes each of size $n!$, so there must be $|X|/n!$ (which is $m/n!$ if $|X| = m$) of these equivalence classes.

(d)

Key here is to read the definition carefully so you are sure you know what is going on. In fact, a possibly even better idea is to think ... How might $Bij(A)$ act on the set of injections $A \hookrightarrow B$? Then you get 6 fairly easy marks. I think there is a background assumption that A and B are both finite.

There are two situations to consider: $|A| \leq |B|$ and $|A| > |B|$. In the second case $Inj(A, B)$ is the empty set.