Dynamics and Relativity, Revision suggestions

Here is a list of all the Part IA Newtonian Dynamics and Part IB Special Relativity Tripos questions available on the Faculty web site, with a brief commentary. I have recommended a subset suitable for revision purposes by means of a dagger. You might like to look at others, if only to check that you could do them.

The schedule was changed in 2009 to include Special Relativity and exclude phase plane analysis, so not all the Newtonian Dynamics questions are relevant.

Dynamics

2008

†Q3 Two body problem. The answer to the last part comes out in terms of \( k \) and the reduced mass.

Q4 Phase plane — off-schedule.

†Q9 This may be the first time an octopus has appeared in a Tripos question. It takes a bit of reading, though the words ‘jet propulsion’ in the first sentence give a good idea of the set-up.

†Q10 The \( u(\theta) \) orbital equation. For the distance of closest approach, you can follow the path laid out in the question (i.e. set \( \cos(\theta - \theta_0) = 1 \) at closest approach); but in fact, conservation of energy and momentum give the result (on solving a quadratic) without recourse to the \( u \) equation.

†Q11 Rotating frame. There is a bookwork derivation of the acceleration in the rotating frame and a coriolis force application. The way to derive the acceleration in the rotating frame is to get first

\[
\frac{dw}{dt} \bigg|_{\text{inertial}} = \frac{dw}{dt} \bigg|_{\text{rotating}} + \omega \times w
\]

which applies to any vector \( w \), then apply it twice to the position vector.

†Q12 Rolling sphere. The motion of the sphere comprises motion of its centre and rotation about it centre. The point of contact with the large sphere is instantaneously at rest (by the no-slip condition). The kinetic energy of the sphere can be found using the fact that it is instantaneously rotating about its point of contact (using parallel axes theorem to find the moment of inertia about the point of contact) or using the result that the KE can be regarded as the ‘KE of the centre’ plus the rotational KE.

2007

Q3 Standard rocket equation — it is on the second examples sheet.

Q4 Phase plane — off-schedule.

Q9 Ring sliding on smooth vertical parabola. There’s some dimensional analysis, but otherwise it is more like an old STEP question. It can be done simply by conservation of energy, the only difficulty being the calculation of the KE of the ring.

†Q10 Analysing a vector equation of motion (Lorentz force). You don’t ever have to go into axes and coordinates. You have to remember that scalar and vector products are differentiated using the usual Leibniz rule (for a product).
†Q11 Orbit equation. Bookwork derivation of the $u$ equation, then a parabolic orbit problem. (A parabolic orbit is one for which the particle has the escape velocity at each point. You go from the geometric equation of the orbit ($u$ as a function of $\theta$) to the kinematic equation using $\dot{\theta} = hu^2$ to introduce time.

In fact, if the object of the question is obtain the last result, then it may be better not to use the $u$ equation at all. You can get it straight from conservation of energy (total energy is zero in this case, since the particle has just enough energy to overcome the potential and get out to infinity) and conservation of momentum (to elimate $\dot{\theta}$) cf 2001 Q11.

†Q12 System of particles; pretty much bookwork, though you will probably find it easier to do it yourself than follow the book calculation.

2006

Q3 Spinning car wheels; on-schedule but off-style. First the wheels spin, giving a force of $\mu N$ ($N$ is normal reaction, so $N = mg$). This makes the car accelerate (the acceleration being improbably independent of the speed of the wheel relative to the ground). When the car reaches the speed such that the wheels could roll without slipping ($V = \Omega r$), the slipping stops.

Q4 Phase plane: off-schedule.

Q9 Motor cycle in a bowl. On schedule, I suppose, and perfectly doable, but pretty unrewarding.

Q10 Nice question (about expansion of a gas, I think — you end up deriving what is called an adiabatic invariant) but it won’t do much to help your understanding of the course. I once set a STEP question along these lines, but no one did it. Save it for a fine summer afternoon.

Q11 Spinning rods. The last paragraph is certainly off-schedule (coefficient of restitution), and you could argue that the rest is too.

Q12 Yuk.

2005

†Q3 Circular orbit. It is not clear what form of the equations is required in the second paragraph, but it doesn’t matter since you only ever calculate a circular orbit. You could therefore write down the acceleration in polar coordinates and equate to the central acceleration due to gravity, or you could write down the $u$ orbital equation and the conservation of angular momentum equation. The result you want is essentially Kepler’s third law.

You should perhaps prove that the orbit you want is circular — you are only given that it is geostationary (i.e. that it rotates at the same speed as the planet).

For the last paragraph, you should just draw the two orbits and use conservation of energy to discuss how the speed varies.

I’m not sure why the planet has a specific name (Zog), since all planets have mass $M$ and days of length $T$. If anything, it should be Earth, since $geo$ is a reference to Earth. Perhaps the intention is make the question more accessible to viewers of MTV.

Q4 Vehicle stopping distances: on-syllabus but a bit off-style. It is all about speed and distance, so you can just use $Mvdv/dx = $ force. Not very interesting.

†Q9 This is a bit of a dog’s dinner; but worth doing if you don’t follow the hints. Note that
there is a misprint: in cylindrical polar coordinates, $r = (r, 0, z)$ not $(r, \theta, z)$ (the latter is just a list of the three coordinates).

The way to do this question is to integrate the equation of motion once in vector form. The constant of integration can be set to zero by using the given initial condition (for the $\mathbf{B}$ component) and by translating the origin ($r \rightarrow r + r_0$) for the components orthogonal to $\mathbf{B}$. That leaves $m\ddot{r} = qr \times \mathbf{B}$. Now dotting with $\mathbf{B}$ and dotting with $r$ shows that the particle moves on the intersection of a plane and a sphere; i.e. in a circle.

You can do exactly the same for the last part, using an exponential integrating factor.

The hint for the first part (‘Choose the origin of cylindrical polar coordinates so that $\dot{r} = \ddot{r} = 0$ at $t = 0$) is interesting. Can such origin really by chosen? The answer is yes: the equations (see Vector Calculus) $\dot{r} = vt$ and $\dot{t} = v\mathbf{n}/\rho$, where $\rho$ is the radius of curvature and $v$ is speed, define the centre of curvature, namely the point a distance $\rho$ along $\mathbf{n}$. The motion, to second approximation, is circular with this point at the centre, so choosing this as the origin gives $\dot{r} = \ddot{r} = 0$.

**Q10** Bicycle wheel on a roller. It is about simple moments of inertia and torque; probably takes longer to read than to do.

BUT: a previous lecturer comments that although this looks easy, very few can get it completely right. There are two problems, the first relatively minor: most students find it hard to give a good explanation of why $K$ should be used instead of some other combination of $I$ and $J$ — the usual answer is something like ‘it’s necessary to take the inertia of both the wheel and the roller into account’. More importantly, the answer to the final paragraph is often wrong, because you need to introduce a factor $S/R$ when considering the torques on the back wheel.

**Q11** Motion of a particle on a smooth surface. It starts with an approximation, which is a bit discouraging.

**Q12** Bouncing ball. The first half involves the coefficient of restitution, which is assumed knowledge but off-schedule (which means that it should not be the focal point of a question, though it could arise); the second half is just the standard quadratic drag model for a falling particle.

2004

**Q3** Cars colliding elastically in the first part (just about on the schedule, I suppose): a car colliding with a wall in the second part and experiencing rapid deceleration. It’s all more like an A-level question than a Tripos question.

**Q4** Bubble rising in water. This is a problem from fluid dynamics. In the second part, the bubble gets smaller, so you have to remember to use the form of Newton’s second law appropriate for varying mass (i.e. rate of change of momentum equals force).

**Q9** A non-inertial, but not rotating, frame. It turns out to be quite straightforward to write down the equation of motion once you have understood the set-up. Thereafter, you just solve not very pleasant differential equations.

**Q10** Gravitational orbit with friction. To show that the orbit without friction is elliptical, it is probably best to derive and solve the $a(\theta)$ equation. For the circular orbit, it is just as easy to use the $r(t)$ equation. For the last part, you can do it in vectors, without choosing axes or coordinates at all (you get $dL/dt = -\mathbf{AL}$).
**Q11** Lorentz magnetic force (cf 2005, Q9), with \( \mathbf{r} \) given as \((r, \theta, z)\) again! You have to assume for the last paragraph that the orbit lies in a plane \( z = \text{constant} \), which is why it makes sense to talk about the angular momentum about the \( z \)-axis (rather than about a point).

The very last part is interesting. You have to find the escape velocity even though there is no potential. Normally, the escape velocity (escape speed, really) derives from the amount of kinetic energy required to overcome the potential and get out to infinity. Here the KE is conserved (because the force is perpendicular to the velocity) so the KE at any point just has to be greater than the KE at \( r = \infty \). The KE can be written as \( \frac{1}{2}m(r^2 + \frac{L^2}{m^2}r^2) \) and since \( \frac{L}{r} \rightarrow -qB_0 \) as \( r \rightarrow \infty \), the escape velocity is \( \frac{qB_0}{m} \).

**Q12** Cylinder rolling up a slope. The first paragraph should ask for the moment of inertia about the axis.

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**2003**

**Q3** This one puzzled me for a bit, until I realised that the rocket stays on the ground — more like a rocket-propelled car. The \( g \) enters the problem because the frictional force is proportional to the normal reaction (which is \( mg \)). This very quick if you write the rocket equation in the form (as is suggested by the answer) \( \frac{dv}{dt} + u\frac{d}{dt}(\log m) = -\mu g \).

**Q4** System of particles: the effect of changing the origin. You don’t need the equations of motion of the particles, just the definitions of total momentum and angular momentum.

**Q9** Particle in a magnetic field. Interesting but a bit elaborate. The important result is that there is still a conserved quantity (other than energy) even though angular momentum is not conserved; this is a result from Lagrangian dynamics (in a Part II course) and follows from the axial symmetry.

**Q10** System of gravitating particles. A majestic question occupying a full page. You are very much led through it. The system represents a Newtonian model of the universe in which the particles (galaxies) are all expanding isotropically away from (e.g.) a big bang. The constant \( k \) that arises in the first integral of the equations (essentially the energy integral) determines whether the universe will recollapse \((k < 0)\) because the particles do not have the required escape velocity, or expand for ever \((k \geq 0)\).

**Q11** A rocking hemisphere. Working out the velocity of the centre of mass is a bit tedious: the sort of thing that one could do if one absolutely had to.

**Q12** Orbits using \( r(\theta) \) instead of the usual \( r(t) \) or \( u(\theta) \). The third displayed equation needs a little thought and a good diagram (use Pythagoras and the second displayed equation) but after that it is plain sailing.

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**2002**

**Q3** Phase plane: off-schedule.

**Q4** Rotating frames. It is 2/3 page long, but essentially trivial once you have understood what is wanted. The idea is that the coriolis force has exactly the same form as the Lorentz magnetic force.

**Q9** System of particles — it was on one of the examples sheets.

**Q10** Orbits — interesting question. You derive the orbit equation in the \( u(\theta) \) form then show that if the orbits include circles passing through (so not centred on) the centre of force, the
force must be quintic. Once you get the equation of a circle in the form \( u = 1/(2R \cos \theta) \) there is some algebra but not much else.

**Q11** Lorentz magnetic force. A pleasant question, doable by vectors (no coordinates), so very much on-message for this year’s course.

**Q12** Jointed rods. Not as hard as it looks; in fact, straightforward once you have the position of the centre of mass of one rod in the form \((l \cos \theta, l \sin \theta)\).

### 2001

**Q3** Bookwork derivation of the \( u(\theta) \) orbital equation.

**Q4** More or less bookwork two-body problem for a central (i.e. in this case \( F \) is parallel to \((x_1 - x_2)\)) force.

**Q9** Rotating frame. I wouldn’t go further than writing down the equation of motion in the rotating frame and putting it into cartesian coordinates (no point in solving the equations).

**Q10** Raindrop with drag. You just have to remember to use the form of Newton’s second law appropriate for variable mass.

†**Q11** Parabolic orbit, using conservation of energy and momentum rather than the \( u(\theta) \) equation as in 2007, Q11. You get a slightly awkward integral which, for a parabolic orbit is doable, but which is more tricky for an elliptical orbit. Notice that there are two arbitrary constants in the given result. One is an arbitrary constant from the \( r \) integration, and the other must come from the angular momentum. The quantity \( h^2/2GM \) arises naturally in the integral, and you could save writing by calling it, e.g. \( r_0 \). Then the other constant is just the time at which \( r = r_0 \).

If you like doing integrals, cover your eyes now. The integral you get is

\[
\int \frac{r \, dr}{\sqrt{r - r_0}}
\]

which can be done either with the substitution \( r = r_0 \cosh^2 \theta \) or (probably easier) by parts, integrating \((r - r_0)^{-\frac{1}{2}}\) and differentiating \( r \), which gets you to the given answer pretty directly.

**Q12** Yet another cylinder on an inclined plane, comparing slipping and rolling (the latter takes more time: conservation of energy means that potential energy has to be converted to both rotational and translational kinetic energy).

### Special Relativity

2008

**1Q4** Bookwork calculation of the effect of a 4-D Lorentz transformation on the 4-momentum of a photon.

†**2Q7** Collision of particle and photon, giving rise to a different (lighter) particle and a photon. This comes out in a line or so using 4-momenta (just take length squared of each side of the conservation equation). You have to show that \( P_\gamma \cdot P \geq 0 \), which follows quickly if you work in the rest frame of the particle. It is not clear from the wording of the question whether you have to show that the displayed result is actually a possible value, or merely a lower bound.
The latter, I assume; for the former you need $P_\gamma \cdot P = 0$, which can be achieved only if the photon has zero energy.

†4Q17 Moving observers (one on either end of a train, one on the platform) and space-time diagrams. There is lot of reading to do; but it is worth it. The statement that ‘C observes …’ is a little unnerving in this context; I’m sure it means that ‘In C’s frame, the events occur at …’.

2007

1Q4 This depends on the definition of the 4-momentum vector given by the lecturer in the relevant years.

For the first paragraph, you can just use $P = MdX/d\tau$. (In this question, the 4-momentum is written $p_a$, the $a$ being a suffix running from 1 to 4, or 0 to 3. No need to use suffices for this.)

For the second paragraph, it is probably intended that you show that $P \cdot P = M^2c^2$ (either directly from the components as given in the question or using invariants), then just say that a photon has zero mass; or maybe note that for a massless particle, $P \cdot P = 0$ because momentum is parallel to the 4-velocity and the 4-velocity of a massless particle is light-like (null).

†2Q7 Velocity addition law.

†4Q17 More particle decay. Use 4-momentum conservation. To obtain $v/c$, you first find $\gamma$ from (e.g.) the relativistic energy.

2006

1Q4 Particle collision. You may feel that you have done enough of these.

†2Q7 Velocity addition for three velocities.

†4Q17 Bookwork ladder and barn.

2005

1Q4 Definitions of 4-vectors. For the last line (‘concept of rest energy’), you just take the time-like component of the 4-momentum (which is said in the question to be the relativistic energy) and set $v = 0$ to get the ‘rest energy’.

2Q7 Bookwork Alice and Bob.

4Q17 Bookwork aberration formula for starlight, by Lorentz transformation of photon 4-vector. The first request is to obtain the Lorentz transformation between two inertial frames. This sounds like a bit of bookwork that was covered in 2005, but which was not covered in 2009; so ignore it.

2004

3Q8 Addition of velocities.

4Q7 4 dimensional Lorentz transform acting on the 4-momentum vector.

†4Q17 Photon striking electron creating electron and positron. You work in what is called here the centre of mass frame, more often referred to as the centre of momentum frame — i.e.
the frame in which the total momentum is zero.

2003

3Q10 Compton scattering (photon incident on electron). It is on the examples sheet.

4Q9 Lorentz transforms by hyperbolic angle. Straightforward algebra.

4Q18 Pion decay into muon and neutrino. All the meat is in the first paragraph (which is similar to 2007/4Q17). You find \( \gamma \) first, by finding the relativistic energy, then the speed.

The second paragraph is pretty impenetrable. It means the following. \( S \) is the rest frame of the pion, and \( S' \) moves with speed \( v \) in the \( x \)-direction relative to \( S \). In \( S \), the muon moves with speed \( u \) in the \( y \) direction. Apply a 4-dimensional Lorentz transformation to find the direction of the muon velocity (or momentum) in \( S' \).

2002

3Q10 Bookwork addition of velocities.

4Q9 Decay of a particle. You can find the energies of each of the decay products quickly using 4-vector invariants; to obtain the momentum, you can then use \( E^2 = p^2c^2 + m^2c^4 \).

4Q18 Bookwork ladder and barn.

2001

3Q10 Speed of a particle in terms of invariants made from its 4-momentum and the observer’s 4-velocity. It was on the examples sheet.

4Q9 Einstein’s principle of relativity did not occur explicitly in the lectures, I think, so ignore the first paragraph; and also the very last sentence which is related to the first paragraph.

For the second paragraph, you just have to substitute for \( x \) and \( t \) in terms of \( x' \) and \( t' \) in the given equation of a spherical shell.

4Q18 The first part is a simple decay: you find quickly that \( E = \frac{1}{2}Mc^2 \), where \( E \) is the energy of either decay particle. You can then find the momentum from \( E^2 = p^2c^2 + m^2c^4 \), and the speed from the identity \( pc^2 = Ev \). In the second part, the factor of \( M/2m \) is just the \( \gamma \) required for the Lorentz transform from the rest from of particle \( A \).