1 A particle moves in a fixed plane and its position vector at time $t$ is $\mathbf{r}$. Let $(r, \theta)$ be plane polar coordinates and let $\hat{r}$ and $\hat{\theta}$ be unit vectors in the direction of increasing $r$ and increasing $\theta$ respectively. Show that

$$\dot{r} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}.$$ 

The particle moves outwards with speed $v$ on the equiangular spiral $r = a \exp(\theta \cot \alpha)$, where $a$ and $\alpha$ are constants, with $0 < \alpha < \frac{\pi}{2}$. Show that

$$v = r \dot{\theta} \csc \alpha$$

and hence that

$$\dot{r} = v \cos \alpha \hat{r} + v \sin \alpha \hat{\theta}.$$ 

Show that

$$\ddot{r} = \dot{v} \hat{t} + v \ddot{\theta} \hat{n}$$

where $\hat{t}$ and $\hat{n}$ are unit vectors in the tangent and normal directions to the curve. Find the radius of curvature in terms of $r$ and $\alpha$.

2 In these (shortish) orbital questions, the particles move in a $-k/r$ gravitational potential ($k > 0$) and you only have to consider energy and angular momentum conservation, and (for the circular orbits) the radial component of the equation of motion.

(i) Show that the radius, $R$, of the orbit of a satellite in geostationary orbit (in the equatorial plane) is approximately

$$R \approx \left(\frac{28}{3}\right) R_m,$$

where $R_m$ is the radius of the moon’s orbit round the Earth.

(ii) A particle moves in a parabolic orbit and another particle moves in a circular orbit. Show that if they pass through the same point then the ratio of their speeds at this point is $\sqrt{2}$. For a satellite orbiting the Earth in a circular orbit, what is the relationship between its orbiting speed and its escape velocity?

If, instead of passing through the same point, the particles have the same angular momentum per unit mass, show that the perihelion distance of the parabola is half the radius of the circle.

What is the relationship between the radius of curvature of a parabolic orbit at its perihelion and the perihelion distance?

(iii) A particle moves with angular momentum $h$ per unit mass in an ellipse, for which the distances from the focus to the periapsis (closest point to focus) and apoapsis (furthest point) are $p$ and $q$, respectively. Show that

$$h^2 \left(\frac{1}{p} + \frac{1}{q}\right) = 2k.$$ 

Show also that the speed $V$ of the particle at the periapsis is related to the speed $v$ of a particle moving in a circular orbit of radius $p$ by

$$V^2 = 2v^2 \left(1 + \frac{p}{q}\right)^{-1}.$$ 

(iv) A particle $P$ is initially at a very large distance from the origin moving with speed $V$ on a trajectory that, in the absence of any force, would be a straight line for which the shortest distance from the origin is $l$. The shortest distance between $P$’s actual trajectory and the origin is $d$. Show that

$$2kd = V^2 (l^2 - d^2).$$

3 For particle subject to an inverse square force given by $\mathbf{F} = -\frac{mk}{r^3} \mathbf{r}$, the vectors $\mathbf{h}$ and $\mathbf{e}$ are defined by

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}, \quad \mathbf{e} = \frac{\dot{\mathbf{r}} \times \mathbf{h}}{k} - \frac{\mathbf{r}}{r}.$$ 

Show that $\mathbf{h}$ is constant and deduce that the particle moves in a plane through the origin.

Show also that $\mathbf{e}$ is constant and that

$$ec \cos \theta = h^2 / k - r,$$

where $e = |\mathbf{e}|$, $h = |\mathbf{h}|$ and $\theta$ is the angle between $\mathbf{r}$ and $\mathbf{e}$. Deduce that the orbit is a conic section.
4 A particle of unit mass moves with speed \( v \) in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are \( \mu/r^2 \) towards the sun and \( kv \) opposing the motion, where \( \mu \) and \( k \) are constants. Write down the vector equation of motion and show that the vector \( \mathbf{L} \), defined by

\[
\mathbf{L} = \epsilon^{kt} \mathbf{r} \times \dot{\mathbf{r}}
\]
is constant. Deduce that the particle moves in a plane through the origin.

Establish the equations

\[
r^2 \dot{\theta} = \mu e^{-kt}, \quad \mu r = \varepsilon^2 e^{-2kt} - r(\dot{r} + k\dot{r}),
\]

where \( r \) and \( \theta \) are plane polar coordinates centred on the Sun and \( \varepsilon \) is a constant.

Show that, when \( k = 0 \), a circular orbit of radius \( a \) exists for any value of \( a \), and find its angular frequency \( \omega \) in terms of \( a \) and \( \mu \).

When \( k/\omega \ll 1 \), \( r \) varies so slowly that \( \dot{r} \) and \( \ddot{r} \) may be neglected in the above equations. Verify that in this case an approximate solution is

\[
r = a e^{-2kt}, \quad \dot{\theta} = \omega e^{3kt}.
\]

Give a brief qualitative description of the behaviour of this solution for \( t > 0 \). Does the speed of the particle increase or decrease?

5 A particle \( P \) of unit mass moves in a plane under a central force

\[
F(r) = -\frac{\lambda}{r^3} - \frac{\mu}{r^2},
\]

where \( \lambda \) and \( \mu \) are positive constants. Write down the differential equation satisfied by \( u(\theta) \), where \( u = 1/r \).

Given that \( P \) is projected with speed \( V \) from the point \( r = r_0 \), \( \theta = 0 \) in the direction perpendicular to \( OP \), find the equation of the orbit under the assumptions

\[
\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda.
\]

Explain the significance of these inequalities.

Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

\[
\pi(1 - \lambda/(V^2 r_0^2))^{-1/2}.
\]

Under what condition is the orbit a closed curve?

6 * A particle \( P \) of mass \( m \) moves under the influence of a central force of magnitude \( mk/r^3 \) directed towards a fixed point \( O \). Initially \( r = a \) and \( P \) has velocity \( V \) perpendicular to \( OP \), where \( V^2 < k/a^2 \). Prove that \( P \) spirals in towards \( O \) (you should give the geometric equation of the spiral). Show also that it reaches \( O \) in a time

\[
T = \frac{a^2}{\sqrt{k - a^2 V^2}}.
\]

7 A particle of mass \( m \) moves in a circular orbit of radius \( R \) under the influence of an attractive central force of magnitude \( F(r) \). Obtain an equation relating \( R \), \( F(R) \), \( m \) and the orbital angular momentum per unit mass \( h \).

The particle experiences a very small radial perturbation of the form \( u(\theta) = U + \epsilon(\theta) \), where \( u = 1/r \) and \( U = 1/R \). The orbital angular momentum is not affected. Obtain the equation for \( \epsilon'(\theta) \). Given that the subsequent orbit is both stable and closed, show that

\[
RF'(R)/F(R) = \beta^2 - 3, \quad \beta \text{ is a rational number.}
\]

where \( \beta \) is a rational number. Deduce that, if \( \beta \) is independent of \( R \), then \( F(r) \) is of the form \( Ar^\alpha \), where \( \alpha \) is rational and greater than \(-3\).

Explain why, if (*) holds, \( \beta \) would be expected to be independent of \( R \).

8 Shortish coriolis force questions. Where required, use \( \omega \) for the angular speed of the Earth, assume that events take place at latitude \( \lambda \) in the northern hemisphere and ignore centrifugal forces.

(i) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?

(ii) On a very calm day, the sea freezes. A particle is projected along the frozen surface. Given that the particle moves in a circle, state whether it is clockwise or anticlockwise.

* Determine the radius of the circle in terms of \( \lambda \), the speed of the particle, and \( \omega \). You may assume that the motion lies in the plane orthogonal to the radius vector; you may wish to think about this assumption.

(iii) A straight river flows with speed \( v \) in a direction \( \alpha \) degrees East of North. Show that the effect of the coriolis force is to undermine the right bank. Does the magnitude of the effect depend on \( \alpha \)?

(iv) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed \( V \) in the north-easterly direction the plumb line hangs at an angle \( \theta \) to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that \( \theta \approx (2\omega V \sin \lambda)/g \). Why can the centrifugal force be ignored?