Computer Science Tripos 2002 Paper 1 Question 8

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Let Ω be a set. Write $\mathcal{P}(\Omega)$ for its power set. Recall the definition of the intersection of $\mathcal{B} \subseteq \mathcal{P}(\Omega)$:

$$\bigcap_{B \in \mathcal{B}} \mathcal{B} = \{x \in \Omega : (\forall B \in \mathcal{B})(x \in B)\}$$

(a) Let $\mathcal{B} \subseteq \mathcal{P}(\Omega)$ and $\mathcal{C} \subseteq \mathcal{P}(\Omega)$.

(i) Prove that

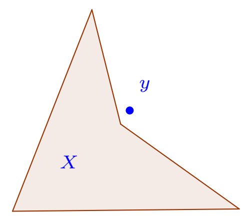
$$(\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C) \subseteq \bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C)$$

(ii) Prove that

$$\bigcap_{(B,C) \in \mathcal{B} \times \mathcal{C}} (B \cup C) \subseteq (\bigcap_{B \in \mathcal{B}} B) \cup (\bigcap_{C \in \mathcal{C}} C)$$

(b) This part seems to have caused problems for some. Let's have a look.

We are contemplating relations that hold between elements of Ω and subsets of Ω. An example of the sort of thing the examiner has in mind is the relation that a point y in the plane bears to a (typically non-convex) region X when y is in the convex hull of X.



The idea is that y is one of the points you have to "add" to obtain something convex. (Check that you know what a convex set is, as i'm going to procede on the assumption that you do, and use it as a—one hopes!—illuminating illustration)

What is $\mathcal{R}$? $\mathcal{A}$ is an intersection-closed family of subsets of Ω. (As it might be, the collection of convex subsets of the plane). We are told that it is the relation that relates y to X whenever anything in A that extends X also contains y. In our illustration—where $\mathcal{A}$ is the collection of convex subsets of the plane—$\mathcal{R}$ is the relation that hold between X and y whenever y is in the convex hull of X. (If you don't already know the meaning of the expression "convex hull" you can probably guess it from the news that, in the picture above, y is in the convex hull of X.) Certainly in this case any set that is $\mathcal{R}$-closed is convex.

Assume C is $\mathcal{R}$-closed. That is to say

$$\forall (X, y) \in \mathcal{R}. X \subseteq C \to y \in C \tag{1}$$

(That's in their notation: i'd've written it $(\forall \langle X, y\rangle \in \mathcal{R})(X \subseteq C \to y \in C)$ which (i think) makes the scoping clearer.)

But $\mathcal{R} = \{(X, y) \in \mathcal{P}(\Omega) \times \Omega : (\forall A \in \mathcal{A})(X \subseteq A \to y \in A)\}$. Substituting this for '$\mathcal{R}$' in (1) we obtain

$$\forall (X, y) \in \{(X, y) \in \mathcal{P}(\Omega) \times \Omega : (\forall A \in \mathcal{A})(X \subseteq A \to y \in A)\}. X \subseteq C \to y \in C \tag{2}$$

which reduces to

$$(\forall X, y)[(\forall A \in \mathcal{A})(X \subseteq A \to y \in A) \land X \subseteq C. \to y \in C] \tag{3}$$

The examiners suggest you should consider the set $\{A \in \mathcal{A} : C \subseteq A\}$. I think they want you to look at $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$.

If you've followed the action this far you would probably think of this anyway, since this is a set that you know must be in $\mathcal{A}$ and it seems to stand an outside chance of being equal to C. So let's look again at (3) to see if it does, in fact, tell us that $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$ is C.

And—of course—it does. First we instantiate 'X' to 'C' in (3) to obtain:

$$\forall y[(\forall A \in \mathcal{A})(C \subseteq A \to y \in A) \to y \in C] \tag{4}$$

Now let y be an arbitrary member of $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$. That means that y satisfies the antecedent of 4. So it satisifes the consequent of 4 as well. So we have proved that $\bigcap\{A \in \mathcal{A} : C \subseteq A\}$ is a subset of C. It was always a superset of C, so it is equal to C. So $C \in \mathcal{A}$ as desired.