The geometry of the Humbert surface of discriminant N^2

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Pretending to be Abel/Jacobi/Legendre

At the start of the 19th century mathematicians were interested in *elliptic integrals* i.e., integrals

$$\int \frac{a\sqrt{f} + b}{c\sqrt{f} + d} dx$$

where $a, b, d, f \in \mathbb{C}[x]$ are polynomials and f has degree 3 or 4.

Liouville (1830s)

Elliptic integrals cannot be given in terms of elementary functions.

Still pretending

Abel (1829), Jacobi (1829), Legendre (1828) ask instead about the "abelian" integral

$$\int \frac{A\sqrt{g} + B}{C\sqrt{g} + D} d\xi.$$

where $A, B, C, D, g \in \mathbb{C}[\xi]$ and g has degree 5 or 6.

They ask: when can this be "reduced" to an elliptic integral (i.e., written as one after writing $x = R(\xi, \sqrt{g})$ and $\sqrt{f} = Q(\xi, \sqrt{g})$ for some rational functions R, Q (of "degree" N).

- $N = 2$ family found by Legendre and Jacobi (1820s & 30s),
- $N = 3$ Hermite, Goursat, Burkhardt, Brioschi and Bolza (1800s),
- $N = 4$ Bolza (1887).

In modern language...

Take the genus 2 curve $C/\mathbb{C}$

$$C : \nu^2 = g(\xi)$$

where $g(\xi)$ has degree 5 or 6 as above.

Question

When does there exist an elliptic curve $E : y^2 = f(x)$ (where $f(x)$ as above has degree $3, 4$) such that C admits a morphism to E?

To help us try to answer this question, lets have a definition.

Definition

We say that a morphism $\phi : C \to E$ is *optimal* if it does not factor through an isogeny (NB. every covering factors through an optimal one).

An example

Example

- If $C : \nu^2 = f(\xi) = g(\xi^2)$ for some g of degree 3, then take $E : y^2 = g(x)$ and the morphism $C \to E$ given by $(x, y) \mapsto (x^2, y)$ is an optimal covering of degree 2.

The moduli space of such gadgets

Write $\mathcal{M}_2$ for the moduli space of genus 2 curves (this is birational over $\mathbb{Q}$ to $\mathbb{A}^3$).

Humbert (1899) considered what we now call the *Humbert surface of discriminant* N^2

$$\mathcal{H}_{N^2} \subset \mathcal{M}_2$$

which parametrises genus 2 curves with an optimal morphism of degree N to an elliptic curve.

Question

What is the geometry of these surfaces??

Computations of $\mathcal{H}_{N^2}$

Birational models for the surface $\mathcal{H}_{N^2}$ (even over $\mathbb{Q}$) have recently been computed in many cases, for example (non-exhaustively):

N	Reference
$2 \leq N \leq 5$	Shaska + Magaard, Völklein, Wijesiri, Wolf, Woodland ('01–'08)
3	Bröker–Lauter–Howe–Stevenhagen('15), Djukanović ('17)
4	Bruin–Doerksen ('11)
$2 \leq N \leq 11$	Kumar ('14)
$12^\dagger$, $14^\dagger$, $15^\dagger$	F. ('22 + forthcoming)
$13^\dagger$, $17^\dagger$	Fisher ('19–'23)

†These calculations do not give the universal genus 2 curve

Simplified Enriques–Kodaira classification

A theorem of Enriques–Kodaira breaks up (smooth projective) algebraic surfaces $X/\mathbb{C}$ into certain classes based on the *Kodaira dimension*.
We only need the *irregularity* $= 0$ case (i.e., $\dim H^1(X, \mathcal{O}_X) = 0$).

Kodaira dimension	Type
$\kappa = -\infty$	Rational (i.e., birational to $\mathbb{P}^2$)
$\kappa = 0$	$\begin{cases} \text{K3} \\ \text{Enriques} \end{cases}$
$\kappa = 1$	Honestly elliptic
$\kappa = 2$	General type

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Rational-pointy-ness

Roughly (with many caveats)

$$\begin{cases} \kappa = -\infty \longleftrightarrow \text{like genus 0 curves,} \\ \kappa = 0, 1 \longleftrightarrow \text{like elliptic curves,} \\ \kappa = 2 \longleftrightarrow \text{like genus} \geq 2 \text{ curves.} \end{cases}$$

E.g., in the $\kappa = 2$ case it is the *Bombieri–Lang conjecture* that if $X/\mathbb{Q}$ the rational points lie on a proper Zariski closed subvariety.

We prove the following theorem which was shown by Hermann ('92) when N is a prime number.

Theorem (F.)

The Humbert surface $\mathcal{H}_{N^2}$ (or more precisely its desingularisation) is:

- rational if $N \leq 16$ and $N = 18, 20, 24$,
- an elliptic K3 surface if $N = 17$,
- an elliptic surface of Kodaira dimension 1 if $N = 19$, and
- of general type if $N \geq 22$ and $N \neq 24$.

Clearly 21 is missing, it should be $\kappa = 1$, but TBC.

The geometry of $\mathcal{H}_{N^2}$

It follows from:

Theorem (F.)

The geometric genus, arithmetic genus (and Chern numbers)[a] of (a smooth projective surface birational to) $\mathcal{H}_{N^2}$ are given by explicit formulae involving (e.g.,):

- continued fraction expansions,
- class numbers of imaginary quadratic orders,
- data associated to the modular curves $X_{\mathrm{ns}}^{+}(N)$ and $X_{\mathrm{s}}^{+}(N)$[b].

[a] After choosing an appropriate model in the birational equivalence class.

[b] In particular we get a very satisfying genus formula for these by computing certain intersection numbers on (a double cover of) $\mathcal{H}_{N^2}$ – the Hilbert modular surface.

Final things

Important inputs

- Techniques of Hirzeburch ('70s), Haussmann ('79, '82), and Bassendowski ('85) for "symmetric Hilbert modular surfaces".
- Work of Kani–Schanz ('98) on "modular diagonal quotient surfaces".

Final things

More generally...

The Humbert surfaces $\mathcal{H}_{N^2}$ are defined more generally for any discriminant D (in our case N^2). The *Hilbert modular surface* $Y_-(D)$ is a double cover of $\mathcal{H}_D$.

As with Hilbert modular surfaces, there is more than one Humbert surface with discriminant D (parametrised by the narrow class group of the quadratic order of discriminant D). We can prove the generalisation of the above theorem[a] for this more general class of Humbert surfaces of discriminant N^2.

[a] Except for in some stubborn elliptic K3 and elliptic surface cases