Computer Science Tripos 1997 Paper 1 Question 7

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(a) Yes: equality is a partial order, and it is tree-like because the set of strict predecessors is always empty.

(b) Yes. The usual order is a partial (indeed *total*) order and every total order is tree-like.

(c) No. This is a partial order but is not tree-like because (for example) 6 has two immediate strict predessors.

(d) This is reflexive and antisymmetrical (if $x\,R\,y$ and yRx—so that x and y are either equal or each is the greatest prime factor of the other—then they are equal). The hard part is to show that it is transitive. Suppose $x\,R\,y$ and $y\,R\,z$. If $x = y$ or $z = z$ we deduce $x\,R\,z$ at once, so consider the case where $x\,R\,y$ and $y\,R\,z$ hold, but *not* in virtue of $x = y$ or $y = z$. But this case cannot arise, because if $y\,R\,z$ and $y \neq z$, then y is a prime, and the only x such that $x\,R\,y$ is y itself. Finally, it's easy to show this relation is tree-like, because no number can have more than one greatest prime factor.

It seem to me that the number of treelike partial orderings of n elements is precisely $n!$. Each treelike partial ordering of n chaps gives rise to n new partial orderings because the extra chap can be stuck on top of any of the n things already there. No new partial ordering gets counted twice.