ON GENUS ONE CURVES OF DEGREE 5 WITH SQUARE-FREE DISCRIMINANT

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ABSTRACT. We study genus one curves of degree 5 defined by Pfaffians. We give new formulae for the invariants, and prove the equivalence of two different definitions of minimality. As an application we show that transformations between models with square-free discriminant are necessarily integral. This result is used by Bhargava and Shankar in their work on the average ranks of elliptic curves.

1. INTRODUCTION

Let E be an elliptic curve over a number field $\mathbb{K}$. An *n-covering* of E is a pair (C, π) where C is a smooth curve of genus one and $\pi : C \to E$ is a morphism such that $\pi = [n] \circ \psi$ for some isomorphism $\psi : C \to E$ defined over $\overline{\mathbb{K}}$. If C is everywhere locally soluble then by [6, Theorem 1.3] there exists a $\mathbb{K}$-rational divisor D on C such that D is linearly equivalent to $\psi^*(n.0_E)$. The linear system $|D|$ defines a morphism $C \to \mathbb{P}^{n-1}$. If $n \geq 3$ then this morphism is an embedding, and the image is called a *genus one normal curve* of degree n. The word "normal" refers to the fact the curve is projectively normal, i.e. the homogeneous co-ordinate ring is integrally closed. This should not be confused with the fact C is normal, which is automatic since C is smooth.

When $n = 2, 3, 4$ the curve C is represented by a binary quartic, ternary cubic, or pair of quadrics in 4 variables. In this paper we take $n = 5$, in which case C is represented by data of the following form.

A *Pfaffian model* Φ over a ring R is a 5×5 alternating matrix of linear forms in $R[x_1, \ldots, x_5]$. We write $X_5(R)$ for the space of all Pfaffian models over R. Two models Φ and Φ' are R-equivalent if $\Phi' = [A, B]\Phi$ for some $A, B \in \mathrm{GL}_5(R)$. The action of A is given by $\Phi \mapsto A\Phi A^T$, and the action of B is given by

$$(\Phi_{ij}(x_1, \ldots, x_5)) \mapsto (\Phi_{ij}(x'_1, \ldots, x'_5))$$

where $x'_j = \sum_{i=1}^5 B_{ij}x_i$. We define $\det[A, B] = (\det A)^2 \det B$. The models Φ and Φ' are *properly R-equivalent* if $\det[A, B] = 1$. The invariants $c_4, c_6, \Delta \in \mathbb{Z}[X_5]$ are

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certain integer coefficient polynomials in the 50 coefficients of a Pfaffian model. We give formulae for these in Section 2.

We work over a discrete valuation field K with valuation ring $\mathcal{O}_K$, normalised valuation $v : K^\times \to \mathbb{Z}$, uniformiser π, and residue field $k = \mathcal{O}_K/\pi\mathcal{O}_K$.

Our main result is the following. It answers a question of Bhargava, and is used in the work of Bhargava and Shankar [3, Proposition 11] on the average size of the 5-Selmer group of an elliptic curve.

Theorem 1.1. *Let $\Phi, \Phi' \in X_5(\mathcal{O}_K)$ be Pfaffian models with $v(\Delta(\Phi)) \leq 1$ and $v(\Delta(\Phi')) \leq 1$. If $\Phi' = [A, B]\Phi$ for some $A, B \in \mathrm{GL}_5(K)$ then $A, B \in K^\times \mathrm{GL}_5(\mathcal{O}_K)$. In particular*

(i) *If Φ and Φ' are K-equivalent then they are $\mathcal{O}_K$-equivalent.*
(ii) *The stabiliser of Φ in $\mathrm{GL}_5(K) \times \mathrm{GL}_5(K)$ is contained in the subgroup generated by $\mathrm{GL}_5(\mathcal{O}_K) \times \mathrm{GL}_5(\mathcal{O}_K)$ and $[\pi^{-1}I_5, \pi^2 I_5]$.*

To indicate how Theorem 1.1 is useful, we give the following global application. We take $\mathbb{K} = \mathbb{Q}$, but note that the result generalises immediately to any number field with class number 1. We say that a Pfaffian model Φ has the same invariants as an elliptic curve E if the invariants $c_4(\Phi)$, $c_6(\Phi)$, $\Delta(\Phi)$ are the same as the invariants c_4, c_6, Δ of a minimal Weierstrass equation for E.

Theorem 1.2. *Let $E/\mathbb{Q}$ be an elliptic curve with square-free minimal discriminant. Then the 5-Selmer group $S^{(5)}(E/\mathbb{Q})$ is in bijection with the set of Pfaffian models over $\mathbb{Z}$ with the same invariants as E, up to proper $\mathbb{Z}$-equivalence.*

In Sections 3 and 4 we introduce two different definitions of minimality, and show that if they agree then Theorem 1.1 is a natural consequence. The agreement of the two definitions is proved in Sections 5, 6 and 7. This extends [17, Theorem 4.1] from genus one curves of degrees 2, 3 and 4, to degree 5. In Section 8 we give a short alternative proof of Theorem 1.1, that is motivated by the ideas in the rest of this paper, but avoids nearly all the scheme-theoretic machinery.

2. Pfaffians and invariants

In this section we briefly describe how the equations for a genus one normal curve of degree 5 can be written in terms of Pfaffians. We then give some new formulae for the invariants of a Pfaffian model, that are simpler than the evaluation algorithms in [9, Section 8].

The Pfaffian of an alternating matrix is an integer coefficient polynomial in the entries of the matrix, whose square is the determinant. We only need to consider

Pfaffians of 4×4 matrices, in which case

$$\operatorname{pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

If Φ is an 5×5 alternating matrix then the row vector of submaximal Pfaffians of Φ is $\operatorname{Pf}(\Phi) = (p_1, \ldots, p_5)$ where $p_i = (-1)^i \operatorname{pf}\left(\Phi^{\{i\}}\right)$ and $\Phi^{\{i\}}$ is the matrix obtained by deleting the ith row and column of Φ. It can be shown, for example by direct calculation, that $\operatorname{Pf}(\Phi)\Phi = 0$, $\operatorname{adj}(\Phi) = \operatorname{Pf}(\Phi)^T \operatorname{Pf}(\Phi)$ and $\operatorname{Pf}(A\Phi A^T) = \operatorname{Pf}(\Phi)\operatorname{adj}(A)$ for all 5×5 matrices A.

In this section we work over any field K. Let $C \subset \mathbb{P}^4_K$ be a genus one normal curve, i.e. a smooth curve of genus one embedded by a complete linear system of degree 5. Let $R = K[x_1, \ldots, x_5] = \oplus_{d \geq 0} R_d$ be the polynomial ring with its usual grading by degree. Let $R(d)$ be the graded free R-module of rank 1 with $R(d)_e = R_{d+e}$. By the Buchsbaum-Eisenbud structure theorem [4], [5], or the treatment specific to this case in [10], the coordinate ring of C has minimal free resolution

$$0 \longrightarrow R(-5) \xrightarrow{\operatorname{Pf}(\Phi)^T} R(-3)^5 \xrightarrow{\Phi} R(-2)^5 \xrightarrow{\operatorname{Pf}(\Phi)} R \tag{1}$$

for some $\Phi \in X_5(K)$. In particular the homogeneous ideal of C is generated by the 4×4 Pfaffians of Φ. More generally, for any $\Phi \in X_5(K)$, we let $C_\Phi \subset \mathbb{P}^4_K$ be the subscheme defined by its 4×4 Pfaffians. We say that Φ is *non-singular* if C_Φ is a smooth curve of genus one. We write $K[X_5]$ for the polynomial ring in the 50 coefficients of a Pfaffian model. A polynomial $F \in K[X_5]$ is an *invariant of weight k* if $F \circ g = (\det g)^k F$ for all $g \in \mathrm{GL}_5 \times \mathrm{GL}_5$.

Theorem 2.1. *There are invariants $c_4, c_6, \Delta \in \mathbb{Z}[X_5]$ of degrees $20, 30, 60$ and weights $4, 6, 12$, satisfying $c_4^3 - c_6^2 = 1728\Delta$, with the following properties.*

(i) *If $\operatorname{char}(K) \neq 2, 3$ then the ring of invariants in $K[X_5]$ is generated by (the images of) c_4 and c_6.*

(ii) *A model $\Phi \in X_5(K)$ is non-singular if and only if $\Delta(\Phi) \neq 0$.*

(iii) *There exist $a_1, a_2, a_3, a_4, a_6, b_2, b_4, b_6 \in \mathbb{Z}[X_5]$ satisfying*

$$b_2 = a_1^2 + 4a_2, \qquad b_4 = a_1a_3 + 2a_4, \qquad b_6 = a_3^2 + 4a_6,$$
$$c_4 = b_2^2 - 24b_4, \qquad c_6 = -b_2^3 + 36b_2b_4 - 216b_6,$$

such that if $\Phi \in X_5(K)$ is non-singular then $\mathcal{C}_\Phi$ has Jacobian

$$y^2 + a_1(\Phi)xy + a_3(\Phi)y = x^3 + a_2(\Phi)x^2 + a_4(\Phi)x + a_6(\Phi). \tag{2}$$

PROOF: This is [9, Theorem 4.4] together with [11, Theorem 1.1]. □

It is shown in [9, Section 5.4] that if $\operatorname{char} K \neq 2$ and $\Phi \in X_5(K)$ is non-singular then there is an invariant differential ω_Φ on C_Φ given by

$$\omega_\Phi = \frac{x_i^2 d(x_j/x_i)}{Q(x_1, \ldots, x_5)}, \text{ where } Q = \frac{\partial P}{\partial x_k}\frac{\partial \Phi}{\partial x_l}\frac{\partial P^T}{\partial x_m}, \tag{3}$$

$P = \operatorname{Pf}(\Phi)$, and (i, j, k, l, m) is any even permutation of $(1, 2, 3, 4, 5)$. In the definition of Q, it is understood that by the partial derivative of a matrix we mean the matrix of partial derivatives. As we show in Remark 7.6, the restriction $\operatorname{char} K \neq 2$ is not needed.

In [12, Section 7] an alternative description of the invariant differential is given in terms of a certain covariant. We now give an explicit construction of this covariant, based in part on ideas in [2, Section 4]. For (i, j, k, l, m) an even permutation of $(1, 2, 3, 4, 5)$ we define

$$\Omega_{ij} = \frac{\partial P}{\partial x_k}\frac{\partial \Phi}{\partial x_l}\frac{\partial P^T}{\partial x_m} + \frac{\partial P}{\partial x_m}\frac{\partial \Phi}{\partial x_k}\frac{\partial P^T}{\partial x_l} + \frac{\partial P}{\partial x_l}\frac{\partial \Phi}{\partial x_m}\frac{\partial P^T}{\partial x_k}.$$

Now $\Omega = (\Omega_{ij})$ is an alternating matrix of quadratic forms. We define an action of $\mathrm{GL}_5 \times \mathrm{GL}_5$ on the space of such matrices via

$$[A, B] : \Omega \mapsto B^{-T}(\Omega_{ij}(x'_1, \ldots, x'_5))B^{-1}$$

where $x'_j = \sum_{i=1}^5 B_{ij}x_i$. In particular the first copy of GL_5 acts trivially. Recall that for $g = [A, B]$ we defined $\det g = (\det A)^2 \det B$.

Lemma 2.2. *The map $\Phi \mapsto \Omega$ is a covariant of weight 1, in the sense that*

$$g\Phi \mapsto (\det g)g\Omega$$

for all $g \in \mathrm{GL}_5 \times \mathrm{GL}_5$.

PROOF: If we replace Φ by $A\Phi A^T$ then P is replaced by $P \operatorname{adj} A$ and Ω is multiplied by $(\det A)^2$. So it suffices to consider $g = [I_5, B]$ for B running over a set of generators for GL_5. Since the cases where B is a diagonal matrix or a permutation matrix are easy, this reduces us to considering $B = I_5 + \lambda E_{12}$, where $\lambda \in K$ and E_{12} is the elementary matrix with a 1 in position $(1, 2)$ and all other entries 0. This corresponds to the substitution $x_2 \leftarrow x_2 + \lambda x_1$. In the definition of Ω_{ij} we replace $\frac{\partial P}{\partial x_1}$ by $\frac{\partial P}{\partial x_1} + \lambda\frac{\partial P}{\partial x_2}$ and $\frac{\partial \Phi}{\partial x_1}$ by $\frac{\partial \Phi}{\partial x_1} + \lambda\frac{\partial \Phi}{\partial x_2}$. This has the effect of replacing Ω_{r2} by $\Omega_{r2} - \lambda\Omega_{r1}$ and Ω_{2r} by $\Omega_{2r} - \lambda\Omega_{1r}$ for $r = 3, 4, 5$. A calculation, using the fact Φ is alternating, shows that the other entries of Ω do not change. Thus Ω changes to $g\Omega$ as required. □

We put

$$M_{ij} = \sum_{r,s=1}^{5} \frac{\partial \Omega_{ir}}{\partial x_s} \frac{\partial \Omega_{js}}{\partial x_r} \quad \text{and} \quad N_{ijk} = \sum_{r=1}^{5} \frac{\partial M_{ij}}{\partial x_r} \Omega_{rk}.$$

Theorem 2.3. *The invariants $c_4, c_6 \in \mathbb{Z}[X_5]$ are given by*

$$c_4(\Phi) = \frac{1}{13440} \sum_{i,j,r,s=1}^{5} \frac{\partial^2 M_{ij}}{\partial x_r \partial x_s} \frac{\partial^2 M_{rs}}{\partial x_i \partial x_j}$$

and

$$c_6(\Phi) = \frac{-1}{1036800} \sum_{i,j,k,r,s,t=1}^{5} \frac{\partial^3 N_{ijk}}{\partial x_r \partial x_s \partial x_t} \frac{\partial^3 N_{rst}}{\partial x_i \partial x_j \partial x_k}.$$

Proof: It may be checked using Lemma 2.2 that these polynomials are invariants of degrees 20 and 30. By Theorem 2.1 it only remains to show they are scaled as specified in [9]. We can do this by computing a single numerical example. □

We may compute the discriminant Δ either as $(c_4^3 - c_6^2)/1728$, or directly using the method at the end of [9, Section 8].

3. Minimal Pfaffian models

In this section we make some remarks about minimal Pfaffian models, and more specifically those with square-free discriminant. We also explain how Theorem 1.2 follows from Theorem 1.1.

From now on K will be a discrete valuation field, with ring of integers $\mathcal{O}_K$, and normalised valuation $v : K^\times \to \mathbb{Z}$. We fix a uniformiser π and write k for the residue field. Let $S = \operatorname{Spec} \mathcal{O}_K$. For the proof of Theorem 1.1 we are free to replace K by any unramified extension. We may therefore assume when convenient that K is complete, and k is algebraically closed.

A Pfaffian model $\Phi \in X_5(K)$ is *integral* if $\Phi \in X_5(\mathcal{O}_K)$, i.e. it has coefficients in $\mathcal{O}_K$. It follows from Theorem 2.1 that if Φ is non-singular and integral then $v(\Delta(\Phi)) = v(\Delta_E) + 12\ell(\Phi)$ where Δ_E is the minimal discriminant of the Jacobian E, and $\ell(\Phi) \geq 0$ is an integer called the *level*. We say that Φ is *minimal* if $v(\Delta(\Phi))$ is minimal among all integral models K-equivalent to Φ. If $\Phi' = g\Phi$ for $g = [A, B]$ with $A, B \in \mathrm{GL}_5(K)$ than $\ell(\Phi') = \ell(\Phi) + v(\det g)$.

Theorem 3.1. *(Minimisation theorem) Let $\Phi \in X_5(K)$ be non-singular. If $C_\Phi(K) \neq \emptyset$ then Φ is K-equivalent to an integral model of level 0.*

Proof: This is [11, Theorem 2.1(i)]. □

The proof of Theorem 3.1 is rather short. In [11] the first author also investigated to what extent the hypothesis $C_\Phi(K) \neq \emptyset$ can be weakened, and gave an algorithm for minimising.

Lemma 3.2. *Let $\Phi \in X_5(\mathcal{O}_K)$ with $v(\Delta(\Phi)) \leq 1$.*

(i) *The Jacobian E of C_Φ has Kodaira symbol I_0 or I_1.*
(ii) *If K is a p-adic field then $C_\Phi(K) \neq \emptyset$.*

PROOF: (i) By Theorem 2.1 we have $v(\Delta_E) \leq 1$. It follows by Tate's algorithm that the Kodaira symbol is either I_0 or I_1.
(ii) Since $v(\Delta(\Phi)) < 12$ we have $\ell(\Phi) = 0$. Then by [11, Theorem 7.1] we have $C_\Phi(K^{\mathrm{nr}}) \neq \emptyset$ where K^{nr} is the maximal unramified extension. By (i) we know that E/K has Tamagawa number 1. Therefore, as explained in [13, Lemma 2.1], solubility over K^{nr} is equivalent to solubility over K. □

Remark 3.3. To prove Theorem 1.1 it suffices to show that $B \in K^\times \mathrm{GL}_5(\mathcal{O}_K)$. The reason for this is as follows. By Lemma 5.2 we know that if Φ is minimal then its 4×4 Pfaffians are linearly independent mod π. So if Φ and Φ' are both minimal and $\Phi' = [A, \lambda I_5]\Phi$, then from $\mathrm{Pf}(\Phi') = \lambda^2 \mathrm{Pf}(\Phi)\,\mathrm{adj}(A)$ we deduce that $A \in K^\times \mathrm{GL}_5(\mathcal{O}_K)$. The final statements (i) and (ii) of Theorem 1.1 are immediate, since $v(\det[A, B]) = 0$ and the transformations $[\lambda I_5, \lambda^{-2} I_5]$ for $\lambda \in K^\times$ act trivially on the space of Pfaffian models.

We now explain how Theorem 1.2 follows from Theorem 1.1.

Theorem 3.4. *Let $E/\mathbb{Q}$ be an elliptic curve. The 5-Selmer group $S^{(5)}(E/\mathbb{Q})$ is in bijection with the proper $\mathbb{Q}$-equivalence classes of Pfaffian models $\Phi \in X_5(\mathbb{Q})$ with the same invariants as E and $C_\Phi(\mathbb{Q}_p) \neq \emptyset$ for all primes p.*

PROOF: This is a special case of [12, Theorem 6.1]. □

PROOF OF THEOREM 1.2: By Theorem 3.1 and strong approximation, each of the classes in Theorem 3.4 contains a model with coefficients in $\mathbb{Z}$. Since Δ_E is square-free, Theorem 1.1 shows that the map from proper $\mathbb{Z}$-equivalence classes to proper $\mathbb{Q}$-equivalence classes is injective. Moreover the condition $C_\Phi(\mathbb{Q}_p) \neq \emptyset$ is automatically satisfied by Lemma 3.2. □

Let $\Phi \in X_5(\mathcal{O}_K)$ have reduction $\phi \in X_5(k)$. We write $\mathcal{C}_\Phi \subset \mathbb{P}^4_S$ for the S-scheme defined by the 4×4 Pfaffians. It has generic fibre C_Φ and special fibre C_ϕ.

Suppose the entries of ϕ span $\langle x_1, \ldots, x_5 \rangle$. If P is k-point on C_ϕ then by an $\mathcal{O}_K$-equivalence we may assume $P = (1 : 0 : \ldots : 0)$. We may further assume $\phi_{12} = x_1$ and all other ϕ_{ij} (for $i < j$) are linear forms in $x_2, \ldots, x_5$. The tangent space to C_ϕ at P is $\{\phi_{34} = \phi_{35} = \phi_{45} = 0\} \subset \mathbb{P}^4_k$.

Lemma 3.5. *Let $P \in C_\phi$ as above. The following are equivalent.*

(i) *The tangent space to $\mathcal{C}_\Phi$ at P has dimension at most 2.*
(ii) *Every linear combination $r\Phi_{34} + s\Phi_{35} + t\Phi_{45}$ (where $r, s, t \in \mathcal{O}_K$, not all in $\pi\mathcal{O}_K$) that vanishes mod π has coefficient of x_1 not divisible by π^2.*

PROOF: By (i) we mean $\dim(\mathfrak{m}_P/\mathfrak{m}_P^2) \leq 2$ where $\mathfrak{m}_P$ is the maximal ideal of the local ring at P. The lemma is proved by a straightforward calculation. □

The following lemma will be used both to show that $\mathcal{C}_\Phi$ is regular, and in the elementary proof of Theorem 1.1 in Section 8.

Lemma 3.6. *If $\Phi \in X_5(\mathcal{O}_K)$ with $v(\Delta(\Phi)) \leq 1$ then every k-point P on C_ϕ satisfies the conditions in Lemma 3.5.*

PROOF: If the entries of ϕ fail to span $\langle x_1, \ldots, x_5\rangle$ then Φ is clearly not minimal and $v(\Delta(\Phi)) \geq 12$. Therefore an $\mathcal{O}_K$-equivalence brings us to the situation considered in Lemma 3.5. Let d be the dimension of the tangent space to C_ϕ at P. If $d = 1$ we are done. If $d \geq 3$ we may assume $\phi_{34} \in \langle x_5\rangle$ and $\phi_{35} = \phi_{45} = 0$. Then

$$[\mathrm{Diag}(\pi^{1/2}, \pi^{1/2}, 1, 1, 1), \pi^{-1/2}\,\mathrm{Diag}(\pi^{-1/2}, 1, 1, 1, \pi^{1/2})]\Phi$$

has coefficients in $\mathcal{O}_K[\pi^{1/2}]$. So in this case $v(\Delta(\Phi)) \geq 6$.

Now suppose $d = 2$. We may assume $\phi_{34} = x_4$, $\phi_{35} = x_5$ and $\phi_{45} = 0$. To complete the proof we show that if Φ_{45} has coefficient of x_1 divisible by π^2 then $v(\Delta(\Phi)) \geq 2$. Checking this directly, using the formulae for the invariants in Section 2, is unfortunately not practical. Instead we argue as follows. By making substitutions of the form $x_4 \leftarrow x_4 + \lambda x_1$ and $x_5 \leftarrow x_5 + \mu x_1$ for suitable $\lambda, \mu \in \pi\mathcal{O}_K$ we may arrange that Φ_{34} and Φ_{35} also have their coefficients of x_1 divisible by π^2. Then substituting for x_1 we have

$$\Phi = \begin{pmatrix} 0 & x_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ & 0 & \beta_1 & \beta_2 & \beta_3 \\ & & 0 & \ell_3 & -\ell_2 \\ & - & & 0 & \ell_1 \\ & & & & 0 \end{pmatrix}$$

where $\ell_1 \equiv 0 \pmod{\pi}$, the coefficient of x_1 in each of the α_i and β_i vanishes mod π, and the coefficient of x_1 in each of the ℓ_i vanishes mod π^2. By subtracting suitable multiples of the first two rows/columns from the last three rows/columns we may further assume that the coefficient of x_1 in each of the α_i and β_i vanishes mod π^2. Since it only matters what the coefficients are mod π^2, we may now

assume that none of the α_i, β_i and ℓ_i involve x_1. By [11, Lemma 2.4], Φ has the same discriminant as the quadric intersection

$$\ell_1\alpha_1 + \ell_2\alpha_2 + \ell_3\alpha_3 = 0$$
$$\ell_1\beta_1 + \ell_2\beta_2 + \ell_3\beta_3 = 0.$$

Since $\ell_1 \equiv 0 \pmod{\pi}$, the reduction of this quadric intersection mod π contains a line. It can then be checked (for example by a brute force calculation) that the discriminant vanishes mod π^2. This completes the proof. □

4. Geometric minimality and an application

In this section we define the notion of *geometric minimality* and explain the role it has to play in the proof of Theorem 1.1. We assume from now on that the residue field k is algebraically closed. Following [15, Definition 8.3.1] we have

Definition 4.1. A *fibred surface* $\mathcal{C}/S$ is an integral projective flat S-scheme of dimension 2.

Lemma 4.2. *Let $C \subset \mathbb{P}^{n-1}_K$ be a smooth projective curve and $\mathcal{C}$ its closure in $\mathbb{P}^{n-1}_S$. Then $\mathcal{C}$ is a fibred surface. Moreover $\mathcal{C}$ is normal if and only if*

(i) *$\mathcal{C}$ is Cohen-Macaulay, and*
(ii) *there are only finitely many non-regular points on the special fibre.*

Proof: The coordinate ring of $\mathcal{C}$ is a subring of that of C. Since C is integral it follows that $\mathcal{C}$ is integral. Then $\mathcal{C} \to S$ is flat and $\dim \mathcal{C} = 2$ by [15, Corollaries 4.3.10 and 4.3.14]. By definition $\mathcal{C}$ is projective. Since $\dim \mathcal{C} = 2$ and the generic fibre is smooth, (i) and (ii) are equivalent to the conditions (S_2) and (R_1) in Serre's criterion [15, Theorem 8.2.23]. □

Let $\mathcal{C}/S$ be a fibred surface. Lipman [1] showed that if K is complete then $\mathcal{C}$ admits a *desingularisation* (i.e. resolution of singularities). If $\mathcal{C}$ has smooth generic fibre then the hypothesis that K is complete may be removed, as described in [15, Corollary 8.3.51]. If in addition $\mathcal{C}$ is normal then by [15, Proposition 9.3.32] it admits a *minimal desingularisation*.

Definition 4.3. Let $C \subset \mathbb{P}^{n-1}_K$ be a genus one normal curve of degree n, with Jacobian E. Let $\mathcal{C}$ be the closure of C in $\mathbb{P}^{n-1}_S$. We say that C is *geometrically minimal* if $\mathcal{C}$ is normal, and the minimal desingularisation of $\mathcal{C}$ is isomorphic (as an S-scheme) to the minimal proper regular model of E.

This definition is *not* invariant under changes of co-ordinates defined over K. We remark that if C is geometrically minimal then $C(K) \neq \emptyset$, and $\mathcal{C}$ is obtained from the minimal proper regular model of E by contracting some of the irreducible components of the special fibre.

Before explaining how geometric minimality is used in the proof of Theorem 1.1, we quote the following lemma.

Lemma 4.4. *Let $\mathcal{C}$ be a projective S-scheme, and $\mathcal{L}$ an invertible sheaf on $\mathcal{C}$.*

(i) *The natural map $H^0(\mathcal{C}, \mathcal{L}) \otimes_{\mathcal{O}_K} K \to H^0(\mathcal{C}_K, \mathcal{L}_K)$ is an isomorphism.*
(ii) *We have the inequality $\dim_k H^0(\mathcal{C}_k, \mathcal{L}_k) \geq \dim_K H^0(\mathcal{C}_K, \mathcal{L}_K)$.*
(iii) *If equality holds in (ii) then $H^0(\mathcal{C}, \mathcal{L})$ is a free $\mathcal{O}_K$-module and the natural map $H^0(\mathcal{C}, \mathcal{L}) \otimes_{\mathcal{O}_K} k \to H^0(\mathcal{C}_k, \mathcal{L}_k)$ is an isomorphism.*

Proof: Part (i) is [15, Corollary 5.2.27] with $A = \mathcal{O}_K$ and $B = K$. The rest is [15, Lemma 5.2.31 and Theorem 5.3.20]. □

Theorem 4.5. *Let $C_1 \subset \mathbb{P}^{n-1}_K$ and $C_2 \subset \mathbb{P}^{n-1}_K$ be genus one normal curves of degree n. Suppose that C_1 and C_2 are isomorphic via a change of coordinates given by $B \in \mathrm{GL}_n(K)$. If C_1 and C_2 are geometrically minimal, and their Jacobian E has Kodaira symbol I_0 or I_1, then $B \in K^\times \mathrm{GL}_n(\mathcal{O}_K)$.*

Proof: Since the Jacobian E has Kodaira symbol I_0 or I_1 the special fibre of $\mathcal{E}$ (the minimal proper regular model of E) is either a smooth curve of genus one, or a rational curve with a node. Let $\mathcal{C}_i$ be the closure of C_i in $\mathbb{P}^{n-1}_S$. Then $\mathcal{C}_i$ is obtained from $\mathcal{E}$ by contracting some of the irreducible components of the special fibre. Since $\mathcal{E}_k$ is irreducible and $\mathcal{C}_{i,k}$ is a curve it follows that $\mathcal{C}_i \cong \mathcal{E}$. We now write $f_i : \mathcal{E} \to \mathbb{P}^{n-1}_S$ for the embedding with image $\mathcal{C}_i$ and let $\mathcal{L}_i = f_i^* \mathcal{O}(1)$.

Since $C_i = \mathcal{C}_{i,K}$ is a genus one curve of degree n we have $\dim_K H^0(E, \mathcal{L}_{i,K}) = n$. Since $\mathcal{C}_{i,k}$ is either a genus one curve or a rational curve with a node, and it has degree n by [14, Chapter III, Corollary 9.10], we have $\dim_k H^0(\mathcal{E}_k, \mathcal{L}_{i,k}) = n$. Then Lemma 4.4 shows that $H^0(\mathcal{E}, \mathcal{L}_i) \cong \mathcal{O}_K^n$. Our choice of co-ordinates on $\mathbb{P}^{n-1}_S$ corresponds to a choice of bases for $H^0(\mathcal{E}, \mathcal{L}_1)$ and $H^0(\mathcal{E}, \mathcal{L}_2)$. By hypothesis $\mathcal{L}_{1,K} \cong \mathcal{L}_{2,K}$, and the isomorphism $H^0(E, \mathcal{L}_{1,K}) \cong H^0(E, \mathcal{L}_{2,K})$ is given, relative to our chosen bases, by some scalar multiple of B.

Let $\mathcal{L} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$. By Lemma 4.4(ii) both $\mathcal{L}_k$ and its dual $\mathcal{L}_k^{-1}$ have non-zero global sections. Since $\mathcal{E}_k$ is irreducible it follows that $\mathcal{L}_k$ is trivial. Then by Lemma 4.4(iii) both $\mathcal{L}$ and $\mathcal{L}^{-1}$ have global sections that are nowhere vanishing on the special fibre. Therefore $\mathcal{L}$ is trivial and so $\mathcal{L}_1 \cong \mathcal{L}_2$. Taking global sections gives an isomorphism of $\mathcal{O}_K$-modules $H^0(\mathcal{E}, \mathcal{L}_1) \cong H^0(\mathcal{E}, \mathcal{L}_2)$. This isomorphism

is again given, relative to our chosen bases, by a scalar multiple of B. It follows that $B \in K^\times \mathrm{GL}_n(\mathcal{O}_K)$. □

Proof of Theorem 1.1: We saw in Lemma 3.2(i) that for $\Phi \in X_5(\mathcal{O}_K)$ with $v(\Delta(\Phi)) \leq 1$, the Jacobian of C_Φ has Kodaira symbol I_0 or I_1. We are free to replace K by an unramified extension. So by [11, Theorem 7.1] we may assume that $C_\Phi(K) \neq \emptyset$ and likewise for Φ'. In Sections 5, 6 and 7 we show that, since Φ and Φ' are minimal, C_Φ and $C_{\Phi'}$ are geometrically minimal. Theorem 4.5 then shows that $B \in K^\times \mathrm{GL}_5(\mathcal{O}_K)$ and we are done by Remark 3.3. □

5. Minimal Pfaffian models are flat

Let $\Phi \in X_5(\mathcal{O}_K)$ with reduction $\phi \in X_5(k)$. In this section we show that if Φ is minimal then $\mathcal{C}_\Phi$ is a fibred surface.

Lemma 5.1. *If $\Phi \in X_5(\mathcal{O}_K)$ is non-singular then the following are equivalent.*

(i) *$\mathcal{C}_\Phi$ is the closure of C_Φ in $\mathbb{P}^4_S$.*
(ii) *$\mathcal{C}_\Phi$ is a fibred surface.*
(iii) *C_ϕ is a curve.*

Proof: (i) ⇒ (ii) ⇒ (iii). See Lemma 4.2 and [15, Corollary 4.3.14].
(iii) ⇒ (i). Let $R = k[x_1, \ldots, x_5]$. With notation as in Section 2, there is a complex of graded free R-modules

$$0 \longrightarrow R(-5) \xrightarrow{\mathrm{Pf}(\phi)^T} R(-3)^5 \xrightarrow{\phi} R(-2)^5 \xrightarrow{\mathrm{Pf}(\phi)} R. \tag{4}$$

Since C_ϕ is a curve, this complex is exact by the Buchsbaum-Eisenbud acyclicity criterion [8, Theorem 20.9].

Let $\mathrm{Pf}(\Phi) = (p_1, \ldots, p_5)$. Let $\mathcal{I}$ be the ideal in $\mathcal{R} = \mathcal{O}_K[x_1, \ldots, x_5]$ generated by $p_1, \ldots, p_5$. We must show that if $f \in \mathcal{R}$ and $\pi f \in \mathcal{I}$ then $f \in \mathcal{I}$. We write $\pi f = \sum_{i=1}^5 f_i p_i$ for some $f_1, \ldots, f_5 \in \mathcal{R}$. Then $\sum_{i=1}^5 f_i p_i \equiv 0 \pmod{\pi}$. Since (4) is exact it follows that $f_i = \pi g_i + \sum_{j=1}^5 \Phi_{ij} h_j$ for some $g_1, \ldots, g_5, h_1, \ldots h_5 \in \mathcal{R}$. Then $\pi f = \sum_{i=1}^5 f_i p_i = \pi \sum_{i=1}^5 g_i p_i$ and so $f \in \mathcal{I}$ as required. □

Lemma 5.2. *If $\Phi \in X_5(\mathcal{O}_K)$ is minimal then*

(i) *The 4×4 Pfaffians of ϕ are linearly independent.*
(ii) *The subscheme $C_\phi \subset \mathbb{P}^4_k$ does not contain a plane.*
(iii) *The entries of ϕ span $\langle x_1, \ldots, x_5 \rangle$.*

Proof: This is [11, Lemma 7.8]. □

Lemma 5.3. *If $\phi \in X_5(k)$ satisfies conditions (i) and (ii) in Lemma 5.2 then C_ϕ is a curve.*

PROOF: By [9, Lemma 5.8] every irreducible component of C_ϕ has dimension at least 1. We must show there are no components of dimension 2 or more. Let $\operatorname{Sing} C_\phi$ be the set of points of C_ϕ with tangent space of dimension at least 2. This contains all components of C_ϕ of dimension 2 or more. If $\operatorname{Sing} C_\phi$ is contained in a line then we are done. So suppose $P_1, P_2, P_3 \in \operatorname{Sing} C_\phi$ span a plane Π. If C_ϕ contains each of the lines P_iP_j then it must contain Π, since C_ϕ is defined by quadrics. But this is impossible by (ii). We may therefore suppose $P_1P_2 \not\subset C_\phi$.

A change of co-ordinates gives $P_1 = (1 : 0 : \ldots : 0)$ and $P_2 = (0 : 1 : \ldots : 0)$. If we write $\phi = \sum x_i M_i$ then M_1 and M_2 have rank 2, but their sum has rank 4. Therefore ϕ is equivalent to a model with $\phi_{12} = x_1$, $\phi_{34} = x_2$ and all other ϕ_{ij} (for $i < j$) linear forms in x_3, x_4, x_5. Since $P_1, P_2 \in \operatorname{Sing} C_\phi$ it follows that ϕ_{35} and ϕ_{45} are linearly dependent, and ϕ_{15} and ϕ_{25} are linearly dependent. Therefore the space of linear forms spanned by the entries of the last row/column of ϕ has dimension at most 2. Replacing ϕ by a k-equivalent model brings us to the case

$$\phi = \begin{pmatrix} 0 & \xi & \alpha_1 & \alpha_2 & \alpha_3 \\ & 0 & \beta_1 & \beta_2 & \beta_3 \\ & & 0 & \eta & 0 \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

where $\xi, \eta, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are linear forms in $x_1, \ldots, x_5$. By (i) the linear forms α_3 and β_3 are linearly independent, and $\eta \neq 0$. Therefore C_ϕ is the union of

$$\Gamma_2 = \{\alpha_3 = \beta_3 = \xi\eta - \alpha_1\beta_2 + \alpha_2\beta_1 = 0\}$$

and

$$\Gamma_3 = \left\{ \operatorname{rank} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix} \leq 1 \right\} \cap \{\eta = 0\}.$$

We may think of Γ_2 as a degenerate conic, and Γ_3 as a degenerate twisted cubic. It remains to show that these degenerations are still curves. In the case of Γ_2 this is clear by (ii). In the case of Γ_3 we use the following lemma. The conditions of the lemma are satisfied by (i) and (ii). □

Lemma 5.4. *Let ψ be a 2×3 matrix of linear forms in $x_1, \ldots, x_4$. Let $\Gamma_3 \subset \mathbb{P}^3$ be defined by $\operatorname{rank} \psi \leq 1$. Suppose that*

(i) *The* 2×2 *minors of* ψ *span a vector space of dimension at least* 2.
(ii) *The subscheme* $\Gamma_3 \subset \mathbb{P}^3$ *does not contain a plane.*

Then Γ_3 *is a curve.*

PROOF: Since Γ_3 is defined by quadrics, any irreducible component of dimension 2 would have degree 1 or 2. These possibilities are ruled out by (ii) and (i). □

Theorem 5.5. *If* $\Phi \in X_5(\mathcal{O}_K)$ *is minimal then* $\mathcal{C}_\Phi$ *is a fibred surface.*

PROOF: This is immediate from Lemmas 5.1, 5.2 and 5.3. □

6. Minimal Pfaffian models are normal

We have seen that if $\Phi \in X_5(\mathcal{O}_K)$ is minimal then $\mathcal{C}_\Phi$ is a fibred surface. In this section we show that $\mathcal{C}_\Phi$ is normal. If $v(\Delta(\Phi)) \leq 1$ then Lemma 3.6 already shows that $\mathcal{C}_\Phi$ is regular, and hence normal. To treat the general case we check the conditions in Lemma 4.2.

Lemma 6.1. *If* $\Phi \in X_5(\mathcal{O}_K)$ *is minimal then*

(i) $\mathcal{C}_\Phi$ *is a local complete intersection,*
(ii) $\mathcal{C}_\Phi$ *is Cohen-Macaulay.*

PROOF: (i) Since $\mathcal{C}_\Phi \subset \mathbb{P}^4_S$ has codimension 3 we must show it is locally defined by 3 equations. Let $\mathrm{Pf}(\Phi) = (p_1, \ldots, p_5)$. Since Φ is alternating, the relations $\sum_{i=1}^5 p_i \Phi_{ij} = 0$ for $j = 4, 5$, show that the intersection $\mathcal{C}_\Phi \cap \{\Phi_{45} \neq 0\}$ is defined by $p_1 = p_2 = p_3 = 0$. By Lemma 5.2(iii) the affine pieces $\{\Phi_{ij} \neq 0\}$ cover $\mathbb{P}^4_S$.
(ii) This follows from (i) and [15, Corollary 8.2.18]. □

We prepare to check the second condition in Lemma 4.2. Recall that we assume k is algebraically closed.

Lemma 6.2. *Let* $\phi \in X_5(k)$ *satisfy the conditions of Lemma 5.2. Suppose* C_ϕ *has a multiple component* Γ. *Then after replacing* ϕ *by a* k*-equivalent model, we are in one of the following cases*

$$(i) \quad \begin{pmatrix} 0 & x_1 & x_2 & * & * \\ & 0 & * & * & 0 \\ & & 0 & * & 0 \\ & - & & 0 & x_5 \\ & & & & 0 \end{pmatrix} \qquad (ii) \quad \begin{pmatrix} 0 & x_1 & x_2 & x_3 & 0 \\ & 0 & x_3 & x_4 & 0 \\ & & 0 & 0 & x_4 \\ & - & & 0 & x_5 \\ & & & & 0 \end{pmatrix}$$

$$(iii)\quad \begin{pmatrix} 0 & x_1 & x_2 & * & * \\ & 0 & * & * & * \\ & & 0 & * & * \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix} \qquad (iv)\quad \begin{pmatrix} 0 & 0 & x_1 & x_2 & x_4 \\ & 0 & x_2 & x_3 & x_5 \\ & & 0 & x_5 & 0 \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

where the entries $*$ are linear forms in x_3, x_4, x_5. Moreover $\Gamma = \{x_3 = x_4 = x_5 = 0\}$ in cases (i),(ii),(iii), and $\Gamma = \{x_1x_3 - x_2^2 = x_4 = x_5 = 0\}$ in case (iv).

Proof: Lemma 5.3 shows that C_ϕ is a curve and so the complex (4) is exact. From this minimal free resolution we compute that C_ϕ has Hilbert polynomial

$$h(t) = \binom{t+4}{4} - 5\binom{t+2}{4} + 5\binom{t+1}{4} - \binom{t-1}{4} = 5t.$$

In particular $C_\phi \subset \mathbb{P}^4$ has degree 5. The multiple component Γ must therefore be a line or a conic.

Case Γ is a line. We may assume $\Gamma = \{x_3 = x_4 = x_5 = 0\}$. Then $\phi = \sum x_i M_i$ where all linear combinations of M_1 and M_2 have rank at most 2. By hypothesis, $M_1, \ldots, M_5$ are linearly independent. So we are either in case (iii), or ϕ takes the form

$$\begin{pmatrix} 0 & x_1 & x_2 & * & * \\ & 0 & * & \alpha & \beta \\ & & 0 & \gamma & \delta \\ & - & & 0 & x_5 \\ & & & & 0 \end{pmatrix}$$

where the entries $\alpha, \beta, \gamma, \delta$ and $*$ are linear forms in x_3, x_4, x_5. By row and column operations (and substitutions for x_1 and x_2) we may suppose $\alpha, \beta, \gamma, \delta$ do not involve x_5. We write $\alpha = \alpha_3 x_3 + \alpha_4 x_4$ and likewise for β, γ, δ. As shown in [11, Section 4], Γ is a multiple component if and only if the determinant of

$$\begin{pmatrix} \gamma_3 s - \alpha_3 t & \gamma_4 s - \alpha_4 t \\ \delta_3 s - \beta_3 t & \delta_4 s - \beta_4 t \end{pmatrix}$$

vanishes as a polynomial in s and t. If the rows of this matrix are linearly dependent (over k) then we may reduce to case (i). Otherwise the columns are linearly dependent, and we may reduce to the case $\alpha_3 = \beta_3 = \gamma_3 = \delta_3 = 0$ yet $\alpha_4\delta_4 - \beta_4\gamma_4 \neq 0$. Since C_ϕ does not contain the plane $\{x_4 = x_5 = 0\}$ it follows

that ϕ_{23}, and at least one of ϕ_{14} and ϕ_{15}, involves x_3. By a substitution for x_3 we may assume $\phi_{23} = x_3$. By row and columns operations (and substitutions for x_1 and x_2) we may assume ϕ_{14} and ϕ_{15} are multiples of x_3. Replacing the 4th and 5th rows/columns by suitable linear combinations, and likewise for the 2nd and 3rd rows/columns, brings us to case (ii).

Case Γ is a conic. We may assume $\Gamma = \{x_1x_3 - x_2^2 = x_4 = x_5 = 0\}$. Let $\mathrm{Pf}(\phi) = (p_1, \ldots, p_5)$. Replacing ϕ by an equivalent model we have $p_i(x_1, x_2, x_3, 0, 0) = 0$ for $i = 1, 2, 3, 4$ and $p_5(x_1, x_2, x_3, 0, 0) = x_1x_3 - x_2^2$. Since $\mathrm{Pf}(\phi)\phi = 0$ and C_ϕ is a curve, we may suppose the last row/column of ϕ has entries $x_4, x_5, 0, 0, 0$. As shown in [11, Section 4], Γ is a multiple component if and only if $\phi_{34}(x_1, x_2, x_3, 0, 0) = 0$. In this case ϕ is equivalent to a model of the form

$$\begin{pmatrix} 0 & \xi & x_1 + \langle x_4, x_5\rangle & x_2 + \langle x_4, x_5\rangle & x_4 \\ & 0 & x_2 + \langle x_4, x_5\rangle & x_3 + \langle x_4, x_5\rangle & x_5 \\ & & 0 & \langle x_4, x_5\rangle & 0 \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

where each $\langle x_4, x_5\rangle$ denotes some linear combination of x_4 and x_5. Subtracting multiples of the last three rows/columns from the first two row/columns we may suppose $\xi = 0$. Since the 4×4 Pfaffians of ϕ are linearly independent we cannot have $\phi_{34} = 0$. So making substitutions for x_4 and x_5 brings us to the case

$$\begin{pmatrix} 0 & 0 & x_1 + \langle x_4, x_5\rangle & x_2 + \langle x_4, x_5\rangle & \langle x_4, x_5\rangle \\ & 0 & x_2 + \langle x_4, x_5\rangle & x_3 + \langle x_4, x_5\rangle & \langle x_4, x_5\rangle \\ & & 0 & x_5 & 0 \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix}.$$

If $P \in \mathrm{GL}_2(k)$ then

$$P \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix} P^T = \begin{pmatrix} x_1' & x_2' \\ x_2' & x_3' \end{pmatrix}$$

where x_1', x_2', x_3' are linear combinations of x_1, x_2, x_3. Acting on ϕ by a matrix of the form $\mathrm{Diag}(P, P, 1)$ we may therefore reduce to the case $\phi_{15} = x_4$ and $\phi_{25} = x_5$. Subtracting multiples of the 5th row/column from the 3rd and 4th rows/columns

we may assume $\phi_{14} = \phi_{23}$. Then making substitutions for x_1, x_2, x_3 brings us to case (iv). □

The following lemma and its proof could also be used to extend the algorithms for testing local solubility in [13] to genus one curves of degree 5.

Lemma 6.3. *If $\Phi \in X_5(\mathcal{O}_K)$ is minimal then each multiple component Γ of the special fibre C_ϕ has at most three non-regular points.*

PROOF: We split into the four cases in Lemma 6.2.
(i) We put

$$\Phi_{25} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}$$
$$\Phi_{35} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \ldots) \pmod{\pi^2}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$. We find that $(s : t : 0 : 0 : 0) \in \Gamma$ is a non-regular point if and only if the linear form $s\phi_{34} - t\phi_{24}$ vanishes, or

$$\beta_1 s^2 + (\alpha_1 - \beta_2)st - \alpha_2 t^2 = 0.$$

If $\phi_{24} = \phi_{34} = 0$ then C_ϕ is not a curve. If the quadratic form in s and t vanishes identically then, after subtracting a multiple of the 1st row/column from the 5th row/column, we may assume $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0$. Since $\phi_{45} = x_5$ we may assume by a substitution for x_5 that $\Phi_{45} = x_5$. Then the transformation

$$[\mathrm{Diag}(\pi, 1, 1, 1, \pi^{-1}), \pi^{-1}\,\mathrm{Diag}(1, 1, \pi, \pi, \pi^2)]$$

shows that Φ is not minimal.
(ii) We put

$$\Phi_{24} \equiv x_4 + \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}$$
$$\Phi_{25} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \ldots) \pmod{\pi^2}$$
$$\Phi_{34} \equiv \pi(\gamma_1 x_1 + \gamma_2 x_2 + \ldots) \pmod{\pi^2}$$
$$\Phi_{35} \equiv x_4 + \pi(\delta_1 x_1 + \delta_2 x_2 + \ldots) \pmod{\pi^2}$$

We find that $(s : t : 0 : 0 : 0) \in \Gamma$ is a non-regular point if and only if

$$\gamma_1 s^3 + (\alpha_1 - \gamma_2 - \delta_1)s^2 t - (\alpha_2 + \beta_1 - \delta_2)st^2 + \beta_2 t^3 = 0.$$

Making a substitution for x_4 we may assume $\alpha_2 = \delta_1 = 0$. Subtracting a multiple of the 1st row/column from the 5th row/column we may assume $\alpha_1 = \delta_2 = 0$. If the cubic form in s and t vanishes identically then $\beta_1 = \beta_2 = \gamma_1 = \gamma_2 = 0$. Since $\phi_{45} = x_5$ we may assume by a substitution for x_5 that $\Phi_{45} = x_5$. Then the transformation

$$[\mathrm{Diag}(\pi, 1, 1, \pi^{-1}, \pi^{-1}), \pi^{-1}\,\mathrm{Diag}(1, 1, \pi, \pi^2, \pi^3)]$$

shows that Φ is not minimal.
(iii) We put

$$\Phi_{45} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \ldots) \pmod{\pi^2}.$$

We find that $(s : t : 0 : 0 : 0) \in \Gamma$ is a non-regular point if and only if $s\phi_{34} - t\phi_{24}$ and $s\phi_{35} - t\phi_{25}$ are linearly dependent, or $\alpha_1 s + \alpha_2 t = 0$. If the first of these possibilities is true for all s and t, then C_ϕ is not a curve. If $\alpha_1 = \alpha_2 = 0$ then the transformation

$$[\mathrm{Diag}(1, 1, 1, \pi^{-1}, \pi^{-1}), \mathrm{Diag}(1, 1, \pi, \pi, \pi)]$$

shows that Φ is not minimal.
(iv) We put

$$\Phi_{35} \equiv \pi(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \ldots) \pmod{\pi^2}$$
$$\Phi_{45} \equiv \pi(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \ldots) \pmod{\pi^2}$$

We find that $(s^2 : st : t^2 : 0 : 0) \in \Gamma$ is a non-regular point if and only if

$$\beta_1 s^3 + (\beta_2 - \alpha_1)s^2 t + (\beta_3 - \alpha_2)st^2 - \alpha_3 t^3 = 0.$$

Subtracting a multiple of the first two rows/columns from the last row/column we may assume $\alpha_1 = \alpha_2 = 0$. If the cubic form in s and t vanishes identically then $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3$. Then the transformation

$$[\mathrm{Diag}(\pi, \pi, 1, 1, \pi^{-1}), \pi^{-1}\,\mathrm{Diag}(1, 1, 1, \pi, \pi)]$$

shows that Φ is not minimal. □

Theorem 6.4. *If $\Phi \in X_5(\mathcal{O}_K)$ is minimal then $\mathcal{C}_\Phi$ is a normal fibred surface.*

Proof: In Section 5 we showed that $\mathcal{C}_\Phi \subset \mathbb{P}^4_S$ is the closure of C_Φ and hence a fibred surface. The conditions for normality in Lemma 4.2 were checked in Lemmas 6.1 and 6.3. □

7. Minimal Pfaffian models are geometrically minimal

In this section we show that if $\Phi \in X_5(\mathcal{O}_K)$ is minimal and $C_\Phi(K) \neq \emptyset$ then $C_\Phi \subset \mathbb{P}^4_K$ is geometrically minimal. This extends [17, Theorem 4.1] from genus one curves of degrees 2, 3 and 4, to degree 5, and could also be used to prove results analogous to those in [16].

Definition 7.1. Let E/K be an elliptic curve with minimal Weierstrass equation $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$. Then

$$\omega_E = \frac{dx}{2y + a_1x + a_3} \in H^0(K, \Omega^1_{E/K})$$

is called a *Néron differential* on E. It is uniquely determined up to multiplication by elements of $\mathcal{O}_K^\times$.

Let $\mathcal{C}$ be a fibred surface over $S = \operatorname{Spec} \mathcal{O}_K$. If $\mathcal{C}/S$ is a local complete intersection then we can define the canonical sheaf $\omega_{\mathcal{C}/S}$ as in [15, Definition 6.4.7]. This is an invertible sheaf on $\mathcal{C}$. If $\mathcal{C}$ has generic fibre E then $H^0(\mathcal{C}, \omega_{\mathcal{C}/S})$ is a sub-$\mathcal{O}_K$-module of the 1-dimensional K-vector space $H^0(E, \Omega^1_{E/K})$.

The following theorem and its proof is closely related to [15, Theorem 9.4.35]. See also [7].

Theorem 7.2. *Let E/K be an elliptic curve, with minimal proper regular model $\mathcal{E}/S$. Let $\mathcal{C}/S$ be a normal fibred surface with generic fibre isomorphic to E, and minimal desingularisation $\widetilde{\mathcal{C}}$. Suppose $\mathcal{C}$ is a local complete intersection and $\omega_{\mathcal{C}/S} = \omega\mathcal{O}_{\mathcal{C}}$ for some $\omega \in H^0(E, \Omega^1_{E/K})$. The following are equivalent.*

(i) *We have $\omega_{\mathcal{C}/S} = \omega_E \mathcal{O}_{\mathcal{C}}$ where ω_E is a Néron differential on E.*
(ii) *The morphism $\widetilde{\mathcal{C}} \to \mathcal{E}$ (which exists by definition of $\mathcal{E}$) is an isomorphism.*

Proof: (i) $\Rightarrow$ (ii). Let $f : \widetilde{\mathcal{C}} \to \mathcal{E}$ be the morphism in (ii) and $g : \widetilde{\mathcal{C}} \to \mathcal{C}$ the minimal desingularisation. We are assuming that $\omega_{\mathcal{C}/S} = \omega_E \mathcal{O}_{\mathcal{C}}$, whereas [15, Theorem 9.4.35] gives that $\omega_{\mathcal{E}/S} = \omega_E \mathcal{O}_{\mathcal{E}}$. Therefore

$$f^*\omega_{\mathcal{E}/S} \cong \omega_E \mathcal{O}_{\widetilde{\mathcal{C}}} \cong g^*\omega_{\mathcal{C}/S}. \tag{5}$$

Let Γ be an exceptional divisor (or (-1)-curve) on $\widetilde{\mathcal{C}}$. Since the desingularisation $g : \widetilde{\mathcal{C}} \to \mathcal{C}$ is minimal, it does not contract Γ. Therefore

$$\omega_{\widetilde{\mathcal{C}}/S}|_\Gamma \cong g^*\omega_{\mathcal{C}/S}|_\Gamma$$

By [15, Corollary 9.3.27] we know that $\omega_{\mathcal{E}/S}$ is globally free. Therefore each of the sheaves in (5) is globally free. Writing $K_{\widetilde{\mathcal{C}}/S}$ for a canonical divisor on $\widetilde{\mathcal{C}}/S$ we have

$$K_{\widetilde{\mathcal{C}}/S} \cdot \Gamma = \deg(\omega_{\widetilde{\mathcal{C}}/S}|_\Gamma) = \deg(g^*\omega_{\mathcal{C}/S}|_\Gamma) = 0.$$

On the other hand [15, Proposition 9.3.10] shows that $K_{\widetilde{\mathcal{C}}/S} \cdot \Gamma < 0$. This is a contradiction. We deduce that $\widetilde{\mathcal{C}}$ has no exceptional divisors. It follows by the factorisation theorem [15, Theorem 9.2.2] that $f : \widetilde{\mathcal{C}} \to \mathcal{E}$ is an isomorphism.
(ii) $\Rightarrow$ (i). Let F be the exceptional locus of the minimal desingularisation $g : \widetilde{\mathcal{C}} \to \mathcal{C}$. Then

$$H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) \subset H^0(\widetilde{\mathcal{C}} \setminus F, \omega_{\widetilde{\mathcal{C}}/S}) = H^0(\mathcal{C} \setminus g(F), \omega_{\mathcal{C}/S}) = H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) \tag{6}$$

where the last equality uses that $\mathcal{C}$ is normal: see [15, Lemma 9.2.17]. We are assuming that $\widetilde{\mathcal{C}} \cong \mathcal{E}$ and $\omega_{\mathcal{C}/S} = \omega\mathcal{O}_{\mathcal{C}}$. Therefore $H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) = \omega_E \mathcal{O}_K$ and $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = \omega\mathcal{O}_K$. The inclusion (6) shows that $\omega_E = h\omega$ for some $h \in \mathcal{O}_K$.

Since the sheaves $\omega_{\widetilde{\mathcal{C}}/S}$ and $g^*\omega_{\mathcal{C}/S}$ are identical on $\widetilde{\mathcal{C}} \setminus F$, the divisor $\mathrm{div}(h)$ on $\widetilde{\mathcal{C}} \cong \mathcal{E}$ is a sum of irreducible components of F. On the other hand, $\mathrm{div}(h)$ is a multiple of the special fibre. Since not all of the irreducible components of the special fibre are contracted, it follows that $h \in \mathcal{O}_K^\times$ as required. □

Remark 7.3. Following the proof of [15, Theorem 9.4.35(a)], we have the following alternative proof of "(ii) ⇒ (i)". Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of the special fibre that are contracted by $g : \widetilde{\mathcal{C}} \to \mathcal{C}$. By [15, Theorem 9.1.27] the intersection matrix $(\Gamma_i \cdot \Gamma_j)$ is negative definite. Since $\widetilde{\mathcal{C}} \cong \mathcal{E}$ is minimal, an argument using Castelnuovo's criterion and the adjunction formula (see [15, Example 9.4.19] or [18, Chapter IV, Theorem 8.1(b)]) shows that $K_{\widetilde{\mathcal{C}}/S} \cdot \Gamma_i = 0$ for all i. Therefore the contraction morphism $g : \widetilde{\mathcal{C}} \to \mathcal{C}$ satisfies the hypotheses of [15, Corollary 9.4.18]. As a consequence $g_*\omega_{\widetilde{\mathcal{C}}/S} = \omega_{\mathcal{C}/S}$ and $g^*\omega_{\mathcal{C}/S} = \omega_{\widetilde{\mathcal{C}}/S}$. Therefore $H^0(\mathcal{C}, \omega_{\mathcal{C}/S}) = H^0(\widetilde{\mathcal{C}}, \omega_{\widetilde{\mathcal{C}}/S}) = H^0(\mathcal{E}, \omega_{\mathcal{E}/S}) = \omega_E \mathcal{O}_K$.

Theorem 7.4. *Let $\Phi \in X_5(\mathcal{O}_K)$ be non-singular with reduction $\phi \in X_5(k)$. Suppose $\mathcal{C} = \mathcal{C}_\Phi$ is a fibred surface, and the entries of ϕ span $\langle x_1, \ldots, x_5 \rangle$. Then $\mathcal{C}$ is a local complete intersection with $\omega_{\mathcal{C}/S} = \omega_\Phi \mathcal{O}_\mathcal{C}$ where ω_Φ is defined by (3).*

Proof: Exactly as in the proof of Lemma 6.1, the affine piece $\mathcal{C} \cap \{\Phi_{45} \neq 0\}$ is defined by $p_1 = p_2 = p_3 = 0$. The restriction of the canonical sheaf to this affine piece is as claimed by [15, Corollary 6.4.14] and the next lemma. Since the definition (3) of ω_Φ is invariant under all even permutations of the subscripts, and the affine pieces $\{\Phi_{ij} \neq 0\}$ cover $\mathbb{P}^4_S$, the theorem follows. □

Lemma 7.5. *Let R be any ring. Let $\phi \in X_5(R)$ with $\mathrm{Pf}(\phi) = (p_1, \ldots, p_5)$. Let I be the ideal in $R[x_1, \ldots, x_5]$ generated by $p_1, \ldots, p_5$. Then*

$$\frac{\partial(p_1, p_2, p_3)}{\partial(x_1, x_2, x_3)} \equiv \phi_{45} \sum_{i,j=1}^{5} \frac{\partial p_i}{\partial x_1} \frac{\partial \phi_{ij}}{\partial x_2} \frac{\partial p_j}{\partial x_3} \pmod{I}. \tag{7}$$

Proof: We have $\sum_{i=1}^5 p_i \phi_{ij} = 0$ for all $1 \leq j \leq 5$. Differentiating with respect to x_k, and working mod I, this gives

$$\sum_{i=1}^{5} \frac{\partial p_i}{\partial x_k} \phi_{ij} = 0. \tag{8}$$

Using first that ϕ is alternating, and then (8), we compute

$$\begin{aligned}\phi_{45}\sum_{i=1}^{3}\sum_{j=4}^{5}\frac{\partial p_i}{\partial x_1}\frac{\partial \phi_{ij}}{\partial x_2}\frac{\partial p_j}{\partial x_3} &= \sum_{i=1}^{3}\sum_{j=4}^{5}\frac{\partial p_i}{\partial x_1}\left(\frac{\partial \phi_{i4}}{\partial x_2}\phi_{j5}-\frac{\partial \phi_{i5}}{\partial x_2}\phi_{j4}\right)\frac{\partial p_j}{\partial x_3}\\ &= -\sum_{i,j=1}^{3}\frac{\partial p_i}{\partial x_1}\left(\frac{\partial \phi_{i4}}{\partial x_2}\phi_{j5}-\frac{\partial \phi_{i5}}{\partial x_2}\phi_{j4}\right)\frac{\partial p_j}{\partial x_3}.\end{aligned}$$

Subtracting the same identity with $\frac{\partial}{\partial x_1}$ and $\frac{\partial}{\partial x_3}$ switched gives

$$\phi_{45}\sum_{i=1}^{3}\sum_{j=4}^{5}\frac{\partial \phi_{ij}}{\partial x_2}\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)} = \sum_{i<j}\frac{\partial}{\partial x_2}(-\phi_{i4}\phi_{j5}+\phi_{i5}\phi_{j4})\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)} \tag{9}$$

where we write $\sum_{i<j}$ for the sum over all $1\le i<j\le 3$. Again using (8),

$$\sum_{i,j=4}^{5}\frac{\partial p_i}{\partial x_1}\phi_{ij}\frac{\partial p_j}{\partial x_3} = -\sum_{i=4}^{5}\sum_{j=1}^{3}\frac{\partial p_i}{\partial x_1}\phi_{ij}\frac{\partial p_j}{\partial x_3} = \sum_{i,j=1}^{3}\frac{\partial p_i}{\partial x_1}\phi_{ij}\frac{\partial p_j}{\partial x_3}.$$

Therefore

$$\phi_{45}\frac{\partial(p_4,p_5)}{\partial(x_1,x_3)} = \sum_{i<j}\phi_{ij}\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)}. \tag{10}$$

We break up the sum on the right of (7) as

$$\sum_{i<j}\frac{\partial \phi_{ij}}{\partial x_2}\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)}+\sum_{i=1}^{3}\sum_{j=4}^{5}\frac{\partial \phi_{ij}}{\partial x_2}\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)}+\frac{\partial \phi_{45}}{\partial x_2}\frac{\partial(p_4,p_5)}{\partial(x_1,x_3)}.$$

Then by (9) and (10), the right hand side of (7) is

$$\sum_{i<j}\frac{\partial}{\partial x_2}(\phi_{ij}\phi_{45}-\phi_{i4}\phi_{j5}+\phi_{i5}\phi_{j4})\frac{\partial(p_i,p_j)}{\partial(x_1,x_3)}.$$

Since for i,j,k an even permutation of $1,2,3$ we have $-p_k=\phi_{ij}\phi_{45}-\phi_{i4}\phi_{j5}+\phi_{i5}\phi_{j4}$ the result follows. □

Remark 7.6. We keep the notation of the lemma. Differentiating the relation $\sum_{j=1}^{5}\phi_{ij}p_j=0$ with respect to x_4 and x_5 we have

$$\sum_{j=1}^{5}\frac{\partial \phi_{ij}}{\partial x_4}\frac{\partial p_j}{\partial x_5}+\sum_{j=1}^{5}\frac{\partial \phi_{ij}}{\partial x_5}\frac{\partial p_j}{\partial x_4}+\sum_{j=1}^{5}\phi_{ij}\frac{\partial^2 p_j}{\partial x_4\partial x_5}=0.$$

We multiply by $\frac{\partial p_i}{\partial x_4}$ and sum over i. By (8) and the fact ϕ is alternating, the second two terms vanish mod I. Therefore

$$\sum_{i,j=1}^{5} \frac{\partial p_i}{\partial x_4}\frac{\partial \phi_{ij}}{\partial x_4}\frac{\partial p_j}{\partial x_5} \equiv 0 \pmod{I}.$$

This shows that the restriction $\operatorname{char} K \neq 2$ in [9, Section 5.4] is not needed.

Lemma 7.7. *Let $\Phi \in X_5(K)$ be non-singular with $C_\Phi(K) \neq \emptyset$. Then Φ has level 0 if and only if ω_Φ is a Néron differential on $C_\Phi \cong E$.*

Proof: Let E/K have minimal Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The complete linear system $|4.0_E|$ defines a morphism $\alpha : E \to \mathbb{P}^3$. It is given by $(x, y) \mapsto (1 : x : y : x^2)$. The image is $C_4 = \{Q_1 = Q_2 = 0\} \subset \mathbb{P}^3$ where

$$\begin{aligned} Q_1 &= x_1x_4 - x_2^2, \\ Q_2 &= x_3^2 + a_1x_2x_3 + a_3x_1x_3 - x_2x_4 - a_2x_2^2 - a_4x_1x_2 - a_6x_1^2, \end{aligned}$$

and an invariant differential ω_4 on C_4 is given by

$$\omega_4 = \frac{x_1^2 d(x_2/x_1)}{\frac{\partial Q_1}{\partial x_4}\frac{\partial Q_2}{\partial x_3} - \frac{\partial Q_1}{\partial x_3}\frac{\partial Q_2}{\partial x_4}}.$$

We claim that (i) $\Delta(Q_1, Q_2) = \Delta_E$ and (ii) ω_4 is a Néron differential on $C_4 \cong E$. Indeed the invariants were scaled in [9] so that (i) is true, whereas for (ii) it is easy to see that $\alpha^*\omega_4 = dx/(2y + a_1x + a_3)$.

Since $C_\Phi(K) \neq \emptyset$ we may identify $C_\Phi \cong E$. The hyperplane section is linearly equivalent to $4.0_E + P$ for some $P \in E(K)$. Let $\Psi \in X_5(K)$ be the Pfaffian model constructed from the quadric intersection (Q_1, Q_2) by "unprojection centred at P" as described in [11, Lemma 2.3]. By [11, Lemma 2.4] and its proof, we have (i) $\Delta(\Psi) = \Delta_E$ and (ii) ω_Ψ is a Néron differential on $C_\Psi \cong E$.

The curves C_Φ and C_Ψ differ by a change of co-ordinates defined over K. So by [10, Theorem 4.1(ii)], the Pfaffian models Φ and Ψ are K-equivalent, say $\Phi = g\Psi$ for some $g \in \mathrm{GL}_5(K) \times \mathrm{GL}_5(K)$. Since Δ is an invariant of weight 12 we have $\Delta(\Phi) = (\det g)^{12}\Delta(\Psi)$. Let $\gamma : C_\Phi \to C_\Psi$ be the isomorphism described by g. By [9, Proposition 5.19] we have $\gamma^*\omega_\Psi = (\det g)\omega_\Phi$. Therefore both the conditions in the statement of the lemma are equivalent to $v(\det g) = 0$. □

Remark 7.8. If $\operatorname{char} K \neq 2, 3$ then [9, Proposition 5.23] shows that (C_Φ, ω_Φ) and (E, ω) are isomorphic over $\overline{K}$, where E is the elliptic curve (2) and $\omega = dx/(2y + a_1(\Phi)x + a_3(\Phi))$. This gives an alternative proof of Lemma 7.7. The isomorphism

$C_\Phi \cong E$ might not be defined over K, but differs from an isomorphism that is defined over K by an automorphism of the curve E. The latter might rescale ω by a root of unity, but won't change whether it is a Néron differential.

Theorem 7.9. *Let $\Phi \in X_5(\mathcal{O}_K)$ be non-singular with $C_\Phi(K) \neq \emptyset$. Suppose $\mathcal{C}_\Phi$ is a fibred surface, and the entries of ϕ span $\langle x_1, \ldots, x_5\rangle$. Then Φ is minimal if and only if C_Φ is geometrically minimal.*

Proof: Lemma 5.1 shows that $\mathcal{C} = \mathcal{C}_\Phi$ is the closure of C_Φ in $\mathbb{P}^4_S$. By either Definition 4.3 or Theorem 6.4 we may suppose $\mathcal{C}$ is normal. Let E be the Jacobian of C_Φ. Since $C_\Phi(K) \neq \emptyset$ we have $C_\Phi \cong E$. Theorem 3.1 and Lemma 7.7 show that Φ is minimal if and only if ω_Φ is a Néron differential on $C_\Phi \cong E$. The theorem now follows from Theorems 7.2 and 7.4. □

By Lemma 5.2, Theorem 5.5 and Theorem 7.9 we have

Corollary 7.10. *If $\Phi \in X_5(\mathcal{O}_K)$ is minimal and $C_\Phi(K) \neq \emptyset$ then C_Φ is geometrically minimal.*

8. An alternative proof of Theorem 1.1

We give a short alternative proof of Theorem 1.1, that avoids using schemes, except for the definition of a regular point. It would however be rather hard to motivate this proof without the work in earlier sections.

By putting the matrices $A, B \in \mathrm{GL}_5(K)$ in Smith normal form (and making use of Remark 3.3), Theorem 1.1 is equivalent to the following.

Theorem 8.1. *Let $\Phi, \Phi' \in X_5(\mathcal{O}_K)$ with $v(\Delta(\Phi)) \leq 1$ and $v(\Delta(\Phi')) \leq 1$. If*

$$\Phi' = [\mathrm{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \mathrm{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})]\Phi$$

for some $r_1, \ldots, r_5, s_1, \ldots, s_5 \in \mathbb{Z}$ then $s_1 = s_2 = \ldots = s_5$.

For the proof we may assume the residue field k is algebraically closed. As before we write $\phi \in X_5(k)$ for the reduction of Φ mod π. For the purposes of this section, a k-point P on C_ϕ is *regular* if it satisfies the conditions in Lemma 3.5, and otherwise *non-regular*. Since $\dim \mathcal{C}_\Phi = 2$ this agrees with the standard terminology, but we don't need to know this.

Lemma 8.2. *If $v(\Delta(\Phi)) \leq 1$ then C_ϕ contains no lines or conics.*

PROOF: If C_ϕ contains a line or conic then, arguing as in the proof of Lemma 6.2, we may assume

$$\phi = \begin{pmatrix} 0 & x_1 & x_2 & * & * \\ & 0 & * & * & * \\ & & 0 & * & * \\ & - & & 0 & * \\ & & & & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & * & * & * \\ & 0 & * & * & * \\ & & 0 & * & 0 \\ & - & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

where the entries $*$ on the left are linear forms in x_3, x_4, x_5, and on the right are linear forms in $x_1, \ldots, x_5$. In the first case we apply the transformation

$$[\mathrm{Diag}(\pi, 1, 1, 1, 1), \pi^{-1}\,\mathrm{Diag}(1, 1, \pi, \pi, \pi)].$$

Then $\phi_{14} = \phi_{15} = 0$ and an $\mathcal{O}_K$-equivalence brings us to the second case. In the second case we may assume $\phi_{34} \in \langle x_1 \rangle$. Applying the transformation

$$[\mathrm{Diag}(\pi, \pi, 1, 1, 1), \pi^{-1}\,\mathrm{Diag}(\pi, 1, 1, 1, 1)]$$

gives a model with a non-regular point at $(1 : 0 : \ldots : 0)$. Since all transformations we have used preserve (the valuation of) the discriminant, we are done by Lemma 3.6. □

Lemma 8.3. *Let $\Phi, \Phi' \in X_5(\mathcal{O}_K)$ be Pfaffian models satisfying*

$$\Phi' = [\mathrm{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \mathrm{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})]\Phi$$

for some $r_1 \le \ldots \le r_5$ and $s_1 \le \ldots \le s_5$.

(i) *If C_ϕ contains no lines then $r_1 + r_4 \le s_2$, $r_2 + r_3 \le s_2$ and $r_2 + r_4 \le s_3$.*
(ii) *If C_ϕ contains no lines or conics then $r_1 + r_5 \le s_3$, $r_2 + r_5 \le s_4$, $r_3 + r_4 \le s_4$ and $r_3 + r_5 \le s_5$.*

PROOF: (i) If $r_1 + r_4 > s_2$ then all entries of ϕ outside the top left 3×3 submatrix are linear forms in x_3, x_4, x_5. So C_ϕ contains the line $\{x_3 = x_4 = x_5 = 0\}$. If $r_2 + r_3 > s_2$ then all entries of ϕ outside the first row/column are linear forms in x_3, x_4, x_5. So C_ϕ contains the line $\{x_3 = x_4 = x_5 = 0\}$. If $r_2 + r_4 > s_3$ then C_ϕ contains the line $\{\phi_{23} = x_4 = x_5 = 0\}$.

(ii) If $r_1 + r_5 > s_3$, $r_2 + r_5 > s_4$ or $r_3 + r_5 > s_5$ then the entries of the last row/column of ϕ are in $\langle x_4, x_5 \rangle$, $\langle \phi_{15}, x_5 \rangle$ or $\langle \phi_{15}, \phi_{25} \rangle$. If $r_3 + r_4 > s_4$ then the bottom right 3×3 submatrix of ϕ has entries in $\langle x_5 \rangle$. So in all these cases ϕ is k-equivalent to a model with $\phi_{35} = \phi_{45} = 0$. Let p_5 be the Pfaffian of the top left

4×4 submatrix. Then C_ϕ contains $\{\phi_{12} = \phi_{25} = p_5 = 0\}$ which is either a conic or contains a line. □

Lemma 8.4. *Let Φ and Φ' be as in Theorem 8.1, and suppose $0 = r_1 \le \ldots \le r_5$ and $s_1 \le \ldots \le s_5$. Then the r_i and s_i are given by*

r_1	r_2	r_3	r_4	r_5	s_1	s_2	s_3	s_4	s_5
0	α	2α	3α	4α	$\le 2\alpha$	3α	4α	5α	$\ge 6\alpha$

for some $\alpha \ge 0$.

Proof: The inequalities in Lemma 8.3 together with the inequalities obtained when we replace $(r_1, \ldots, r_5; s_1, \ldots, s_5)$ by $(-r_5, \ldots, -r_1; -s_5, \ldots, -s_1)$ give

$$\begin{aligned} s_2 = r_1 + r_4 = r_2 + r_3 \quad &\Longrightarrow\ r_2 - r_1 = r_4 - r_3 \\ s_3 = r_1 + r_5 = r_2 + r_4 \quad &\Longrightarrow\ r_2 - r_1 = r_5 - r_4 \\ s_4 = r_2 + r_5 = r_3 + r_4 \quad &\Longrightarrow\ r_3 - r_2 = r_5 - r_4 \end{aligned}$$

Therefore $r_1, \ldots, r_5$ are in arithmetic progression. The other statements follow. □

Lemma 8.5. *Let $\Phi, \Phi' \in X_5(\mathcal{O}_K)$ be Pfaffian models satisfying*

$$\Phi' = [\mathrm{Diag}(\pi^{-r_1}, \ldots, \pi^{-r_5}), \mathrm{Diag}(\pi^{s_1}, \ldots, \pi^{s_5})]\Phi$$

for some $r_1 \le \ldots \le r_5$ and $s_1 \le \ldots \le s_5$.

(i) *If $r_1 + r_4 > s_1$ and $r_4 + r_5 > s_5 > s_1$ then $\mathcal{C}_\Phi$ has a non-regular point.*
(ii) *If $r_1 + r_3 > s_1$ and $r_3 + r_4 > s_3 > s_1$ then $\mathcal{C}_\Phi$ has a non-regular point.*
(iii) *If $r_2 + r_5 < s_5$ and $r_1 + r_2 < s_1 < s_5$ then $\mathcal{C}_{\Phi'}$ has a non-regular point.*
(iv) *If $r_3 + r_5 < s_5$ and $r_2 + r_3 < s_3 < s_5$ then $\mathcal{C}_{\Phi'}$ has a non-regular point.*

Proof: (i) Since $r_1 + r_4 > s_1$ the only entries of ϕ involving x_1 are in the top left 3×3 submatrix. So $P = (1 : 0 : \ldots : 0)$ is a point on C_ϕ. Since $r_4 + r_5 > s_5$ we have $\phi_{45} = 0$ and so P is a singular point. Since $r_4 + r_5 > s_1 + 1$ the coefficient of x_1 in Φ_{45} vanishes mod π^2. Therefore P is a non-regular point.

(ii) Since $r_1 + r_3 > s_1$ the only entries of ϕ involving x_1 are in the top left 2×2 submatrix. So $P = (1 : 0 : \ldots : 0)$ is a point on C_ϕ. Since $r_3 + r_4 > s_3$ we have $\phi_{34}, \phi_{35}, \phi_{45} \in \langle x_4, x_5 \rangle$ and so P is a singular point. Since $r_3 + r_4 > s_1 + 1$ the coefficient of x_1 in each of Φ_{34}, Φ_{35} and Φ_{45} vanishes mod π^2. Therefore P is a non-regular point.

(iii), (iv) These follow from (i) and (ii) by switching the roles of Φ and Φ'. □

PROOF OF THEOREM 8.1: We may assume $r_1 \leq \ldots \leq r_5$ and $s_1 \leq \ldots \leq s_5$. Replacing r_i by $r_i + \lambda$ and s_i by $s_i + 2\lambda$ still gives the same transformation. So we may assume $r_1 = 0$. Then the r_i and s_i are as given in Lemma 8.4.

If $\alpha = 0$ then $r_1 = \ldots = r_5$ and the conclusion $s_1 = \ldots = s_5$ follows from the fact Φ and Φ' are minimal. We assume for a contradiction that $\alpha \geq 1$. Since $r_1 + r_4 = 3\alpha > s_1$ it follows by Lemmas 3.6 and 8.5(i) that $r_4 + r_5 \leq s_5$. Since $r_2 + r_3 = 3\alpha < s_3$ it follows by Lemmas 3.6 and 8.5(iv) that $r_3 + r_5 \geq s_5$. Putting these together we have

$$r_4 + r_5 \leq s_5 \leq r_3 + r_5.$$

Therefore $r_3 = r_4$ and this contradicts our assumption that $\alpha \geq 1$. □

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