A WEISS–WILLIAMS THEOREM FOR SPACES OF EMBEDDINGS AND THE HOMOTOPY TYPE OF SPACES OF LONG KNOTS

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ABSTRACT. We establish a pseudoisotopy result for embedding spaces in the line of that of Weiss and Williams for diffeomorphism groups. In other words, for $P \subset M$ a codimension at least three embedding, we describe the difference in a range of homotopical degrees between the spaces of block and ordinary embeddings of P into M as a certain infinite loop space involving the relative algebraic K-theory of the pair $(M, M - P)$. This range of degrees is the so-called concordance embedding stable range, which, by recent developments of Goodwillie–Krannich–Kupers, is far beyond that of the aforementioned theorem of Weiss–Williams.

As an application, we give a full description of the homotopy type (away from 2 and up to the concordance embedding stable range) of the space of long knots of codimension at least 3.

CONTENTS

1. INTRODUCTION

The classical approach to study the homotopy type of the diffeomorphism group $\mathrm{Diff}_\partial(M)$ of a high-dimensional manifold M^d (i.e. $d \geq 5$) is based on the so called *surgery-pseudoisotopy* program. Recall that there is a homotopy fibre sequence

$$(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \longrightarrow B\mathrm{Diff}_\partial(M) \xrightarrow{\ i\ } B\widetilde{\mathrm{Diff}}_\partial(M), \tag{1.1}$$

where $B\widetilde{\mathrm{Diff}}_\partial(M)$ denotes the classifying space for the simplicial group $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$ of *block diffeomorphisms* of M, and $(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M)$ is by definition the homotopy fibre of the map i. *Surgery theory*, as developed by Browder, Novikov, Ranicki, Sullivan, Wall, et al., roughly studies the difference between $\widetilde{\mathrm{Diff}}_\partial(M)$ and the topological monoid $h\mathrm{Aut}_\partial(M)$ of homotopy automorphisms of M in terms of the mapping space $(G/O)_*^{(M,\partial M)}$ and the L-theory of the group ring $\mathbb{Z}[\pi_1(M)]$. Therefore, one of the advantages of this theory is that it makes the homotopy type of $\widetilde{\mathrm{Diff}}_\partial(M)$ theoretically accessible through homotopy theory and L-theory.

It is in understanding the homotopy type of $(\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M)$ where *pseudoisotopy theory* comes into play. Two diffeomorphisms $\phi_0, \phi_1 \in \mathrm{Diff}(M)$ are said to be *pseudoisotopic* if there exists a diffeomorphism $\psi : M \times I \cong M \times I$ with $\psi\,|_{M\times\{i\}} = \phi_i$ for $i = 0, 1$. Originally, pseudoisotopy theory [Igu88, HW73] was

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concerned with the study of the topological group $C(M)$ of *concordances* or *pseudoisotopies* of M consisting of diffeomorphisms $\psi \in \mathrm{Diff}(M \times I)$ that restrict to the identity on a neighbourhood of $M \times \{0\} \cup \partial M \times I$ (i.e. pseudoisotopies starting at the identity). The spaces $C(M)$ and $\widetilde{\mathrm{Diff}}/\mathrm{Diff}(M)$ are intimately related, as was first made precise by Hatcher through a spectral sequence (cf. [Hat78, Prop. 2.1]). Hatcher's realisation eventually evolved into the following celebrated theorem of Weiss and Williams [WW88, Thm. A], which will be of central importance all throughout this paper.

Theorem 1.1 (Weiss–Williams)**.** *Let M^d be a compact smooth d-manifold. There exists a map*

$$\Phi^{\mathrm{Diff}} : (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \longrightarrow \Omega^\infty \left(\Sigma^{-1}\mathbf{Wh}_s^{\mathrm{Diff}}(M)_{hC_2}\right)$$

which is $(\phi(d)+1)$-connected, where $\phi(d)$ denotes the concordance stable range of dimension d (which by Igusa's theorem [Igu88] *is at least* $\min(\frac{d-4}{3}, \frac{d-7}{2})$*).*

In this theorem, the C_2-spectrum $\mathbf{Wh}_s^{\mathrm{Diff}}(M)$ is known as the *simple smooth Whitehead spectrum* of M; it is the 2-connective cover of the *smooth Whitehead spectrum* $\mathbf{Wh}^{\mathrm{Diff}}(M)$, which itself is closely related to algebraic K-theory by the splitting of spectra

$$\mathbf{A}(M) \simeq \Sigma^\infty_+ M \vee \mathbf{Wh}^{\mathrm{Diff}}(M). \tag{1.2}$$

Here $\mathbf{A}(M)$ denotes Waldhausen's A-theory spectrum of M (cf. [Wal85]). Even though these spectra and their corresponding involutions are difficult to understand in general, much can be said when M is homotopy equivalent to a point [Rog02, Rog03, BM19] or the circle [Hes09], or when working rationally [BF86].

The study of the homotopy type of embedding spaces is intimately tied to that of diffeomorphism groups as seen via the *isotopy extension theorem*. More precisely, let M^d be as before and let $\iota : P \hookrightarrow M$ be a closed (codimension zero for simplicity) submanifold that meets ∂M transversely. Write $\mathrm{Emb}_{\partial_0}(P, M)$ for the space of embeddings of P into M which agree with ι in a neighbourhood of $\partial_0 M := P \cap \partial M$ and send $\partial P - \partial_0 P$ to the interior of M. Then there is a homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P, M) \longrightarrow B\mathrm{Diff}_\partial(\overline{M - P}) \longrightarrow B\mathrm{Diff}_\partial(M), \tag{1.3}$$

where $\langle\iota\rangle$ stands for the components of the embedding space in the orbit of the action of $\mathrm{Diff}_\partial(M)$ on the standard embedding ι by postcomposition. In this sense, embedding spaces are the corresponding "relative analogues" of diffeomorphism groups, and often their homotopy type becomes easier to study.

In this paper we would like to advertise a direct approach for studying the homotopy type of embedding spaces (up to a range of degrees) which is analogous to the one for diffeomorphism groups that we just surveyed. As before, it begins by analysing the space of block embeddings $\widetilde{\mathrm{Emb}}_{\partial_0}(P, M)$ via relative surgery methods; the main result in this direction is due to Browder–Casson–Sullivan–Wall (cf. [GKW01, Thm. 2.2.1]), and asserts that the space of block embeddings appears as the homotopy pullback of a diagram involving so called *Poincaré block embeddings* and *immersions* and ordinary block immersions. Due to the Smale–Hirsch immersion theorem, all the ingredients that come into the mix are accessible through homotopy theory and thus, in theory, so are block embeddings.

It remains to understand the difference between ordinary and block embeddings, i.e. the homotopy fibre

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) := \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P, M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P, M)), \tag{1.4}$$

by means of pseudoisotopy theory. This space also fits in another homotopy fibre sequence

$$\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M) \longrightarrow (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(\overline{M - P}) \xrightarrow{\ \alpha\ } (\widetilde{\mathrm{Diff}}/\mathrm{Diff})_\partial(M) \tag{1.5}$$

obtained as the fibre of the map from (1.3) to its block analogue (see (3.4)). What was previously fulfilled by Theorem 1.1 for the pseudoisotopy part of diffeomorphism groups seems to be missing in the case of embedding spaces; the best result known in this direction is *Morlet's lemma of disjunction* [BLR06, Thm. 3.1] which, in that reformulation, determines the connectivity of the map α.

The main result of this paper fills in this gap in the surgery-pseudoisotopy program for embedding spaces, and describes the homotopy type of $\mathrm{Emb}^{(\sim)}_{\partial_0}(P, M)$ in a range outside of the connectivity of α. For this reason, one could think of it as a second order Morlet's lemma. There are two features about this result that we would like to emphasise: firstly, as expected from Theorem 1.1, the description involves relative algebraic K-theory, which can be fully analysed using so-called *trace methods* (see Section 1.1.2 below). Secondly, recent developments of Goodwillie–Krannich–Kupers [GKK23] imply that the range of degrees in which our description holds is, surprisingly, much greater than that of Weiss–Williams' (see Section 1.1.1).

As a proof of concept, we will run this program to fully describe the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$—also known as a *space of long knots* or *long embeddings*—for $p \leq d-3$ and $d \geq 5$, localised away from the prime two and up to the range of degrees in which our analogue of Theorem 1.1 holds.

1.1. **Statement of the main result.** Let M^d be a compact smooth manifold (possibly with boundary) of dimension d and let $\iota : P \hookrightarrow M$ be a closed submanifold that meets ∂M transversely. Write $\mathrm{Emb}_{\partial_0}(P,M)$ (resp. $\widetilde{\mathrm{Emb}}_{\partial_0}(P,M)$) for the geometric realisation of the simplicial set of embeddings (resp. block embeddings) of P into M which agree with ι in a neighbourhood of $\partial_0 P := P \cap \partial M$ and send $\partial P - \partial_0 P$ to the interior of M. As in (1.4), write $\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M)$ for the homotopy fibre at ι of the natural inclusion $\mathrm{Emb}_{\partial_0}(P,M) \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)$.

Theorem A. *There exists a map*

$$\Phi^{\mathrm{Emb}} : \mathrm{Emb}^{(\sim)}_{\partial}(P,M) \longrightarrow \Omega^\infty \left(\Sigma^{-2}\mathbf{Wh}^{\mathrm{Diff}}(M, M-P)_{hC_2}\right)$$

which is $\phi_{\mathrm{CEmb}}(d,p)$-connected if the handle dimension p of P relative to $\partial_0 P$ satisfies $p \leq d-3$. Here ϕ_{CEmb} is the concordance embedding stable range (see (1.6)) and

$$\mathbf{Wh}^{\mathrm{Diff}}(M, M-P) := \Sigma\,\mathrm{hocofib}(\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M-P) \to \Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)).$$

Remark 1.2. The involutions in the (once desuspended) Whitehead spectra involved in the statement of Theorem A are exactly those of Theorem 1.1 coming from Weiss' orthogonal calculus (see Section 2.1 and Remark 2.1). When M is stably paralellisable, we relate these involutions to more algebraic ones in Theorem 5.11 and Corollary 5.15 (cf. Notation 5.1 for conventions used). See also Warning 5.18 for the effect of these involutions on $\pi_0^s(\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)) = \mathrm{Wh}(\pi_1(M))$ in terms of Milnor's involution.

Remark 1.3 (Topological version of Theorem A). Even though we have stated Theorem A in the smooth setting for notational convenience, an analogous statement in $CAT = \mathrm{Top}$ also holds after some adjustments: Firstly, the smooth embedding spaces should be replaced by their analogues in terms of locally flat topological embeddings, and the smooth Whitehead spectra by their topological versions. Moreover, the result is only valid when P has geometric codimension zero in M; this is because a topological analogue of Proposition 3.3 cannot be true, as we explain in Remark 3.4. One should bear in mind, however, that the bound (1.7) is a priori only valid for smoothable topological manifolds. See also Remarks 3.6, 3.2 and A.5 for modified arguments in the topological setting.

1.1.1. *The concordance embedding stable range.* A *concordance embedding* of P into M is an embedding $\varphi : P \times I \hookrightarrow M \times I$ such that

(*a*) $\varphi^{-1}(M \times \{i\}) = P \times \{i\}$ for $i = 0, 1$ and
(*b*) φ agrees with the inclusion $\iota \times \mathrm{Id}_I$ on a neighbourhood of $P \times \{0\} \cup \partial_0 P \times I$.

We denote by $C\mathrm{Emb}(P,M)$ the space of all such embeddings, topologised as a subspace of $\mathrm{Emb}(P \times I, M \times I)$. There are stabilisation maps $\Sigma_{(M,P)} : C\mathrm{Emb}(P,M) \to C\mathrm{Emb}(P \times I, M \times I)$ given by taking the product of an embedding with I and unbending corners appropriately (see [GKK23, Fig. 1]), and the *concordance embedding stable range* for (d,p) is

$$\phi_{\mathrm{CEmb}}(d,p) := \max\left\{ k \in \mathbb{N} \;\middle|\; \begin{array}{l} \Sigma_{(M,P)} \text{ is } k\text{-connected for all } (M,P) \\ \text{with } \dim M = d \text{ and } h\text{-}\dim(P, \partial_0 P) = p \end{array} \right\}. \tag{1.6}$$

Goodwillie–Krannich–Kupers [GKK23] have recently shown that if $p \leq d-3$, then

$$\phi_{\mathrm{CEmb}}(d,p) \geq 2d - p - 5, \tag{1.7}$$

which is far beyond Igusa's lower bound for the concordance stable range $\phi(d)$. In the concordance stable range $\phi(d)$, Theorem A is a consequence of Theorem 1.1 and the isotopy extension sequence (1.5), so our main contribution is improving the connectivity of the map Φ^{Emb} to the concordance embedding stable range $\phi_{\mathrm{CEmb}}(d,p)$. We will also see in Remark 6.6 that the lower bound of (1.7) is the best one could do (this could already be seen in [GKK23]).

1.1.2. *Relative algebraic K-theory.* The handle codimension condition on the embedding $\iota : P \subset M$ in the statement of Theorem A guarantees that the inclusion $M - P \to M$ is 2-connected. This observation can be used to our advantage, as when $Y \to X$ is 2-connected, the relative Whitehead spectrum $\mathbf{Wh}^{\mathrm{Diff}}(X, Y)$ is far more accessible than $\mathbf{Wh}^{\mathrm{Diff}}(X)$ and $\mathbf{Wh}^{\mathrm{Diff}}(Y)$ on their own: by (1.2), there is an equivalence of spectra

$$\mathbf{A}(X, Y) \simeq \mathbf{Wh}^{\mathrm{Diff}}(X, Y) \vee \Sigma^{\infty}(X/Y),$$

where by X/Y we really mean the mapping cone of $Y \to X$. The spectrum $\mathbf{A}(X, Y)$ is in turn equivalent to the relative topological cyclic homology spectrum $\mathbf{TC}(X, Y)$ [NS18] by the seminal work of Dundas–Goodwillie–McCarthy [DM94, Dun97, DGM12], and $\mathbf{TC}(X, Y)$ is in general accessible through trace methods or the study of the stable homotopy of the *free loop space* functor $L(-) := \mathrm{Map}(S^1, -)$. If working over the rational numbers, there is an even simpler description of $\mathbf{A}(X, Y)$ thanks to the isomorphism [Goo86]

$$\pi_*^s(\mathbf{A}(X, Y)) \otimes \mathbb{Q} \cong HC^-_{*-1}(\Omega X, \Omega Y; \mathbb{Q}) \cong H^{S^1}_{*-1}(LX, LY; \mathbb{Q}), \tag{1.8}$$

where HC^-_* denotes Connes' negative cyclic homology, and $H^{S^1}_*$ stands for the S^1-equivariant homology with respect to the S^1-action on $L(-)$ given by rotating loops. Moreover in this case, the involution of $\Sigma^{-2}\mathbf{Wh}^{\mathrm{Diff}}(X, Y)$ appearing in Theorem A can be related to one on the right hand side of (1.8) using the work of Bustamante–Farrell–Jiang [BFJ20]. Integrally, however, one has to be more careful with involutions; our analysis in Section 5 is an example of how one can proceed.

1.2. **The homotopy type of spaces of long knots.** The homology and homotopy of spaces of long knots $\mathrm{Emb}_\partial(D^p, D^d)$ has been subject to extensive reasearch in recent years, especially through the lens of embedding calculus and its relation to the little disks operad and graph complexes. See for instance Volić [Vol06], Watanabe [Wat07], Sinha [Sin09], Budney and Cohen [Bud08, BC09] for when $p = 1$ and $d = 3, 4$ mainly, or more modern treatments as in Arone–Turchin [AT14, AT15], Dwyer–Hess [DH12], Boavida de Brito–Weiss [BdBW18], at last culminating in the work of Fresse–Turchin–Willwacher [FTW17] where a complete answer of $\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \otimes \mathbb{Q}$ is given in terms of the homology of the *hairy graph complex*.

The second main result of this paper is a full description of the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d - p \geq 3$ roughly up to the concordance embedding stable range (1.6) and localised at odd primes. This is done via the surgery-pseudoisotopy program for embedding spaces surveyed in the introduction, a crucial step of which is Theorem A. For a real k-dimensional G-representation ρ, denote by $\mathrm{Th}(\rho)$ the Thom space of the associated vector bundle $EG \times_G \mathbb{R}^k \to BG$. Let ψ_m denote the real m-dimensional representation of the dihedral group D_m (seen as a subgroup of the symmetric group Σ_m) given by permuting the factors of $\mathbb{R}^m$, and let $\sigma : C_2 = \{\pm 1\} \hookrightarrow \mathbb{R}$ be the sign representation (also regarded as a D_m-representation by restriction along the determinant $D_m \hookrightarrow O(2) \xrightarrow{\det} \{\pm 1\} = C_2$).

Theorem B. *For $p \leq d - 3$ and $d \geq 5$, there exists a homotopy fibre sequence which, after localising away from 2 and taking $(\phi_{\mathrm{CEmb}}(d, p) - 1)$-th Postnikov sections, takes the form*

$$\begin{tikzcd} \prod_{m \geq 2} \Omega^{\infty}\Big(\big(\mathbb{S}^{(d+1)\cdot(\sigma - 1)} \wedge \mathrm{Th}(\psi_m \otimes (d - p - 3 + \sigma) \mid_{C_m})\big)_{hC_2}\Big) \arrow[r] & \mathrm{Emb}_\partial(D^p, D^d) \arrow[d] \\ & \Omega^p \operatorname{hofib}\big(G(d-p)/O(d-p) \to G/O\big). \end{tikzcd} \tag{1.9}$$

It is split if $p \geq 2$, and splits after being looped once if $p = 1$.

Remark 1.4. (i) The map $G(d - p)/O(d - p) \to G/O$ in the above statement is the natural stabilisation map, and $C_2 = D_m/C_m$ acts on $\mathrm{Th}(\psi_m \otimes (d - p - 3 + \sigma) \mid_{C_m})$ by its residual action.

(ii) The space $\mathrm{Emb}_\partial(D^p, D^d)$ is an $\mathbb{E}_p$-algebra, and so it can indeed be localised. When $d - p \geq 3$, it is exactly $(2d - 3p - 4)$-connected by work of Budney [Bud08, Prop. 3.9]. So in the cases when $2d - 3p - 4 < 0$, by *localising* $\mathrm{Emb}_\partial(D^p, D^d)$ we really mean localising each of its connected components one at a time (and *not* localising all together $\pi_0(\mathrm{Emb}_\partial(D^p, D^d))$, which will in general be an abelian group).

(iii) When $p = 1$ and $d = 4$ (i.e. the lowest dimensional case of interest if $d - p \geq 3$), (1.9) exists after looping once and is a split homotopy fibre sequence (see Remark 6.1). Using this to study the homotopy groups of $\mathrm{Emb}_\partial(D^1, D^4)$, however, yields weaker results than the ones in [Bud08, Prop. 3.9].

We will compute some of the homotopy groups of the fibre in (1.9) in Section 6.2, and hence deduce new torsion information about the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$ in high-dimensions (i.e. $d \geq 5$). We will also compare our results to those of Fresse–Turchin–Willwacher [FTW17] in Remark 6.6.

Structure of the paper. Sections 2 and 3 will be devoted to the proof of Theorem A. We start by briefly reviewing Weiss' theory of orthogonal calculus and then, in Section 2.2, we present the orthogonal functors that will play a role in the proof of Theorem A. In doing this, we will have to carefully describe the topology on spaces of bounded diffeomorphisms in such a way that we can employ the machinery of orthogonal calculus in this setting. After reducing to the codimension zero case in Section 3.1, we use the results in the preceding section to define the map Φ^{Emb} in Section 3.3 and analyse its connectivity in 3.4.

As a consequence of Theorem A, in Section 4 we establish a splitting result (Theorem 4.2) for embedding spaces of manifolds containing interval factors reminiscent of work of Burghelea–Lashof [BL82, Cor. E].

Section 5 deals with the analysis of the C_2-spectra involved in the statements of Theorems 1.1 and A. The main results in this direction are Theorem 5.11 and Corollary 5.15, where the involutions on these spectra are expressed (up to homotopy) in terms of the standard involution in algebraic K-theory.

Section 6 is devoted to Theorem B, whose proof is a formal consequence of the results in the preceeding sections. We then draw some conclusions on the homotopy groups of spaces of long knots in Section 6.2.

In Appendix A we explore certain aspects related to spaces of bounded diffeomorphisms and embeddings. Namely in Section A.1 we show that the topological models for these spaces introduced in Section 2.2 coincide (up to weak equivalence) with the simplicial ones of Definition 2.7. In Section A.2 we give a "moduli space of manifolds" description for the classifying space of the bounded diffeomorphism group.

In Appendix B we show that the h-cobordism stabilisation map anti-commutes with the involutions in these spaces. This is analogous to a result of Hatcher [Hat78, Appendix I, Lem.] and Burghelea–Lashof [BL82, Cor. A7] for spaces of concordance diffeomorphisms.

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2. Orthogonal calculus and spaces of bounded diffeomorphisms

Much of the proof of Theorem 1.1 in [WW88] is an application of Weiss' *orthogonal calculus* but in disguise, as this theory was not yet formalised at the time. In this section we briefly review the main aspects of this theory and develop some necessary tools required for the proof of Theorem A.

2.1. A quick tour through orthogonal calculus.

Weiss' *orthogonal calculus* [Wei95] is a calculus of functors useful to understand objects of geometric flavour. It studies *continuous* functors from the category $\mathcal{J}$ of real finite-dimensional inner product vector spaces and linear isometries to the category of (compactly generated weakly Hausdorff) spaces Top. Such a functor $F : \mathcal{J} \to \mathrm{Top}$ is said to be *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}}(U,V) \times F(U) \longrightarrow F(V)$$

is continuous for all $U, V \in \mathcal{J}$. Here $\mathrm{mor}_{\mathcal{J}}(U,V)$ denotes the *Stiefel manifold* of linear isometries from U to V, so that $\mathcal{J}$ is enriched over Top. We will work in a slightly different setup, where Top is replaced by the category Top_* of pointed spaces and $\mathcal{J}$ is replaced by the pointed topological category $\mathcal{J}_0$ with the same objects and with

$$\mathrm{mor}_{\mathcal{J}_0}(U,V) := \mathrm{mor}_{\mathcal{J}}(U,V)_+,$$

as morphism spaces. Similarly, a functor $F : \mathcal{J}_0 \to \mathrm{Top}_*$ is *continuous* if the evaluation map

$$\mathrm{mor}_{\mathcal{J}_0}(U,V) \wedge F(U) \longrightarrow F(V)$$

is continuous for all $U, V \in \mathcal{J}_0$. Such a functor $F(-)$ is also sometimes called an *orthogonal functor*.

The machinery of orthogonal calculus associates to each such orthogonal functor $F(-)$ a sequence of (naïve) $O(k)$-spectra $\Theta F^{(k)}$ for $k \geq 1$—the *derivatives of* F—which fit in a tower

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & T_2F(-) \arrow[d] & \Omega^\infty\left((S^{2\cdot(-)} \wedge \Theta F^{(2)})_{hO(2)}\right) \arrow[l] \\ & T_1F(-) \arrow[d] & \Omega^\infty\left((S^{1\cdot(-)} \wedge \Theta F^{(1)})_{hO(1)}\right) \arrow[l] \\ F(-) \arrow[r, "\eta_0"'] \arrow[ur] \arrow[uur] \arrow[uuur] & T_0F(-) \arrow[r, equal] & F(\mathbb{R}^\infty) \end{tikzcd} \tag{2.1}$$

of orthogonal functors—the *Taylor tower*. Here

- $S^{k\cdot V}$ is the one point compactification of $k \cdot V := \mathbb{R}^k \otimes V$, which is acted upon $O(k)$ in the $\mathbb{R}^k$ component, and diagonally on the smash $S^{k\cdot V} \wedge \Theta F^{(k)}$,
- the right hand horizontal maps—the *layers*—indicate the inclusions of the homotopy fibres of the subsequent vertical maps between the *stages* $T_kF(-)$ of the tower,
- the zeroth stage $T_0F(-)$ is given by $T_0F(V) := \mathrm{hocolim}_k\, F(V \oplus \mathbb{R}^k)$, and thus admits a canonical equivalence from the constant orthogonal functor with *value at infinity* $F(\mathbb{R}^\infty) := \mathrm{hocolim}_k\, F(\mathbb{R}^k)$. The map $\eta_0 : F(V) \to T_0F(V)$ is simply the inclusion map.

In the proof of Theorem A we will analyse the Taylor tower (2.1) only up to the first layer, so we shall now describe the spectrum $\Theta F^{(1)}$ in detail. For $V \in \mathcal{J}_0$, consider $F^{(1)}(V) := \mathrm{hofib}(F(V) \to F(V \oplus \mathbb{R}))$, the homotopy fibre of the map induced by the standard inclusion $V \to V \oplus \mathbb{R}$. These spaces inherit an $O(1)$-action by declaring $-1 \in O(1)$ to act on V and $V \oplus \mathbb{R}$ by -1 on *all* coordinates. There are $O(1)$-equivariant maps

$$s_V : S^1 \wedge F^{(1)}(V) \longrightarrow F^{(1)}(V \oplus \mathbb{R}), \tag{2.2}$$

given by rotating $V \oplus \mathbb{R}^2$ about the 2-plane $0 \oplus \mathbb{R}^2$ (as notation suggests, here $O(1)$ acts trivially on the suspension coordinate. In general, we adopt the convention that S^n denotes the n-sphere with the trivial $O(1)$-action). Then the $O(1)$-spectrum $\Theta F^{(1)}$ has $F^{(1)}(\mathbb{R}^n)$ as its n-th space, and $s_{\mathbb{R}^n}$ as the structure map $S^1 \wedge \Theta F^{(1)}_n \to \Theta F^{(1)}_{n+1}$.

Remark 2.1. This is not quite the $O(1)$-action described in [Wei95, Prop. 3.1]; $O(1)$ there acts on $F^{(1)}(V) = \mathrm{hofib}(F(V) \to F(V \oplus \mathbb{R}))$ by declaring the action of $-1 \in O(1)$ on V to be trivial and by -1 on the $\mathbb{R}$-summand of $V \oplus \mathbb{R}$. If we write $\underline{F}^{(1)}(V)$ for this $O(1)$-space, then the maps

$$s_V : S^\sigma \wedge \underline{F}^{(1)}(V) \longrightarrow \underline{F}^{(1)}(V \oplus \mathbb{R}) \quad \text{and} \quad \underline{F}^{(1)}(V) \to F(V) \tag{2.3}$$

are $O(1)$-equivariant, where σ stands for the (1-dimensional) sign $O(1)$-representation and S^σ for its associated representation sphere. The corresponding (sequential) spectrum, call it $\underline{\Theta} F^{(1)}$, is *not* a naïve $O(1)$-spectrum in the usual sense anymore, as $O(1)$ acts non-trivially on the suspension coordinates. To solve this issue, Weiss introduces in [Wei95, p. 17] the $O(1)$-spectrum $\Theta^\# F^{(1)}$ with n-th $O(1)$-space

$$\Theta^\# F^{(1)}_n := \Omega^{\infty\cdot\sigma}(S^n \wedge \underline{\Theta} F^{(1)}) = \operatorname*{hocolim}_k \Omega^{k\cdot\sigma}(S^n \wedge \underline{F}^{(1)}(\mathbb{R}^k)), \tag{2.4}$$

where $\Omega^{k\cdot\sigma}(-) := \mathrm{Map}_*(S^{k\cdot\sigma}, -)$ and $O(1)$ acts by conjugation on this mapping space.

In order to relate the $O(1)$-spectra $\Theta F^{(1)}$ and $\Theta^\# F^{(1)}$, we observe that F can be naturally upgraded to a functor $\underline{F} : \mathcal{J}_0^{O(1)} \to \mathsf{Top}_*^{O(1)}$ enriched over Top_*, where $\mathcal{J}_0^{O(1)} := \mathrm{Fun}(O(1), \mathcal{J}_0)$ is regarded as the pointed topological category of inner product finite-dimensional $O(1)$-representations. We likewise define for $V \in \mathcal{J}_0^{O(1)}$ the $O(1)$-space $\underline{F}^{(1)}(V) := \mathrm{hofib}(\underline{F}(V) \to \underline{F}(V \oplus \sigma))$. Now tensoring such an $O(1)$-representation with the sign representation σ gives a self-isomorphism of $\mathcal{J}_0^{O(1)}$ denoted by $- \cdot \sigma$. One could stabilise $\underline{F}^{(1)}(-)$ with respect to $\mathbb{R}$ as in (2.3), or with the sign representation σ, giving rise to maps

$$s_{a,b} : S^{a\cdot\sigma+b} \wedge \underline{F}^{(1)}(V) \longrightarrow \underline{F}^{(1)}(V \oplus \mathbb{R}^{a,b}), \quad a, b \geq 0, \quad V \in \mathcal{J}_0^{O(1)},$$

where $S^{a\cdot\sigma+b} := S^{a\cdot\sigma}\wedge S^b$ and $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b\cdot\sigma$. We then obtain a zig-zag of maps of $O(1)$-spectra

$$\begin{tikzcd}[column sep=small] \Theta^\# F^{(1)} := \operatorname{hocolim}_{a\geq 0} \mathbb{S}^{-a\cdot\sigma}\wedge \underline{F}^{(1)}(\mathbb{R}^a) \arrow[dr, "b=0", "\sim"'] & & \operatorname{hocolim}_{b\geq 0} \mathbb{S}^{-b}\wedge \underline{F}^{(1)}(b\cdot\sigma) =: \Theta F^{(1)} \arrow[dl, "a=0"', "\sim"] \\ & \operatorname{hocolim}_{a,b\geq 0} \mathbb{S}^{-a\cdot\sigma-b}\wedge \underline{F}^{(1)}(\mathbb{R}^{a,b}), & \end{tikzcd} \tag{2.5}$$

where the maps in the colimit of the middle spectrum are induced by $s_{1,0}$ and $s_{0,1}$. Non-equivariantly, both of the maps in the zig-zag are equivalences by Fubinni's theorem. This establishes the desired $O(1)$-equivariant equivalence[1] $\Theta F^{(1)} \simeq \Theta^\# F^{(1)}$.

For $V \in \mathcal{J}_0$, let $S(V)$ denote the unit sphere of V, seen as an unbased $O(1)$-space by the antipodal action. The following proposition will be the main ingredient for the construction of the map Φ^{Emb} of Theorem A.

Proposition 2.2. *Let $F : \mathcal{J}_0 \to \mathsf{Top}_*$ be an orthogonal functor. For each $n \geq 0$, there are maps*

$$\Phi^F_n : \operatorname{hofib}(F(0)\to F(\mathbb{R}^n)) \xrightarrow{\eta_1} \operatorname{hofib}(T_1F(0)\to T_1F(\mathbb{R}^n)) \simeq \Omega^\infty(S(\mathbb{R}^n)_+\wedge_{O(1)}\Theta F^{(1)}) \tag{2.6}$$

giving rise to a map of homotopy fibre sequences

$$\begin{tikzcd} \operatorname{hofib}(F(0)\to F(\mathbb{R}^n)) \arrow[r] \arrow[d, "\Phi^F_n"] & \operatorname{hofib}(F(0)\to F(\mathbb{R}^{n+1})) \arrow[r] \arrow[d, "\Phi^F_{n+1}"] & \operatorname{hofib}(F(\mathbb{R}^n)\to F(\mathbb{R}^{n+1})) =: \Theta F^{(1)}_n \arrow[d, "\mathrm{stab.}"] \\ \Omega^\infty(S(\mathbb{R}^n)_+\wedge_{O(1)}\Theta F^{(1)}) \arrow[r] & \Omega^\infty(S(\mathbb{R}^{n+1})_+\wedge_{O(1)}\Theta F^{(1)}) \arrow[r] & \Omega^\infty(\Sigma^n\Theta F^{(1)}), \end{tikzcd}$$

where the vertical map "stab." is $\Theta F^{(1)}_n \hookrightarrow \operatorname{hocolim}_k \Omega^k(\Theta F^{(1)}_{n+k})$. Letting $n\to\infty$ in (2.6), we get

$$\Phi^F_\infty : \operatorname{hofib}(F(0)\to F(\mathbb{R}^\infty)) \longrightarrow \Omega^\infty(EO(1)_+\wedge_{O(1)}\Theta F^{(1)}) =: \Omega^\infty(\Theta F^{(1)}_{hO(1)}). \tag{2.7}$$

Proof. We first need to argue that there is a natural weak equivalence between $\operatorname{hofib}(T_1F(0)\to T_1F(V))$ and $\Omega^\infty(S(V)_+\wedge_{O(1)}\Theta F^{(1)})$ for $V\in\mathcal{J}_0$. Observe that there is an $O(1)$-equivariant homotopy cofibre sequence

$$S(V)_+ \longrightarrow *_+ \longrightarrow \Sigma S(V) \cong S^{V\cdot\sigma},$$

where Σ here stands for the unreduced suspension. For X an $O(1)$-spectrum we thus obtain a fibre sequence

$$\Omega^\infty\left((S(V)_+\wedge X)_{hO(1)}\right) \simeq \Omega^\infty(S(V)_+\wedge_{O(1)} X) \longrightarrow \Omega^\infty(X_{hO(1)}) \longrightarrow \Omega^\infty\left((S^{V\cdot\sigma}\wedge X)_{hO(1)}\right),$$

where the first equivalence follows since the pointed $O(1)$-action on $S(V)_+$ is free. The above sequence when $X = \Theta F^{(1)}$ is the left vertical fibration in the following commutative diagram of fibre sequences:

$$\begin{tikzcd} \Omega^\infty\left(S(V)_+\wedge_{O(1)}\Theta F^{(1)}\right) \arrow[r, "\simeq"] \arrow[d] & \operatorname{hofib}(T_1F(0)\to T_1F(V)) \arrow[r] \arrow[d] & * \arrow[d] \\ \Omega^\infty(\Theta F^{(1)}_{hO(1)}) \arrow[r] \arrow[d] & T_1F(0) \arrow[r] \arrow[d] & T_0F(0) = F(\mathbb{R}^\infty) \arrow[d, "\wr"] \\ \Omega^\infty\left((S^{V\cdot\sigma}\wedge\Theta F^{(1)})_{hO(1)}\right) \arrow[r] & T_1F(V) \arrow[r] & T_0F(V) = F(\mathbb{R}^\infty). \end{tikzcd}$$

This gives rise to the required weak equivalence, which is visibly natural in $V\in\mathcal{J}_0$.

Clearly because of this naturality, the left square of the diagram in the statement commutes, so it remains to show that the right one is homotopy commutative too. A similar claim (but without proof) is made in [Wei95, Ex. 10.1], so let us give an argument of our own: without loss of generality, we may replace F by $\operatorname{hofib}(\eta_0 : F\to T_0F)$ so that T_1F becomes homogeneous of degree 1 (see [Wei95, Defn. 7.1]). Because of the $O(1)$-equivalence $\eta_1 : \Theta F^{(1)} \simeq \Theta(T_1F)^{(1)}$ proved in [Wei95, Thm. 6.3(bis)], we may also replace F by T_1F so that F itself is homogeneus of degree 1. Then it was shown in [Wei95, Thm. 7.3] that there is a natural zig-zag of equivalences between $F(V)$ and $\Omega^\infty((S^{V\cdot\sigma}\wedge\Theta^\# F^{(1)})_{hO(1)})$ for $V\in\mathcal{J}_0$, where $\Theta^\# F^{(1)}$ is the $O(1)$-spectrum of (2.4). But we have seen in Remark 2.1 that there is a zig-zag (2.5) of $O(1)$-equivariant equivalences

[1] In the sense of Borel (see Notation 5.2(i)).

between $\Theta F^{(1)}$ and $\Theta^{\#}F^{(1)}$. Therefore, we obtain a zig-zag of equivalences between $\mathrm{hofib}(F(\mathbb{R}^n) \to F(\mathbb{R}^{n+1}))$ and the homotopy fibre of the map $i_n : \Omega^\infty((S^{n\cdot\sigma} \wedge \Theta F^{(1)}_{hO(1)}) \to \Omega^\infty((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)}_{hO(1)})$. This is the zig-zag showing up in the top part of the diagram

$$\begin{tikzcd}
F^{(1)}(\mathbb{R}^n) \arrow[d, "\text{stab.}"] & \cdot \arrow[l, "\sim"'] \arrow[r, "\sim"] & \mathrm{hofib}\begin{pmatrix} \Omega^\infty((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}) \\ \downarrow i_n \\ \Omega^\infty((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}) \end{pmatrix} \arrow[d] & \\
\Omega^\infty(\Sigma^n \Theta F^{(1)}) \arrow[rr, "\lambda"] \arrow[urr, dashed, "\widehat{\lambda}", "\simeq"'] \arrow[rrr, bend right=15, "\simeq *"'] & & \Omega^\infty\left((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\right) \arrow[r, "i_n"] & \Omega^\infty\left((S^{(n+1)\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\right),
\end{tikzcd} \tag{2.8}$$

where λ is defined as follows: given two copies S^n_i of the n-sphere for $i = 0, 1$ (with the trivial $O(1)$-action), there is an $O(1)$-map $a : S^n_0 \vee S^n_1 \to S^{n\cdot\sigma}$ that sends $v \in S^n_i$ to $(-1)^i v \in S^{n\cdot\sigma}$. Then

$$\Sigma^n \Theta F^{(1)} \xleftarrow{\sim} S^n \wedge \Theta F^{(1)} \xleftarrow{\sim} ((S^n_0 \vee S^n_1) \wedge \Theta F^{(1)})_{hO(1)} \xrightarrow{(a\wedge \mathrm{Id})_{hO(1)}} (S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}$$

gives rise to (the zig-zag) λ upon taking infinite loop spaces. Identifying $S^{(n+1)\cdot\sigma}$ as the homotopy cofibre of the map a (really $S^{n+1} \vee S^{n+1}$ is the cofibre of the inclusion $S^{n\cdot\sigma} \to S^{(n+1)\cdot\sigma}$), we see that λ lifts, uniquely up to homotopy, to the weak equivalence $\widehat{\lambda}$. As the top zig-zag of (2.8) in place of the map "stab." in the diagram from the statement would make its right square commutive by construction, we must show that the upper triangle of (2.8) is homotopy commutative. For this, it suffices to argue that the bigger square that contains it commutes up to homotopy.

One verifies from the proofs of [Wei95, Prop. 7.2 & Thm. 7.3] and the construction of (2.5) that the top-right composition of the square in (2.8) becomes, up to homotopy, the top-right composition of

$$\begin{tikzcd}
F^{(1)}(\mathbb{R}^n) \overset{(!)}{=} \underline{F}^{(1)}_{n,0} \arrow[r] \arrow[d, hook, "k=0"] & (\underline{F}^{(1)}_{n,0})_{hO(1)} \arrow[r, hook, "k=0", "\sim"'] & \left(\operatorname{hocolim}_k \Omega^k(\underline{F}^{(1)}_{n,k})\right)_{hO(1)} & \left(\Omega^\infty(S^{n\cdot\sigma} \wedge \Theta F^{(1)})\right)_{hO(1)} \arrow[l, hook', "s_{n,0}"', "\sim"] \arrow[d, hook] \\
\operatorname{hocolim}_k \Omega^k(\underline{F}^{(1)}_{n,k}) \arrow[urr] \arrow[d, equal, "(!)"] & \Omega^\infty(S^{n\cdot\sigma} \wedge \Theta F^{(1)}) \arrow[l, hook', "\sim"'] \arrow[urr] \arrow[rr, "q"] \arrow[d, equal, "(!)"] \arrow[drr, phantom, "(*)"] & & \Omega^\infty\left((S^{n\cdot\sigma} \wedge \Theta F^{(1)})_{hO(1)}\right) \\
\Omega^\infty(\Sigma^n \Theta F^{(1)}) & \Omega^\infty(S^n \wedge \Theta F^{(1)}) \arrow[l, "\sim"'] & \Omega^\infty\left(((S^n \vee S^n) \wedge \Theta F^{(1)})_{hO(1)}\right) . \arrow[l, hook', "\sim"'] \arrow[ur, "(a\wedge \mathrm{Id})_{hO(1)}"] &
\end{tikzcd}$$

Here $\underline{F}^{(1)}_{a,b} = \underline{F}^{(1)}(\mathbb{R}^{a,b}) = \underline{F}^{(1)}(\mathbb{R}^a \oplus b \cdot \sigma)$ and the decoration (!) indicates that the equality is not (supposed to be) $O(1)$-equivariant. All subtriangles and squares, except possibly the bottom right one labelled by $(*)$, are visibly commutative. Moreover the left-bottom composition is the left-bottom composition of the square in (2.8) by definition. The bottom right subsquare $(*)$ is commutative because the following is for every $O(1)$-spectrum X:

$$\begin{tikzcd}
S^{n\cdot\sigma} \wedge X \arrow[rrr, "q"] \arrow[d, equal, "(!)"] \arrow[dr, "\cong"] & & & (S^{n\cdot\sigma} \wedge X)_{hO(1)} \\
S^n \wedge X \arrow[dr, "\cong"] & (S^{n\cdot\sigma}_0 \vee S^{n\cdot\sigma}_1) \wedge_{O(1)} X & ((S^{n\cdot\sigma}_0 \vee S^{n\cdot\sigma}_1) \wedge X)_{hO(1)} \arrow[l, "\sim"'] \arrow[ur, "(c\wedge \mathrm{Id})_{hO(1)}"] & \\
& (S^n_0 \vee S^n_1) \wedge_{O(1)} X \arrow[u, "b\wedge \mathrm{Id}", "\cong"'] & ((S^n_0 \vee S^n_1) \wedge X)_{hO(1)} \arrow[l, "\sim"'] \arrow[u, "b\wedge \mathrm{Id}", "\cong"'] \arrow[uur, bend right, "(a\wedge \mathrm{Id})_{hO(1)}"'] &
\end{tikzcd}$$

Here the $O(1)$-equivariant isomorphism $b : S^n_0 \wedge S^n_1 \to S^{n\cdot\sigma}_0 \vee S^{n\cdot\sigma}_1$ sends $v \in S^n_i$ to $(-1)^i v \in S^{n\cdot\sigma}_i$, and $c : S^{n\cdot\sigma}_0 \vee S^{n\cdot\sigma}_1 \to S^{n\cdot\sigma}$ is the identity in each wedge summand. This finishes the proof of the proposition. $\square$

Convention 2.3. *From now on, $F^{(1)}(V)$ stands for $\underline{F}^{(1)}(V \cdot \sigma) := \mathrm{hofib}(\underline{F}(V \cdot \sigma) \to \underline{F}((V \oplus \mathbb{R}) \cdot \sigma))$ in the notation of Remark 2.1, unless we explictly say otherwise. This way (2.2) is $O(1)$-equivariant.*

2.2. **The orthogonal functors of bounded diffeomorphisms.** All throughout, let $\iota : P \hookrightarrow M$ be as in the statement of Theorem A. In this section we present the orthogonal functors that will play a role in the proof of Theorem A. These are built out of spaces of bounded diffeomorphisms, for which we will present point-set topological models that agree up to weak equivalence with the more classical simplicial ones.

Let $V \in \mathcal{J}$ be an inner product finite-dimensional real vector space with associated norm $\|-\|_V$, and let Q and Q' be smooth (possibly non-compact) manifolds equipped with proper maps $\pi : Q \to V$ and $\pi' : Q' \to V$. For $t \geq 0$, a smooth map $f : Q \to Q'$ is said to be *t-bounded* if the set $\{\|\pi'(f(q)) - \pi(q)\|_V : q \in Q\} \subset \mathbb{R}$ is bounded by t. More generally, f is *bounded* if it is t-bounded for some $t \geq 0$. If $Q = N \times V$ for some compact manifold N, π will be assumed to be the projection to V.

Definition 2.4. *Let $V \in \mathcal{J}$. The* ***space of bounded diffeomorphisms*** *of $M \times V$ relative to $\partial M \times V$ is*

$$\mathrm{Diff}^b_\partial(M \times V) := \{(t, \phi) \in [0, \infty) \times \mathrm{Diff}_\partial(M \times V) : \phi \text{ is } t\text{-bounded}\},$$

endowed with the subspace topology inhereted from the product $[0, \infty) \times \mathrm{Diff}_\partial(M \times V)$. Here $\mathrm{Diff}_\partial(M \times V)$ is endowed with the Whitney C^∞-topology. It is a group-like topological monoid under the rule

$$(t, \phi) \cdot (t', \phi') := (t + t', \phi \circ \phi').$$

Example 2.5. Orthogonal calculus was largely inspired by the work of Weiss–Williams in [WW88], as can be seen in [WW88, Digr. 3.8]. For $U \in \mathcal{J}_0$, let $B\mathrm{Diff}^b_\partial(M \times U)$ be the classifying space of the group-like topological monoid that we have just introduced. Denote by $B(-)$ the orthogonal functor given by $B(U) := B\mathrm{Diff}^b_\partial(M \times U)$ and, for $i : U \to V$ a morphism in $\mathcal{J}_0$, write $V = U \oplus U^\perp$ and let $B(i)$ be induced by the monoid homomorphism that sends a pair $(t, \phi) \in \mathrm{Diff}^b_\partial(M \times U)$ to $(t, \phi \oplus \mathrm{Id}_{U^\perp}) \in \mathrm{Diff}^b_\partial(M \times U \oplus U^\perp)$. By Corollaries 1.13 and 5.3 of [WW88] (and Proposition A.1 below), the spectrum $\Theta B^{(1)}$ is equivalent to $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$, and hence the latter inherits a homotopy involution—the *Weiss–Williams involution* (see Notation 5.1(*ii*))—from the $O(1)$-action on $\Theta B^{(1)}$. Then

$$\Phi^B_\infty : \mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty)/\mathrm{Diff}_\partial(M) \longrightarrow \Omega^\infty(\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)_{hC_2})$$

is the map from [WW88, Thm. C], and Proposition 2.2 exactly recovers [WW88, Prop. 3.1] in this case.

Notation 2.6. *In the remainder of Section 2, we will denote by $E(-)$ and $B(-)$ the orthogonal functors given on objects by*

$$E(V) := B\mathrm{Diff}^b_\partial(\overline{M - P} \times V), \qquad B(V) := B\mathrm{Diff}^b_\partial(M \times V), \tag{2.9}$$

and on morphisms as in Example 2.5. There is a natural transformation $E(-) \to B(-)$ given by extending a diffeomorphism by the identity on $P \times (-)$, and the orthogonal functor $F(-) := \mathrm{hofib}(E(-) \to B(-))$ will play a especially important role in the proof of Theorem A.

These spaces of bounded diffeomorphisms are usually defined as the geometric realisations of certain simplicial group/set. Before we recall these simplicial models in Definition 2.7 below, let us fix some notation first. For a subset $S \subset \mathbb{R}^{p+1}$ and $\epsilon > 0$, let $B_\epsilon(S) \subset \mathbb{R}^{p+1}$ denote the open ϵ-ball around S. For any face $\sigma \subset \Delta^p \subset \mathbb{R}^{p+1}$ and $0 < \epsilon < 1/2$, fix identifications $B_\epsilon(\partial\sigma) \cap \sigma \cong \partial\sigma \times [0, \times\epsilon)$ which are radial with respect to the barycenter for σ. We will say that a continuous map $f : X \times \Delta^p \to Y \times \Delta^p$ over Δ^p (i.e. $\mathrm{proj}_{\Delta^p} = \mathrm{proj}_{\Delta^p} \circ f$) satisfies the *$\epsilon$-collaring condition* if for every face $\sigma \subset \Delta^p$,

$$f\,|_{X \times (B_\epsilon(\sigma) \cap \sigma)} \equiv f\,|_{X \times \partial\sigma} \times \mathrm{Id}_{[0,\epsilon)}.$$

Definition 2.7. *Let $V \in \mathcal{J}$. The* ***simplicial group*** *$\mathrm{Diff}^b_\partial(M \times V)_\bullet$* ***of bounded diffeomorphisms*** *of $M \times V$ relative to $\partial M \times V$ has as p-simplices the set of diffeomorphisms of $\Delta^p \times M \times V$ over Δ^p which are bounded (with respect to V), that are the identity in a neighbourhood of $\Delta^p \times \partial M \times V$, and that satisfy the ϵ-collaring condition for some $0 < \epsilon < 1/2$. Face and degeneracy maps are determined by the coface and codegeneracy maps of the cosimplicial space $\Delta^\bullet$. If we relax the condition on diffeomorphisms to be over Δ^p to only face-preserving (i.e. diffeomorphisms that send $\sigma \times M \times V$ to itself for every face $\sigma \subset \Delta^p$), we obtain the simplicial group $\widetilde{\mathrm{Diff}}^b_\partial(M \times V)_\bullet$ of* ***bounded block diffeomorphisms*** *of $M \times V$.*

Warning 2.8. One could have defined the orthogonal functor $B(-)$ of Notation 2.6, for instance, to be

$$\mathcal{J}_0 \longrightarrow \mathsf{Top}_*, \quad U \longmapsto B|\mathrm{Diff}^b_\partial(M \times U)_\bullet|.$$

This latter rule, however, does not give rise to a continuous functor in the sense of orthogonal calculus, i.e., it is not enriched over Top_*. A way to fix this is to replace Top_* by sSet_*, $\mathcal{J}_0$ by a category $\mathcal{J}_0^\Delta$ enriched now over sSet_*, and doing orthogonal calculus to sSet_*-enriched functors $\mathcal{J}_0^\Delta \to \mathsf{sSet}_*$. This is morally the point of view taken by Weiss and Williams in [WW88], but orthogonal calculus for simplicially enriched functors has not yet been carried out rigorously, so we prefer to not pursue this approach.

The simplicial models of Definition 2.7 are more convenient to work with than the point-set topological ones of Definition 2.4. Moreover, we will need some results of [WW88] that are stated in the simplicial setting, and so we will have to argue that both models share the same weak homotopy type.

Proposition A.1. *There is a zig-zag of weak equivalences of simplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M\times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M\times V)).$$

In particular, there is a zig-zag of weak equivalences of group-like topological monoids connecting $|\mathrm{Diff}^b_\partial(M\times V)_\bullet|$ *and* $\mathrm{Diff}^b_\partial(M\times V)$.

We defer the proof of this proposition to Section A.1 in the appendix.

3. Proof of Theorem A

We now prove Theorem A. Section 3.1 will first reduce it to the case when P is a codimension zero submanifold of M. Some necessary preliminaries will be presented in Section 3.2. Finally the map Φ^{Emb} of Theorem A and its connectivity will be analysed in Sections 3.3 and 3.4.

Before we move on to the next section, let us record a disjunction result for concordance embeddings known as *Hudson's concordance-implies-isotopy theorem* [Hud70, Thm. 2.1, Addendum 2.1.2].

Theorem 3.1 (Hudson)**.** *The space* $C\mathrm{Emb}(P,M)$ *is connected if* $p\le d-3$. *Equivalently, the natural map* $\pi_0(\mathrm{Emb}_{\partial_0}(P,M))\to\pi_0(\widetilde{\mathrm{Emb}}_{\partial_0}(P,M))$ *is an isomorphism.*

Remark 3.2. Hudson's theorem also holds in the piecewise linear setting (cf. [Hud70, Thm. 1.5]).

3.1. Reduction to geometric codimension zero embeddings.

Let $\iota: P\hookrightarrow M$ be as in the statement of Theorem A. It will be convenient to be able to assume that $P\subset M$ is a codimension zero submanifold (though of handle codimension at least 3). The following result deals with this technicality, and shows that the difference between block and ordinary *smooth* embeddings is insensitive to the geometric codimension.

Proposition 3.3. *Let* M^d *be a compact smooth Riemannian manifold and* $\iota: P^p\hookrightarrow M^d$ *a neat submanifold that is closed as a subspace. Let* $D(\nu_\iota)$ *be the closed disk bundle of the normal bundle of the embedding* ι, *and let* $\hat\iota: D(\nu_\iota)\hookrightarrow M$ *be the induced embedding. Then the square*

$$\begin{tikzcd} \mathrm{Emb}_{\partial_0,\hat\iota}(D(\nu_\iota),M) \arrow[r, "\mathrm{res}_P"] \arrow[d, hook] & \mathrm{Emb}_{\partial_0,\iota}(P,M) \arrow[d, hook] \\ \widetilde{\mathrm{Emb}}_{\partial_0,\hat\iota}(D(\nu_\iota),M) \arrow[r, "\widetilde{\mathrm{res}}_P"] & \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M). \end{tikzcd} \tag{3.1}$$

is homotopy cartesian. Here the subscripts ι *or* $\hat\iota$ *in the embedding spaces stand for the path component consisting of embeddings isotopic to* ι *or* $\hat\iota$ *(relative to* ∂_0P*).*

Equivalently, by taking vertical homotopy fibres in (3.1) and noting Hudson's Theorem 3.1 and that res_P *and* $\widetilde{\mathrm{res}}_P$ *are surjective, there is a weak equivalence*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M)\hookrightarrow\widetilde{\mathrm{Emb}}_{\partial_0}(P,M))\simeq\mathrm{hofib}_{\hat\iota}(\mathrm{Emb}_{\partial_0}(D(\nu_\iota),M)\hookrightarrow\widetilde{\mathrm{Emb}}_{\partial_0}(D(\nu_\iota),M)).$$

Proof. We will show that the horizontal homotopy fibre of the vertical inclusions in (3.1) can be identified, up to equivalence, with the identity map of the topological group $\mathrm{Aut}_{\partial_0}(\nu_\iota)$ of bundle automorphisms of ν_ι which are standard near ∂_0P. In particular, the total homotopy fibre of (3.1) will be weakly contractible.

We first deal with the top horizontal homotopy fibre. Consider the fibration

$$E := \left\{ \begin{array}{c|l} \begin{tikzcd} \nu_\iota \arrow[r, hook, "G"] \arrow[d] & \tau_M \arrow[d] \\ P \arrow[r, hook, "\varphi"] & M \end{tikzcd} & \begin{array}{l} \varphi \in \mathrm{Emb}_{\partial_0,\iota}(P,M), \\ G \in \mathrm{BunInj}_{\partial_0}(\nu_\iota, \tau_M), \\ D\varphi \oplus G : \tau_P \oplus \nu_\iota \cong \varphi^*\tau_M. \end{array} \end{array} \right\} \xrightarrow{r} \mathrm{Emb}_{\partial_0,\iota}(P,M), \quad (G,\varphi) \longmapsto \varphi.$$

Taking derivatives at the zero section of $D(\nu_\iota)$ defines a map $D : \mathrm{Emb}_{\partial_0,\hat{\iota}}(D(\nu_\iota),M) \to E$ over $\mathrm{Emb}_{\partial_0,\iota}(P,M)$. A homotopy inverse $E \to \mathrm{Emb}_{\partial_0,\hat{\iota}}(V,M)$ to D can be defined using the exponential map. Therefore the homotopy fibre of res_P is equivalent to the fibre of r (observe that r is a fibration). Now $\iota^*\tau_M$ is already identified with $\tau_P \oplus \nu_\iota$, so the fibre $F := r^{-1}(\iota)$ can be described as the subspace of bundle automorphisms of $\tau_P \oplus \nu_\iota$ over P which are the identity on the tangent summand τ_P (and near $\partial_0 P$). As the space of bundle maps $\nu_\iota \to \tau_P$ over P is contractible, it follows that the inclusion $\mathrm{Aut}_{\partial_0}(\nu_\iota) \hookrightarrow F$ is a homotopy equivalence.

The argument for the bottom map of (3.1) is similar but trickier; we work with the simplicial model of block embeddings of Definition 2.7. First let ξ and π be vector bundles over spaces B and B', respectively, and fix some bundle map $I : \xi \to \pi$. For any closed subset $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi,\pi)_\bullet$ denote the semi-simplicial set whose n-simplices consist of bundle maps $G : \Delta^n \times \xi \to \tau_{\Delta^n} \boxplus \pi := (\tau_{\Delta^n} \times B') \oplus (\Delta^n \times \pi)$ such that

- G agrees with $\mathbf{0}_{\Delta^n} \boxplus I$ near $\Delta^n \times \partial_0$, where $\mathbf{0}_{\Delta^n} : \epsilon^0_{\Delta^n} \cong \Delta^n \to \tau_{\Delta^n}$ is the inclusion as the zero section, and
- for every face $\sigma \subset \Delta^n$, $G(\sigma \times \xi) \subset \tau_\sigma \boxplus \pi \subset \tau_{\Delta^n} \boxplus \pi$.

Given a map $i : B \to B'$ which agrees with the underlying map of I on $\partial_0 \subset B$, let $\widetilde{\mathrm{BunMap}}_{\partial_0}(\xi,\pi;i)_\bullet$ the semi-simplicial subset consisting of those bundle maps G whose underlying map on the base spaces $\Delta^n \times B \to \Delta^n \times B'$ is $\mathrm{Id}_{\Delta^n} \times i$. Let $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi,\pi)_\bullet$ and $\widetilde{\mathrm{BunInj}}_{\partial_0}(\xi,\pi;i)_\bullet$ be the semi-simplicial subsets of those bundle maps that are fibrewise injective. Then again, taking derivatives at the zero section of $\Delta^\bullet \times D(\nu_\iota)$ yields a simplicial map $\widetilde{D}_\bullet$ from $\widetilde{\mathrm{Emb}}_{\partial_0,\hat{\iota}}(D(\nu_\iota),M)_\bullet$ to a semi-simplicial set $\widetilde{E}_\bullet$ whose n-simplices are

$$\widetilde{E}_n := \left\{ \begin{array}{c|c} \begin{tikzcd} \Delta^n \times \nu_\iota \arrow[r, hook, "G"] \arrow[d] & \tau_{\Delta^n} \boxplus \tau_M \arrow[d] \\ \Delta^n \times P \arrow[r, hook, "\varphi"] & \Delta^n \times M \end{tikzcd} & \begin{array}{c} \varphi \in \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)_n, \\ G \in \widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota,\tau_M)_n, \\ D\varphi \oplus G : \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) \cong \varphi^*(\tau_{\Delta^n} \boxplus \tau_M). \end{array} \end{array} \right\},$$

and whose face maps are given by restriction to face strata. The map $\widetilde{r}_\bullet : \widetilde{E}_\bullet \to \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)_\bullet$ given by $\widetilde{r}(G,\varphi) := \varphi$ is now a Kan fibration, and $\widetilde{r}_\bullet \circ \widetilde{D}_\bullet = \widetilde{\mathrm{res}}_P$. By a similar argument as in the previous case, the homotopy fibre of $\widetilde{\mathrm{res}}_P$ is equivalent to the fibre of $\widetilde{r}_\bullet$. Using the canonical identification

$$(\mathrm{Id}_{\Delta^n} \times \iota)^*(\tau_{\Delta^n} \boxplus \tau_M) = \tau_{\Delta^n} \boxplus \iota^*\tau_M \cong \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota),$$

the fibre $\widetilde{F}_\bullet := \widetilde{r}_\bullet^{-1}(\iota)$ is isomorphic to the semi-simplicial subset of $\widetilde{\mathrm{BunInj}}_{\partial_0}(\nu_\iota, \iota^*\tau_M; \mathrm{Id}_P)_\bullet$ of bundle maps

$$G = G_{\Delta^n} \oplus G_\tau \oplus G_\nu : \Delta^n \times \nu_\iota \longrightarrow \tau_{\Delta^n} \boxplus (\tau_P \oplus \nu_\iota) = (\tau_{\Delta^n} \times P) \oplus (\Delta^n \times \tau_P) \oplus (\Delta^n \times \nu_\iota)$$

for which G_ν is an isomorphism. Thus it follows that

$$\widetilde{F}_\bullet = \widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet \times \mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet,$$

where the boundary condition on $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ forces bundle maps to be zero near $\Delta^\bullet \times \partial_0 P$. Clearly $\widetilde{\mathrm{BunMap}}_{\partial_0}(\nu_\iota, \tau_P; \mathrm{Id}_P)_\bullet$ is weakly contractible: indeed given an n-cycle G in this semi-simplicial set, a nullhomotopy of G is roughly given by regarding Δ^{n+1} as $(\Delta^n \times [0,1], \Delta^n \times \{0\})$ and applying $t \cdot G$ on $\Delta^n \times \{t\}$, for $0 \leq t \leq 1$. Therefore $|\widetilde{F}_\bullet| \simeq |\mathrm{Aut}_{\partial_0}(\nu_\iota)_\bullet| = \mathrm{Aut}_{\partial_0}(\nu_\iota)$, as required. □

Remark 3.4. Proposition 3.3 is false in the topological setting. First of all, a locally flat embedding $\iota : P^p \hookrightarrow M^d$ does not always admit a normal microbundle (cf. [RS67]; they do admit one stably though [Hir66, Thm. B]). But even if it did, the statement would still not hold in general: consider the case $(M,P) = (D^d, D^p)$ for $p \leq d-3$. Then both $\mathrm{Emb}^{\mathrm{Top}}_\partial(D^p,D^d)$ and $\widetilde{\mathrm{Emb}}^{\mathrm{Top}}_\partial(D^p,D^d)$ are contractible by the

Alexander trick. However, using the topological version of Theorem A (cf. Remark 1.3), we will see in Remark 6.3 that the homotopy fibre of the map

$$\mathrm{Emb}^{\mathrm{Top}}_{\partial_0}(D^p \times D^{d-p}, D^d) \longrightarrow \widetilde{\mathrm{Emb}}^{\mathrm{Top}}_{\partial_0}(D^d \times D^{d-p}, D^d)$$

is not contractible. In particular, the topological analogue of the square (3.1) cannot possibly be homotopy cartesian in this case.

3.2. **Last ingrediants.** From now on, let $\iota : P^d \hookrightarrow M^d$ be a codimension zero closed embedding that meets ∂M transversally in $\partial_0 P$, and write p for the handle dimension of P relative to $\partial_0 P$; it suffices to prove Theorem A in this case by Proposition 3.3. We now present the last prelimary results that we will need.

3.2.1. *Parametrised isotopy extension theorem.* The *parametrised isotopy extension theorem* states that for $\varphi_t : P \hookrightarrow M$ any continuous family of embeddings parametrised by $t \in \Delta^k$ (with P *compact*), there exists a continuous family of diffeomorphisms $\{\phi_t\}_{t\in\Delta^k}$ of M (which are the identity away from a compact set of M) such that $\phi_0 = \mathrm{Id}_M$ and $\phi_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times \Delta^k$. Moreover, if $K \subset \Delta^k$ is some contractible subcomplex containing the 0-th vertex and $\{\phi'_t\}_{t\in K}$ is another continuous family of diffeomorphisms of M parametrised by K such that $\phi'_0 = \mathrm{Id}_M$ and $\phi'_t(\varphi_0(x)) = \varphi_t(x)$ for all $(x,t) \in P \times K$, then we can arrange $\{\phi_t\}_{t\in\Delta^k}$ as above to agree with $\{\phi'_t\}_{t\in K}$ on K. A consequence of this fact due to Palais [Pal60] (see [Lim64] for a simple proof) is that the restriction map $\mathrm{Diff}_\partial(M) \to \mathrm{Emb}_{\partial_0}(P,M)$ is a locally trivial fibre bundle with $\mathrm{Diff}_\partial(\overline{M-P})$ as fibre. Such fibration can be delooped to the homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \longrightarrow B\mathrm{Diff}_\partial(\overline{M-P}) \longrightarrow B\mathrm{Diff}_\partial(M), \tag{3.2}$$

where the subscript $\langle\iota\rangle$ stands for the union of all the components in $\mathrm{Emb}_{\partial_0}(P,M)$ that contain embeddings of the form $\phi\circ\iota$ for $\phi \in \mathrm{Diff}_\partial(M)$. By replacing P and M in (3.2) by $P\times I$ and $M \times I$, and modifying the boundary conditions, we get a similar homotopy fibre sequence

$$C\mathrm{Emb}(P,M) \longrightarrow BC(\overline{M-P}) \longrightarrow BC(M). \tag{3.3}$$

Note that $C\mathrm{Emb}(M,P)$ is connected by Hudson's Theorem 3.1. Finally, there is a block analogue of (3.2).

Proposition 3.5. *There is a homotopy fibre sequence*

$$\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(\overline{M-P}) \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(M). \tag{3.4}$$

Proof. There is a right action of the simplicial group $\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ on $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet$; we will write $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ for the simplicial set of (level-wise) cosets of this right action. The geometric realisation $|\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet|$ of this simplicial set is homotopy equivalent to the homotopy fibre of the right map of (3.4), so it suffices to show that the action map

$$a : \widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet \longrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)_\bullet, \quad [\phi] \longmapsto \phi\circ\iota$$

is an isomorphism. It is visibly injective, for if $\phi\circ\iota = \psi\circ\iota$ for $\phi,\psi \in \widetilde{\mathrm{Diff}}_\partial(M)_\bullet$, then $\psi^{-1}\circ\phi \in \widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$ and hence $[\psi] = [\psi\circ\psi^{-1}\circ\phi] = [\phi]$ in $\widetilde{\mathrm{Diff}}_\partial(M)_\bullet/\widetilde{\mathrm{Diff}}_\partial(\overline{M-P})_\bullet$.

For surjectivity, let φ be some k-simplex in $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)_\bullet$. Then there exists some $\phi \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ for which φ and $\phi\circ\iota$ lie in the same component in $\widetilde{\mathrm{Emb}}_{\partial_0}(P,M)_\bullet$. Then $\varphi' := \phi^{-1}\circ\varphi \in \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M)_k$ and, in fact, we can arrange that its restriction to the zero-th vertex φ'_0 is ι by rechoosing ϕ (if necessary) using the isotopy extension theorem. Then applying the isotopy extension theorem to φ' restricted to each of the faces that contains the 0-th vertex, inductively on the dimension of the face, we obtain some $\Phi' \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ such that $\Phi' |_{P\times\Delta^k} \equiv \varphi'$. Then $\Phi := \phi\circ\Phi' \in \widetilde{\mathrm{Diff}}_\partial(M)_k$ is such that $\Phi |_{P\times\Delta^k} \equiv \varphi$, as desired. □

Remark 3.6. There also exists a topological version of the isotopy extension theorem [EK71, Cor. 1.4]. The same proof as above also works in the topological setting.

Remark 3.7 (Speculative). Weiss and Williams point out in [WW88, §1] that an analogue of the (parametrised) isotopy extension theorem in the bounded setting does not hold (see [Hir76, Ch. 8, Ex. 9] for a counterexample in codimension 2). However, we believe that a weaker version of the theorem should still hold: namely, for $V \in \mathcal{J}_0$ define the bounded embedding space $\mathrm{Emb}^b_{\partial_0}(P \times V, M \times V)$ as in Definition 2.4. Then, there should be a homotopy fibre sequence

$$\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times V, M \times V) \longrightarrow E(V) = B\mathrm{Diff}^b_\partial(\overline{M - P} \times V) \longrightarrow B(V) = B\mathrm{Diff}^b_\partial(M \times V),$$

where $E(-)$ and $B(-)$ are as in Notation 2.6. We will not give a proof of this claim, as it seems rather technical and we will not need it for the argument of Theorem A. The reader may however find it useful to think of the orthogonal functor $F(-) := \mathrm{hofib}(E(-) \to B(-))$ as $\mathrm{Emb}^b_{\partial_0,\langle\iota\rangle}(P \times (-), M \times (-))$.

3.2.2. *Alexander trick-like equivalences.* Let $\simeq_0$ denote the equivalence relation in Top_* given by weak equivalence after passing to basepoint components. For $V \in \mathcal{J}_0$, let $D(V) \subset V$ denote the correspoding closed unit disk (so that $D^k = D(\mathbb{R}^k)$). The following is proved in Propositions 1.8, 1.10 and 1.12 of [WW88]. Even though we state it for the orthogonal functor $B(-)$ of Notation 2.6, it of course holds for $E(-)$ too.

Proposition 3.8. *For $V \in \mathcal{J}_0$, the (delooped) Alexander trick-like map*

$$B(\mathrm{alex}) : BC(M \times D(V)) \xrightarrow{\simeq_0} \Omega^V B^{(1)}(V) = \Omega^V \left(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)\right)$$

induces a weak equivalence on basepoint components. Moreover, there is a homotopy commutative diagram

$$\begin{tikzcd} BC(M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\simeq_0"', "B(\mathrm{alex})"] & BC(M \times D(V) \times D^1) \arrow[r, equal, "\sim"] & BC(M \times D(V \oplus \mathbb{R})) \arrow[d, "\simeq_0"', "B(\mathrm{alex})"] \\ \Omega^V B^{(1)}(V) \arrow[rr, "s_V^\vee"] & & \Omega^{V\oplus\mathbb{R}} B^{(1)}(V \oplus \mathbb{R}), \end{tikzcd} \tag{3.5}$$

where Σ denotes the usual concordance stabilisation map and $s_V^\vee$ is the adjoint of the structure map (2.2) for the orthogonal spectrum $\Theta B^{(1)}$.

Proof. The Alexander trick-like map

$$\mathrm{alex} : C(M \times D(V)) \longrightarrow \Omega^{V\oplus\mathbb{R}} \left(\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)\right)$$

is defined as follows: given a concordance diffeomorphism $\phi : M \times D(V) \times I \cong M \times D(V) \times I$, extend it by $\phi\mid_{M \times D(V) \times \{1\}} \times \mathrm{Id}_{[1,+\infty)}$ on $M \times D(V) \times [1,+\infty)$ and by the identity elsewhere to obtain a bounded self-diffeomorphism $\widehat{\phi}$ of $M \times V \oplus \mathbb{R}$; then shift it along $V \oplus \mathbb{R}$ to obtain a $(V \oplus \mathbb{R})$-fold loop in $\mathrm{Diff}^b_\partial(M \times V \oplus \mathbb{R})/\mathrm{Diff}^b_\partial(M \times V)$. This map was shown to be an equivalence in [WW88, Props. 1.8 & 1.10]. Moreover, up to homotopy, it respects the monoidal structures of both the domain and codomain; here $C(M \times D(V))$ is treated as a topological group with respect to composition, whilst $\Omega^{V\oplus\mathbb{R}} B^{(1)}(V \oplus \mathbb{R})$ is a group-like $\mathbb{E}_1$-space under composition of paths in the $\mathbb{R}$-direction. Therefore alex can be delooped to a weak equivalence on basepoint components (of course alex can also be delooped in the V-direction, but we will not need this).

The homotopy commutative diagram of [WW88, Prop. 1.12] can be delooped (by the same reason as above) to the one in the statement. □

We now establish an analogue of Proposition 3.8 for the orthogonal functor $F(-)$.

Proposition 3.9. *For $V \in \mathcal{J}_0$, there are weak equivalences*

$$\mathrm{alex} : C\mathrm{Emb}(P \times D(V), M \times D(V)) \xrightarrow{\sim} \Omega^V F^{(1)}(V),$$

making the following diagram commute up to homotopy:

$$\begin{tikzcd} C\mathrm{Emb}(P \times D(V), M \times D(V)) \arrow[r, "\Sigma"] \arrow[d, "\wr"', "\mathrm{alex}"] & C\mathrm{Emb}(P \times D(V \oplus \mathbb{R}), M \times D(V \oplus \mathbb{R})) \arrow[d, "\wr"', "\mathrm{alex}"] \\ \Omega^V F^{(1)}(V) \arrow[r, "s_V^\vee"] & \Omega^{V\oplus\mathbb{R}} F^{(1)}(V \oplus \mathbb{R}), \end{tikzcd} \tag{3.6}$$

where Σ here is the concordance embedding stabilisation map of Section 1.1.1. In particular,

$$\Omega^\infty(\Theta F^{(1)}) \simeq C\mathcal{E}\mathrm{mb}(P,M) := \operatorname*{hocolim}_k C\mathrm{Emb}(P\times D^k, M\times D^k). \tag{3.7}$$

For the proof of this proposition, we will first need the following result.

Lemma 3.10. *For $p\le d-3$ and $n\ge 0$, the space $\Theta F^{(1)}_n = F^{(1)}(\mathbb{R}^n)$ is n-connected.*

Proof. It suffices to show that the natural map $\Theta E^{(1)}_n \to \Theta B^{(1)}_n$, call it λ, is such that $\pi_*(\lambda)$ is

(a) surjective if $* = n+1$, (b) injective if $*=0$, (c) an isomorphism if $1\le * \le n$.

For (a), observe that the map $\Omega^{n+1}\lambda$ is, up to equivalence, the natural map of concordance spaces $C(\overline{M-P}\times D^n)\to C(M\times D^n)$ by Proposition 3.8. Then by exactness of

$$\dots \longrightarrow \pi_0(C(\overline{M-P}\times D^n)) \xrightarrow{\pi_{n+1}(\lambda)} \pi_0(C(M\times D^n)) \longrightarrow \pi_0(C\mathrm{Emb}(P\times D^n, M\times D^n)) = *,$$

where the equality on the right is the statement of Hudson's Theorem 3.1, it follows that $\pi_{n+1}(\lambda)$ is surjective.

For (b) and (c), consider the commutative diagram

$$\begin{tikzcd} \Theta E^{(1)}_n \arrow[r, "\lambda"] \arrow[d, "\text{stab.}"] & \Theta B^{(1)}_n \arrow[d, "\text{stab.}"] \\ \Omega^\infty(\Sigma^n\Theta E^{(1)}) \arrow[r, "\nu"] & \Omega^\infty(\Sigma^n \Theta B^{(1)}). \end{tikzcd}$$

Since the inclusion $M-P\hookrightarrow M$ is 2-connected, the induced map on Whitehead spectra $\Sigma\Theta E^{(1)} = \mathbf{Wh}^{\mathrm{Diff}}(M-P)\to \mathbf{Wh}^{\mathrm{Diff}}(M) = \Sigma\Theta B^{(1)}$ is also 2-connected, and so ν is $(n+1)$-connected. Now by [WW88, Cor. 5.8], both vertical maps are injective in $\pi_0(-)$ and isomorphisms in $\pi_{1\le *\le n}(-)$. Claims (b) and (c) now follow immediately. □

Proof of Proposition 3.9. The map alex in the statement is the induced map of homotopy fibres in

$$\begin{tikzcd} C\mathrm{Emb}(P\times D(V), M\times D(V)) \arrow[r] \arrow[d, dashed, "\text{alex}"] & BC(\overline{M-P}\times D(V)) \arrow[r] \arrow[d, "B(\text{alex})", "\simeq_0"'] & BC(M\times D(V)) \arrow[d, "B(\text{alex})", "\simeq_0"'] \\ \Omega^V F^{(1)}(V) \arrow[r] & \Omega^V E^{(1)}(V) \arrow[r] & \Omega^V B^{(1)}(V). \end{tikzcd}$$

The top row is a homotopy fibre sequence by the isotopy extension sequence (3.3) with P and M replaced by $P\times D(V)$ and $M\times D(V)$, respectively. Therefore, by the five lemma, the left vertical map is an equivalence on basepoint components. But as both the domain and codomain of this map are connected—the domain is so by Hudson's Theorem 3.1, and the codomain by Lemma 3.10—it is a weak equivalence, as claimed.

Finally, the commutative square resulting from replacing M by $\overline{M-P}$ (and $E(-)$ by $B(-)$) in (3.5) maps to the original square (3.5), and (3.6) is the homotopy fibre of such map. □

3.3. **The map Φ^{Emb} of Theorem A.** In order to define Φ^{Emb}, we will analyse the map Φ^F_∞ of Proposition 2.6 for $F(-)$. Observe that by definition

$$\Theta F^{(1)} := \mathrm{hofib}(\Theta E^{(1)}\to\Theta B^{(1)}) \simeq \Sigma^{-2}\mathbf{Wh}^{\mathrm{Diff}}(M, M-P). \tag{3.8}$$

Therefore by the isotopy extension sequence (3.2), the map Φ^F_∞ is given, up to homotopy, by

$$\Phi^F_\infty : \mathrm{hofib}(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M)\to F(\mathbb{R}^\infty)) \longrightarrow \Omega^\infty(\Sigma^{-2}\mathbf{Wh}^{\mathrm{Diff}}(M,M-P)_{hC_2}).$$

We wish to replace $F(\mathbb{R}^\infty)$ by $\widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)$ in the domain of this map. This turns out to be possible by a similar principle to that of [WW88, Rem. 3.5].

Proposition 3.11. *If $p\le d-3$, then there is a (zig-zag of) weak equivalence*

$$\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M)\to\widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) \simeq \mathrm{hofib}(F(0)\to F(\mathbb{R}^\infty)).$$

Proof. First observe that because $C\mathrm{Emb}(P,M)$ is connected (by Hudson's Theorem 3.1), we have that

$$\begin{aligned}\mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) &= \mathrm{hofib}(\mathrm{Emb}_{\partial_0,\iota}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\iota}(P,M))\\ &= \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M)).\end{aligned}$$

Therefore, it suffices to construct a homotopy commutative diagram

$$\begin{tikzcd} F(0) \arrow[r] & F(\mathbb{R}^\infty) \arrow[r, "i", "\sim"'] & \widetilde{F}(\mathbb{R}^\infty) \\ \mathrm{Emb}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[u, "\wr", "(3.2)"'] \arrow[rr, hook] & & \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M). \arrow[u, "\wr", "j"'] \end{tikzcd} \tag{3.9}$$

To that end, let $\mathcal{J}_0^\delta$ denote the underlying ordinary category of the topological category $\mathcal{J}_0$, and write $\widetilde{E}(-)$ and $\widetilde{B}(-)$ for the functors $\mathcal{J}_0^\delta \to \mathsf{Top}_*$ given by

$$\widetilde{E}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V)_\bullet|, \qquad \widetilde{B}(V) := B|\widetilde{\mathrm{Diff}}^b_\partial(M \times V)_\bullet|.$$

Set $\widetilde{F}(-) := \mathrm{hofib}(\widetilde{E}(-) \to \widetilde{B}(-))$. Then the map i of (3.9) arises as the map on homotopy fibres in

$$\begin{tikzcd} F(\mathbb{R}^\infty) \arrow[r] \arrow[d, dashed, "i"] & E(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] \arrow[d, hook, "\wr"] & B(\mathbb{R}^\infty) = B\mathrm{Diff}^b_\partial(M \times \mathbb{R}^\infty) \arrow[d, hook, "\wr"] \\ \widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times \mathbb{R}^\infty) \arrow[r] & \widetilde{B}(\mathbb{R}^\infty) = B\widetilde{\mathrm{Diff}}^b_\partial(M \times \mathbb{R}^\infty). \end{tikzcd}$$

The middle and right vertical maps are equivalences by [WW88, Thm. B], so i is too by the five lemma.

The map j of (3.9) arises as the map on homotopy fibres in

$$\begin{tikzcd} \widetilde{F}(\mathbb{R}^\infty) \arrow[r] & \widetilde{E}(\mathbb{R}^\infty) \arrow[r] \arrow[dr, phantom, "(\dagger)"] & \widetilde{B}(\mathbb{R}^\infty) \\ \widetilde{F}(0) \simeq \widetilde{\mathrm{Emb}}_{\partial_0,\langle\iota\rangle}(P,M) \arrow[r] \arrow[u, dashed, "j"] & \widetilde{E}(0) = B\widetilde{\mathrm{Diff}}_\partial(\overline{M-P}) \arrow[r] \arrow[u, hook] & \widetilde{B}(0) = B\widetilde{\mathrm{Diff}}_\partial(M). \arrow[u, hook] \end{tikzcd}$$

Then the square (3.9) is the homotopy fibre of the map between the similar (strictly commutative) squares associated to $E(-)$ and $B(-)$, and so it is homotopy commutative by construction.

It remains to show that j is an equivalence or, equivalently, that the square $(\dagger)$ is homotopy cartesian. Write $\widetilde{F}^{(1)}(V) := \mathrm{hofib}(\widetilde{F}(V) \to \widetilde{F}(V \oplus \mathbb{R}))$, and similarly for $\widetilde{E}^{(1)}(V)$ and $\widetilde{B}^{(1)}(V)$. In other words,

$$\widetilde{E}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}^b_\partial(\overline{M-P} \times V)}, \qquad \widetilde{B}^{(1)}(V) := \frac{\widetilde{\mathrm{Diff}}^b_\partial(M \times V \oplus \mathbb{R})}{\widetilde{\mathrm{Diff}}^b_\partial(M \times V)}.$$

For a group π and an integer $j \leq 1$, set $\kappa_j(\pi) := \pi_j^s(\mathbf{Wh}^{\mathrm{Diff}}(B\pi))$. More explicitly,

$$\kappa_j(\pi) = \begin{cases} \mathrm{Wh}_1(\pi), & j = 1, \\ \widetilde{K}_0(\mathbb{Z}\pi), & j = 0, \\ K_j(\mathbb{Z}\pi), & j \leq -1. \end{cases}$$

It was shown in [WW88, Cor. 5.5] (see also [AP83]) that, for a certain C_2-action on $\kappa_j(\pi)$, there are maps

$$\beta : \pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \to H_*(C_2; \kappa_{1-n}(\pi_1(M-P))), \quad \beta : \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \to H_*(C_2; \kappa_{1-n}(\pi_1(M))), \quad n \geq 0,$$

which are injective if $* = 0$ and isomorphisms if $* \geq 1$. Moreover it is not difficult to see from its proof that these are compatible, in the sense that the square

$$\begin{tikzcd} \pi_*(\widetilde{E}^{(1)}(\mathbb{R}^n)) \arrow[r] \arrow[d, "\beta"] & \pi_*(\widetilde{B}^{(1)}(\mathbb{R}^n)) \arrow[d, "\beta"] \\ H_*(C_2; \kappa_{1-n}(\pi_1(M-P))) \arrow[r, "\cong"] & H_*(C_2; \kappa_{1-n}(\pi_1(M))) \end{tikzcd}$$

is commutative. The lower horizontal map is an isomorphism because the fundamental groups of $M-P$ and M can be identified under the obvious inclusion, by the assumption that $p \leq d-3$. Hence, as

$\widetilde{F}^{(1)}(V) \to \widetilde{E}^{(1)}(V) \to \widetilde{B}^{(1)}(V)$ is a homotopy fibre sequence for all V, it follows that $\widetilde{F}^{(1)}(\mathbb{R}^n)$ is weakly contractible for all $n \geq 0$. Using the homotopy fibre sequences

$$\mathrm{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^n)) \longrightarrow \mathrm{hofib}(\widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^{n+1})) \longrightarrow \widetilde{F}^{(1)}(\mathbb{R}^n) \simeq *$$

for $n \geq 0$, we must have by induction that $\mathrm{hofib}(j : \widetilde{F}(0) \to \widetilde{F}(\mathbb{R}^\infty))$ is contractible, i.e., that j is a weak equivalence, as required. □

Definition 3.12. *The map Φ^{Emb} of Theorem A is the zig-zag*

$$\begin{aligned}\Phi^{\mathrm{Emb}} : \mathrm{hofib}_\iota(\mathrm{Emb}_{\partial_0}(P,M) \hookrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}(P,M)) &\simeq \mathrm{hofib}(F(0) \to F(\mathbb{R}^\infty)) && \textit{by Prop. 3.11,}\\ &\xrightarrow{\Phi^F_\infty} \Omega^\infty(\Theta F^{(1)}_{hO(1)}) \simeq \Omega^\infty(\Sigma^{-2}\mathbf{Wh}^{\mathrm{Diff}}(M, M-P)_{hC_2}) && \textit{by (2.7) and (3.8).}\end{aligned}$$

3.4. **Connectivity of the map Φ^{Emb}.** In this section we show that the map Φ^{Emb} just defined is $\phi_{C\mathrm{Emb}}(d,p)$-connected, at last establishing Theorem A (modulo the proof of Proposition A.1). The connectivity of Φ^{Emb} is that of Φ^F_∞; we show by induction on $n \geq 0$ that the maps Φ^F_n of Proposition 2.2 are at least $\phi_{C\mathrm{Emb}}(d,p)$-connected. Note that this is clear for $n = 0$, as both the domain and codomain are contractible.

Suppose now that Φ^F_n is $\phi_{C\mathrm{Emb}}(d,p)$-connected for some $n \geq 0$. To show that Φ^F_{n+1} has this connectivity, it suffices to show that the map stab. : $\Theta F^{(1)}_n \to \Omega^\infty(\Sigma^n \Theta F^{(1)})$ of Proposition 2.2 is $(\phi_{C\mathrm{Emb}}(d,p)+n)$-connected. But $\Theta F^{(1)}_n$ is n-connected by Lemma 3.10 and $\Sigma^n\Theta F^{(1)}$ is $(n+1)$-connective. So it suffices to show that $\Omega^n(\mathrm{stab.})$ is $\phi_{C\mathrm{Emb}}(d,p)$-connected. This follows from the homotopy commutative diagram

$$\begin{tikzcd} C\mathrm{Emb}(P \times D^n, M \times D^n) \arrow[r, hook] \arrow[d, "\mathrm{alex}", "\wr"'] & \mathcal{C}\mathcal{E}\mathrm{mb}(P,M) \arrow[d, "(3.7)", "\wr"'] \\ \Omega^n\Theta F^{(1)}_n \arrow[r, "\Omega^n(\mathrm{stab.})"] & \Omega^\infty(\Theta F^{(1)}). \end{tikzcd}$$

The top horizontal map is $\phi_{C\mathrm{Emb}}(d+n,p) \geq \phi_{C\mathrm{Emb}}(d,p)$-connected by definition. One obtains the above homotopy commutative diagram from stacking together squares of the form (3.6) with $V = \mathbb{R}^k$ for $k \geq n$. This finishes the proof of Theorem A. □

4. A SPLITTING RESULT FOR EMBEDDING SPACES

In this section we derive, as a consequence of Theorem A, a general splitting result[2] for embedding spaces of manifolds with interval factors. This will then be used for the splitting part of Theorem B. All throughout, let $\iota : P^p \subset M^d$ be as in the statement of Theorem A.

For $D(-)$ any of $\mathrm{Diff}^b_\partial(- \times V)$ with $V \in \mathcal{J}_0$, $\widetilde{\mathrm{Diff}}_\partial(-)$ or $C(-)$, there are *graphing maps*

$$\Gamma : \Omega D(M) \longrightarrow D(M \times I)$$

given (roughly) by regarding a 1-parameter family of automorphisms of M as an automorphism of $M \times I$ itself. These are natural with respect to codimension zero embeddings. Moreover, these maps can be delooped as, up to homotopy, they intertwine the (group-like) $\mathbb{E}_1$-structures of concatenating loops for the domain, and stacking automorphisms in the I-direction for the codomain. There are similar maps

$$\Gamma : \Omega E(P,M) \longrightarrow E(P \times I, M \times I) \tag{4.1}$$

for $E(-,-)$ any of $\mathrm{Emb}_{\partial_0}(-,-)$, $\widetilde{\mathrm{Emb}}_{\partial_0}(-,-)$, $\mathrm{Emb}^{(\sim)}_{\partial_0}(-,-)$ (see (1.4)) or $C\mathrm{Emb}(-,-)$. In what follows, we will write Γ for any map of this same nature.

Remark 4.1. Most of the functors $D(-)$ and $E(-,-)$ above either admit a point-set topological model or a simplicial model. In the first case, the graphing maps just introduced are really zig-zags of maps

$$\Omega E(P,M) \xleftarrow{\sim} \Omega^{\mathsf{col,sm}} E(P,M) \xrightarrow{\Gamma} E(P \times I, M \times I),$$

where, for X a pointed (Fréchet) manifold, here $\Omega^{\mathsf{col,sm}}X$ stands for the space of smooth loops $\gamma : S^1 \to X$ which are *collared* in the sense that there exists some neighbourhood of $1 \in S^1$ which is sent by γ to the

[2]This should be compared to the analogous result of Burghelea and Lashof [BL82, Cor. E].

basepoint in X. The inclusion $\Omega^{\mathrm{col,sm}}E(P,M) \hookrightarrow \Omega E(P,M)$ is an equivalence by smooth approximation of continuous functions.

In the simplicial case, the graphing maps Γ are the geometric realisations of the simplicial maps

$$\Gamma_\bullet : (\Omega E(P,M))_\bullet \longrightarrow E(P\times I, M\times I)_\bullet$$

that send a q-simplex in $(\Omega E(P,M))_\bullet$ (seen as a $(q+1)$-simplex $g \in E(P,M)_\bullet$ whose 0-th face and vertex are the basepoint $* \in E(P,M)_\bullet$) to the q-simplex in $E(P\times I, M\times I)_\bullet$ obtained from g by expanding out the 0-th vertex of Δ^{q+1} to a q-dimensional simplex (i.e. regarding Δ^{q+1} as $(\Delta^q \times I, \Delta^q\times\{0\})$).

In the cases when $D(-)$ or $E(-,-)$ admit both models, one verifies that these two graphing maps agree up to homotopy. We ignore both of these technicalities in most of what follows.

Theorem 4.2. *Let I and J both denote closed intervals. For $p \le d-3$, $N := \phi_{C\mathrm{Emb}}(d+1,p+1)-1$ and $N' := \phi_{C\mathrm{Emb}}(d+2,p+2)-1$, there are equivalences away from 2*

$$\Omega\tau_{\le N}\,\mathrm{Emb}_{\partial_0}(P\times I, M\times I) \simeq_{[\frac{1}{2}]} \Omega\tau_{\le N}\left(\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I, M\times I)\times \mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I)\right),$$

$$\tau_{\le N'}\,\mathrm{Emb}_{\partial_0}(P\times I\times J, M\times I\times J) \simeq_{[\frac{1}{2}]} \tau_{\le N'}\begin{pmatrix}\widetilde{\mathrm{Emb}}_{\partial_0}(P\times I\times J, M\times I\times J)\\ \times \\ \mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I\times J, M\times I\times J)\end{pmatrix}.$$

Proof. Suppose given a map of fibration sequences

$$\begin{tikzcd} F' \arrow[r] & E' \arrow[r, "p'"] & B' \\ F \arrow[r] \arrow[u, "f\simeq *"] & E \arrow[r, "p"] \arrow[u, "e"] & B \arrow[u, "b", "\wr"'] \end{tikzcd}$$

such that f is nullhomotopic and b is an equivalence. If $\delta : \Omega B \to F$ (and similarly for δ') denotes the connecting map, then it follows that $\delta'\circ\Omega b \simeq f\circ\delta \simeq *$, and thus Ωb lifts, up to homotopy, to a map $\tilde\sigma : \Omega B \to \Omega E'$. Then for $(\Omega b)^{-1}$ any homotopy inverse to Ωb, the map $\sigma := \tilde\sigma\circ(\Omega b)^{-1} : \Omega B' \to \Omega E'$ is a homotopy section of the fibration $\Omega F' \to \Omega E' \to \Omega B'$ and so provides a splitting $\Omega E' \simeq \Omega B' \times \Omega F'$.

For now, let us focus solely on the first equivalence of the statement. By the previous argument, it suffices to show that the leftmost vertical graphing map in the diagram of fibration sequences

$$\begin{tikzcd} \mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I) \arrow[r] & \mathrm{Emb}_{\partial_0}(P\times I, M\times I) \arrow[r] & \widetilde{\mathrm{Emb}}_{\partial_0}(P\times I, M\times I) \\ \Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) \arrow[r] \arrow[u, "\Gamma"] & \Omega\,\mathrm{Emb}_{\partial_0}(P,M) \arrow[r] \arrow[u, "\Gamma"] & \Omega\widetilde{\mathrm{Emb}}_{\partial_0}(P,M). \arrow[u, "\Gamma", "\wr"'] \end{tikzcd}$$

is nullhomotopic after localising away from 2 and taking $(\phi_{C\mathrm{Emb}}(d+1,p+1)-1)$-th Postnikov sections.

By Proposition 3.3, we may assume that $\dim P = \dim M = d$. Then, similar to Notation 2.6, let $\Omega F(-)$ and $FI(-)$ denote the orthogonal functors given by

$$\Omega F(V) := \Omega\,\mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M-P}\times V) \to B\mathrm{Diff}^b_\partial(M\times V)),$$
$$FI(V) := \mathrm{hofib}(B\mathrm{Diff}^b_\partial(\overline{M-P}\times I\times V) \to B\mathrm{Diff}^b_\partial(M\times I\times V)).$$

By taking fibres of the (delooped) graphing maps introduced at the beginning of the section, we obtain a natural transformation $\Gamma : \Omega F(-) \to FI(-)$ of orthogonal functors, giving rise to a map of $O(1)$-spectra $\Gamma : \Theta(\Omega F)^{(1)} \longrightarrow \Theta FI^{(1)}$ and a commutative diagram

$$\begin{tikzcd} \mathrm{Emb}^{(\sim)}_{\partial_0}(P\times I, M\times I) \arrow[r, "\Phi^{FI}_\infty"] & \Omega^\infty(\Theta FI^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] & \Omega^\infty\Theta FI^{(1)} \\ \Omega\,\mathrm{Emb}^{(\sim)}_{\partial_0}(P,M) \arrow[r, "\Phi^{\Omega F}_\infty"] \arrow[u, "\Gamma"] & \Omega^\infty(\Theta(\Omega F)^{(1)}_{hO(1)}) \arrow[r, "\mathrm{Trf}_{O(1)}"] \arrow[u, "\Gamma"] & \Omega^\infty\Theta(\Omega F)^{(1)}, \arrow[u, "\Gamma"] \end{tikzcd} \tag{4.2}$$

where $\mathrm{Trf}_{O(1)}$ is the $O(1)$-*transfer map*. It is well known that this map is injective in the homotopy category of infinite loop spaces at odd primes. Therefore, since the map Φ^{FI}_∞ is $\phi_{C\mathrm{Emb}}(d+1,p+1)$-connected by

Section 3.4 (and thus becomes an equivalence after taking $(\phi_{\mathrm{CEmb}}(d+1,p+1)-1)$-th Postnikov sections), it will suffice to show that the rightmost vertical map in (4.2) is nullhomotopic. By (3.7), we have that $\Omega^\infty\Theta(\Omega F)^{(1)} \simeq \Omega\mathcal{C}\mathcal{E}\mathrm{mb}(P,M)$ and $\Omega^\infty\Theta FI^{(1)} \simeq \mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I)$ and, under these equivalences, the right vertical map in (4.2) then becomes the graphing map

$$\Gamma : \Omega\mathcal{C}\mathcal{E}\mathrm{mb}(P,M) \longrightarrow \mathcal{C}\mathcal{E}\mathrm{mb}(P\times I, M\times I). \tag{4.3}$$

This is because both the concordance stabilisation map and the Alexander trick-like map of Proposition 3.9 that give rise to the previous equivalences commute on the nose with the graphing maps, i.e., the following diagrams comute:

$$\begin{tikzcd}
\Omega C\mathrm{Emb}(P\times D^{n+1}, M\times D^{n+1}) \arrow[r, "\Gamma"] & C\mathrm{Emb}(P\times I\times D^{n+1}, M\times I\times D^{n+1}) \\
\Omega C\mathrm{Emb}(P\times D^{n}, M\times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\Sigma"] & C\mathrm{Emb}(P\times I\times D^{n}, M\times I\times D^{n}),\arrow[u, "\Sigma"]
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^n\Theta(\Omega F)^{(1)}_n \arrow[r, "\Gamma"] & \Omega^n\Theta FI^{(1)}_n \\
\Omega C\mathrm{Emb}(P\times D^{n}, M\times D^{n}) \arrow[r, "\Gamma"] \arrow[u, "\mathrm{alex}", "\wr"'] & C\mathrm{Emb}(P\times I\times D^{n}, M\times I\times D^{n}). \arrow[u, "\mathrm{alex}", "\wr"']
\end{tikzcd}$$

So in order to show that (4.3) is nullhomotopic, it suffices to argue that it is so unstably, i.e. that the graphing maps

$$\Gamma : \Omega C\mathrm{Emb}_{\partial_0}(P\times D^n, M\times D^n) \longrightarrow C\mathrm{Emb}_{\partial_0}(P\times I\times D^n, M\times I\times D^n)$$

are nullhomotopic for all $n\geq 0$. Replacing $M\times D^n$ by M, we may assume $n=0$. This claim is a consequence of the following trick, which I owe to Oscar Randal-Williams: there is a "U-shaped graphing map"

$$U\Gamma : \Omega C\mathrm{Emb}(P,M) \longrightarrow C\mathrm{Emb}(P\times I, M\times I),$$

which is homotopic to the standard Γ by pulling down the U-shape to the base of the concordance. This homotopy is illustrated in Figure 1, where we replace $C\mathrm{Emb}(-,-)$ by standard concordances $C(-)$ because it is easier to depict, but the idea is the same. Observe that, throughout the homotopy, there are no issues about smoothness in the upper corners because the concordances are equal to the identity near these. Here we are explicitly using the collared condition imposed by the functor $\Omega^{\mathrm{col,sm}}(-)$ (see Remark 4.1).

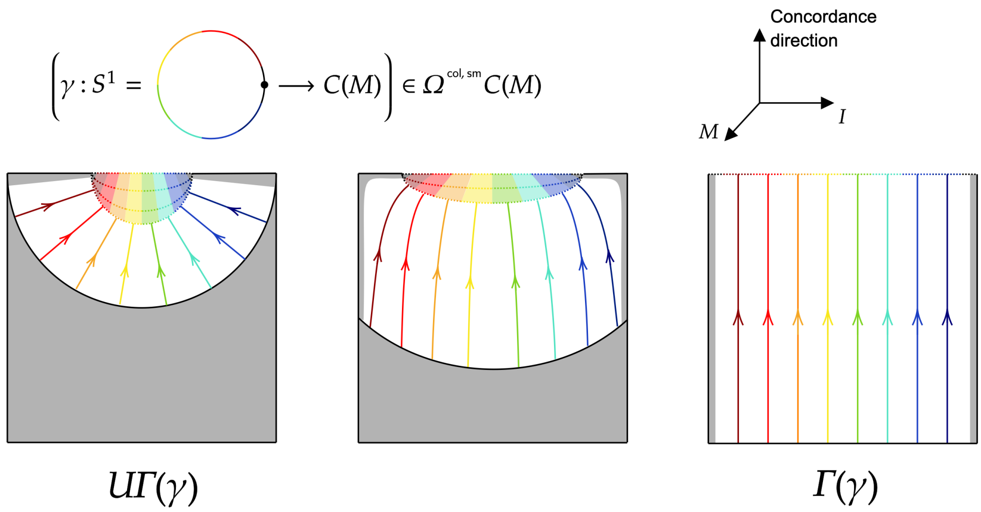

FIGURE 1. Images of $\gamma\in\Omega C(M)$ under the graphing maps Γ and $U\Gamma$, and the homotopy between them. The concordances are equal to the identity on grey shaded regions.

But clearly $U\Gamma$ factors through the path space $\mathrm{Map}(I, C\mathrm{Emb}(P,M))$, and hence there is a homotopy commutative diagram

$$\begin{tikzcd}
C\mathrm{Emb}(P\times I, M\times I) \arrow[r, equal] & C\mathrm{Emb}(P\times I, M\times I) & & \\
& & \mathrm{Map}(I, C\mathrm{Emb}(P,M)) \arrow[ul, "U\Gamma"'] \arrow[r, "\mathrm{ev}_0", "\sim"'] & C\mathrm{Emb}(P,M), \\
\Omega C\mathrm{Emb}(P,M) \arrow[uu, "\Gamma"] \arrow[r, equal] & \Omega C\mathrm{Emb}(P,M) \arrow[uu, "U\Gamma"] \arrow[ur, hook] \arrow[urr, "\mathrm{ev}_0 = *"'] & &
\end{tikzcd}$$

which exhibits the leftmost vertical map as nullhomotopic, as desired. This establishes the first equivalence of the statement.

Observe now that for $E(-,-)$ any of the mapping spaces involved in the proof up until now, the space $E(P\times J, M\times J)$ is a group-like topological monoid with respect to stacking in the J-direction. Then replacing (M,P) by $(M\times J, P\times J)$, one checks that each of the steps in the previous argument can be delooped with respect to this $\mathbb{E}_1$-structure. This results in getting rid of the loopings in the first equivalence of the statement, thus yielding the second one. This finishes the proof of the theorem. $\square$

The following result will be used to establish the splitting part of Theorem B.

Corollary 4.3. *For $2 \le p \le d-3$ and $N := \phi_{C\mathrm{Emb}}(d,p) - 1$, there is an equivalence away from 2*

$$\tau_{\le N}\,\mathrm{Emb}_\partial(D^p, D^d) \simeq_{[\frac{1}{2}]} \tau_{\le N}\left(\widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \times \mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)\right).$$

For $p=1$, such equivalence just exists after looping, i.e.,

$$\Omega\tau_{\le N}\,\mathrm{Emb}_\partial(D^1, D^d) \simeq_{[\frac{1}{2}]} \Omega\tau_{\le N}\left(\widetilde{\mathrm{Emb}}_\partial(D^1, D^d) \times \mathrm{Emb}_\partial^{(\sim)}(D^1, D^d)\right).$$

Proof. When $p \ge 2$, set $(M,P) = (D^{d-2}, D^{p-2})$ in the second equivalence of Theorem 4.2. For $p=1$, set $(M,P) = (D^{d-1}, D^0)$ in the first equivalence of the same theorem. $\square$

5. Involutions in algebraic K-theory

The aim of this section is to explore the involutions of the C_2-spectra involved in the statements of Theorems 1.1 and A, and to identify them in terms of more simple and computable involutions coming from algebraic K-theory—the main result in this direction is Theorem 5.11, which is further simplified by Proposition 5.19 in the case of a suspension. This will then be used in Section 6 to study the case $(M,P) = (D^d, D^p)$. As we will shortly see in Section 5.1, it will be significantly helpful to invert the prime 2 in the analysis of these involutions. Let us now introduce the notation that will be relevant in this section.

Notation 5.1. *(i) Given a smooth manifold M, let $\Theta(M)$ denote the C_2-spectrum $\Theta B^{(1)}$, where $B(-)$ is the orthogonal functor $B(-) := B\mathrm{Diff}^b_\partial(M\times -)$ of Notation 2.6. We recall that the underlying spectrum of $\Theta(M)$ is equivalent to $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M)$. We will refer to $\Theta(M)$ as the* ***Weiss–Williams spectrum*** *of M and denote its involution by τ_{WW}. If $Q \subset M$ is a closed (as a subspace) codimension zero submanifold, we will denote by $\Theta(M,Q)$ the homotopy cofibre of the C_2-equivariant map $\Theta(Q) \to \Theta(M)$.*

(ii) Given a space X and a spherical fibration ξ over X equipped with a section, Vogell defined in [Vog85, p. 300] *an involution τ_ξ on*[3] *$\mathbf{A}(X)$ by means of Spanier–Whitehead duality with respect to the Thom spectrum of ξ; we will write $\mathbf{A}(X;\xi)$ for the corresponding C_2-spectrum. Under the well-known splitting*

$$\mathbf{A}(X) \simeq \Sigma^\infty_+ X \vee \mathbf{Wh}^{\mathrm{Diff}}(X), \tag{5.1}$$

the involution τ_ξ descends to one on $\mathbf{Wh}^{\mathrm{Diff}}(X)$; call it τ_ξ too. We will write $\mathbf{Wh}^{\mathrm{Diff}}(X;\xi)$ for the corresponding C_2-spectrum. When $\xi = \epsilon := X \times S^0$ is the trivial 0-dimensional sphere bundle, we will refer to τ_ϵ as the ***canonical involution*** *of K-theory, and sometimes write $\mathbf{A}(X)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X)$ for $\mathbf{A}(X;\epsilon)$ and $\mathbf{Wh}^{\mathrm{Diff}}(X;\epsilon)$. We will recall a construction of τ_ϵ in terms of Spanier–Whitehead duality in Section 5.4.*

[3] Vogell defined τ_ξ on the A-theory space $A(X)$, but this involution can be upgraded to $\mathbf{A}(X)$ by specifying it on the Waldhausen category of retractive spaces over X "with ξ-duality" and appealing to the definition of algebraic K-theory via the $S_\bullet$-construction.

5.1. **Homotopy involutions.** A *homotopy involution* τ on a space or infinite loop space or spectrum X is a self-map $\tau : X \to X$ whose square τ^2 is homotopic to the identity Id_X. In this section we explain why, in the stable setting and once the prime 2 is inverted, an involution carries the same amount of information as its underlying homotopy involution. This will be very useful when comparing the C_2-spectra $\Theta(M)$ and $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon)$ (see Proposition 5.15). Let us fix some notation first.

Notation 5.2. *(i) Let C denote any of Top_*, Ω^∞-Top or Sp, and let $X, X' \in \mathsf{C}$ be equipped with homotopy involutions τ and τ', respectively. A map $f : X \to X'$ will be said to be **homotopy C_2-equivariant**, or **C_2-equivariant up to homotopy**, if $f\tau \simeq \tau' f$. If X and X' can be connected by a zig-zag of homotopy C_2-equivariant weak equivalences, we will say that X and X' are **homotopy C_2-equivariantly equivalent** and write*

$$X \approx X'.$$

*A C_2**-equivariant equivalence** will always mean a zig-zag of weak equivalences which are C_2-equivariant.*

*(ii) An **H-group** (X, μ) is a group-like $\mathbb{A}_3$-space (i.e. a homotopy associative H-space such that $\pi_0(X)$ is a group with respect to μ). Given H-spaces (X, μ) and (X', μ'), a based map $f : X \to X'$ will be said to be **monoidal up to homotopy**, or simply an **H-map**, if the following diagram is homotopy commutative:*

$$\begin{array}{ccc} X \times X & \xrightarrow{f\times f} & X' \times X' \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu'} \\ X & \xrightarrow{\quad f \quad} & X'. \end{array}$$

*An **equivalence of H-groups** will mean a zig-zag of H-maps that are additionally weak equivalences. In practice, all H-groups we will consider are actually $\mathbb{E}_1$-groups (i.e. group-like $\mathbb{E}_1$-spaces), and all H-maps can be upgraded to $\mathbb{E}_1$-maps even though we will not need this.*

(iii) The functor $(-)_{hC_2}$ will always stand for pointed homotopy C_2-orbits.

In the cases of interest to us and once the prime 2 is inverted, taking homotopy C_2-orbits with respect to a homotopy involution turns out to make sense.

Proposition 5.3. *Let X and X' both denote a spectrum, infinite loop space or an H-group, and let τ be a homotopy involution on X. Suppose that multiplication by two is invertible on X, i.e. $2 : X \xrightarrow{\sim} X$ is an equivalence, and define*

$$E(X,\tau) := \operatorname{hocolim}\big(X \xrightarrow{\frac{1+\tau}{2}} X \xrightarrow{\frac{1+\tau}{2}} \dots \big),$$

where $\frac{1+\tau}{2}$ really stands for the zig-zag $X \xrightarrow{1+\tau} X \xleftarrow[\sim]{2} X$. Then if τ is an actual involution on X, there is a natural equivalence away from two

$$X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau).$$

Therefore if τ and τ' are involutions on X and X', respectively, for which there is a homotopy C_2-equivariant equivalence $X \approx X'$ (in the category of X and X'), then we have an equivalence away from two

$$X_{hC_2} \simeq_{[\frac{1}{2}]} X'_{hC_2}.$$

Proof. Let us just deal with the case when X is an H-group, as the other cases are completely analogous (and easier). We also assume that 2 is inverted. Observe now that as $t \cdot \frac{1+t}{2} = \frac{1+t}{2}$ in $\mathbb{Z}[C_2]$, then the following commutes up to homotopy

$$\begin{tikzcd} X \arrow[r, "\frac{1+\tau}{2}"] \arrow[dr, "i"'] & X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "i"] & \dots \arrow[dl, "i"] \\ & X_{hC_2}, & \end{tikzcd}$$

where $i : X \to X_{hC_2} := \operatorname{hocoeq}\big(X \overset{\tau}{\underset{1}{\rightrightarrows}} X \big)$ is the inclusion into the first stage of the coequaliser. We thus obtain a map $\eta_{(X,\tau)} : E(X,\tau) \to X_{hC_2}$. The Bousfield–Kan homotopy orbits spectral sequence for X (which is no longer fringed as X is group-like), together with the assumption that 2 is inverted, gives a natural isomorphism $\pi_*(X_{hC_2}) \cong H_0(C_2; \pi_*(X)) \cong \pi_*(X)_{C_2}$. Also by definition, we have that

$\pi_*(E(X,\tau)) \cong \mathrm{Im}(\frac{1+\tau}{2} : \pi_*(X) \to \pi_*(X))$. Then under these identifications, the map $\pi_*(\eta_{(X,\tau)})$ is seen to be the natural isomorphism (away from 2) that sends an element $\beta = \frac{1+\tau}{2}\alpha \in \pi_*(E(X,\tau))$ to $[(1+\tau)\alpha] \in \pi_*(X)_{C_2}$. So $\eta_{(X,\tau)}$ is the desired equivalence $X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau)$.

For the second part of the statement, we may assume without loss of generality that the homotopy C_2-equivariant equivalence $X \approx X'$ is induced by a single equivalence $f : X \xrightarrow{\sim} X'$ (which is an H-map in the case when X and X' are H-groups). Then, the diagram

$$\begin{tikzcd} X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "f", "\wr"'] & X \arrow[r, "\frac{1+\tau}{2}"] \arrow[d, "f", "\wr"'] & \dots \\ X' \arrow[r, "\frac{1+\tau'}{2}"] & X' \arrow[r, "\frac{1+\tau'}{2}"] & \dots \end{tikzcd}$$

commutes up to homotopy, yielding the middle equivalence in

$$X_{hC_2} \simeq_{[\frac{1}{2}]} E(X,\tau) \simeq E(X',\tau') \simeq_{[\frac{1}{2}]} X'_{hC_2}.$$

□

Hence, when working with 2 inverted, we will often simply write $\simeq_{[\frac{1}{2}]}$ instead of $\approx$. Even though we are not going to use it explicitly, it is good to have the following result as a rule of thumb, as it captures pretty well the effect that localising away from 2 has in the C_2-equivariant stable setting.

Corollary 5.4. *Let X be a C_2-spectrum and let the prime 2 be inverted. Then the homotopy type of $\Omega^\infty(X_{hC_2})$ (as a space) is completely determined by the homotopy type of the space $\Omega^\infty X$ and the homotopy classes of the maps $t : \Omega^\infty X \to \Omega^\infty X$ and $+ : \Omega^\infty X \times \Omega^\infty X \to \Omega^\infty X$, where $t \in C_2$ is the generator.*

Proof. By Proposition 5.3, the homotopy type away from 2 of $(\Omega^\infty X)_{hC_2} \simeq_{[\frac{1}{2}]} E(\Omega^\infty X, t)$ is of course determined by such data. The claim now follows because the natural map $(\Omega^\infty X)_{hC_2} \to \Omega^\infty(X_{hC_2})$ is an equivalence away from 2 (this is a consequence of the Bousfield–Kan homotopy orbits spectral sequence). □

The following result, though unrelated to what has been discussed so far in this section, will be useful later on. Given an $\mathbb{E}_1$-space X, we will write X^{op} for X equipped with the opposite $\mathbb{E}_1$-structure. An *anti-involution* τ on an $\mathbb{E}_1$-space X is an $\mathbb{E}_1$-map $\tau : X \to X^{\mathrm{op}}$ whose square equals the identity of X (noting that $(X^{\mathrm{op}})^{\mathrm{op}} \cong X$). Up to equivalence, there is a canonical way of delooping such an anti-involution.

Lemma 5.5. *Let X be an $\mathbb{E}_1$-space. There is a natural equivalence*

$$\iota : B(X^{\mathrm{op}}) \simeq BX.$$

such that, for any anti-involution τ on X, the composition

$$\overline{B}\tau : BX \xrightarrow{B\tau} B(X^{\mathrm{op}}) \overset{\iota}{\simeq} BX$$

is an involution on BX.

Proof. For each $k \geq 0$, the map $\mathbb{E}_1(k) \to \pi_0(\mathbb{E}_1(k))$ is an equivalence, and hence there is a natural zig-zag of equivalences of $\mathbb{E}_1$-algebras

$$B(\pi_0(\mathbb{E}_1), \mathbb{E}_1, X) \xleftarrow{\sim} B(\mathbb{E}_1, \mathbb{E}_1, X) \xrightarrow{\sim} X.$$

But the $\mathbb{E}_1$-structure on the left hand side factors through the associative operad $\mathcal{A}ss := \pi_0(\mathbb{E}_1)$, so for simplicity, we may assume that X is strictly associative. The equivalence ι is then induced on the realisation of the nerve $N_\bullet X$ by the maps

$$X^q \times \Delta^q \longrightarrow X^q \times \Delta^q, \quad (x_1, \dots, x_q, r) \longmapsto (x_q, \dots, x_1, \Phi_q(r)),$$

where $\Phi_q : \Delta^q \cong \Delta^q$ is the linear homeomorphism induced by reversing the order of the vertices. It is easy to check that the map $\overline{B}\tau$ indeed defines an involution on BX. □

5.2. **From the Weiss–Williams spectrum to spaces of h-cobordisms.** We now recall Vogell's model for spaces of h-cobordisms (cf. [Vog85, p. 296]). A *partition* of a manifold M^d is a triple (W, F, V), where W is a codimension zero submanifold of $M \times [-1, 1]$, V is the closure of the complement of W and $F^d := W \cap V$. For technical reasons (see Construction 5.9 below), we require F to be standard near $\partial M \times [-1, 1]$, and that it intersects it in $\partial M \times \{0\}$. Let $H(M)_\bullet$ denote the simplicial set a p-simplex of which is a (locally trivial smooth) family of partitions of M parametrised by Δ^p such that W is an h-cobordism from $M \times \{-1\} \times \Delta^p$ to F. Set $H(M) := |H(M)_\bullet|$ and write $H^s(M) \subset H(M)$ for the connected component containing the trivial partition $* = (M \times [-1, 0], M \times \{0\}, M \times [0, 1])$. There is a canonical involution ι_H given by turning upside down partitions. Namely

$$\iota_H : H(M) \longrightarrow H(M), \quad \rho = (W, F, V) \longrightarrow \rho^* := (V^*, F^*, W^*),$$

where W^*, F^* and V^* are respectively the images of W, F and V under the reflection $r = \mathrm{Id}_M \times -1$. For the smooth case, we will also need a small variant of this h-cobordism space, denoted $H_{\mathsf{col}}(M)$, a point of which consists of a partition $\rho = (W, F, V) \in H(M)$ together with a bicollar of F for W and V which is standard near $\partial M \times [-1, 1]$. The forgetful map $H_{\mathsf{col}}(M) \to H(M)$ is a weak equivalence by the contractibility of the space of collars.

Fix an embedding $M^d \subset \mathbb{R}^N \subset \mathbb{R}^\infty$ and recall that $B\mathrm{Diff}_\partial(M)$ admits a model as the moduli space of manifolds embedded in $\mathbb{R}^\infty$ which are abstractly diffeomorphic to M relative to the boundary ∂M. Similarly $B\mathrm{Diff}^b_\partial(M \times \mathbb{R})$ is the moduli space of manifolds embedded in $\mathbb{R}^\infty \times \mathbb{R}$ which are abstractly diffeomorphic to $M \times \mathbb{R} \subset \mathbb{R}^\infty \times \mathbb{R}$ *boundedly* with respect to the $\mathbb{R}$-direction and relative to the boundary $\partial M \times \mathbb{R}$ (this is proved in Appendix A.2). For the remaining of this section, we will denote by $\underline{\mathbb{R}}$ the bounded direction, i.e., the last coordinate in $\mathbb{R}^\infty \times \mathbb{R} =: \mathbb{R}^\infty \times \underline{\mathbb{R}}$. There is a natural map $- \times \underline{\mathbb{R}} : B\mathrm{Diff}_\partial(M) \hookrightarrow B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ given by sending a manifold $N \subset \mathbb{R}^\infty$ to $N \times \underline{\mathbb{R}} \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$.

Suppose now we are given some partition $\rho = (W, F, V) \in H^s(M)_0$ of $M \times [-1, 1] \subset \mathbb{R}^N \times \underline{\mathbb{R}}$. Then, in particular, as W is an s-cobordism from M to F rel boundary, the manifold $F \subset \mathbb{R}^N \times \mathbb{R} \subset \mathbb{R}^\infty$ gives rise to a point in $B\mathrm{Diff}_\partial(M)$; more precisely, the image of the embedding

$$i_\rho : F \subset M \times I \subset \mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1} \subset \mathbb{R}^\infty,$$

is a point in $B\mathrm{Diff}_\partial(M)$, where the isomorphism $\mathbb{R}^N \times \underline{\mathbb{R}} \cong \mathbb{R}^{N+1}$ identifies $\underline{\mathbb{R}}$ with the last coordinate in $\mathbb{R}^{N+1}$. We now construct a point in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$ by "extending W towards infinity": Consider the embedding of $F \times [0, 1]$ into $\mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}$ given by

$$R : F \times [0, 1] \hookrightarrow \mathbb{R}^N \times \mathbb{R} \times \underline{\mathbb{R}}, \quad (x, t) \longmapsto \underline{e} + (\mathrm{Id}_{\mathbb{R}^N} \times Q_{-\pi t/2})(x - \underline{e}),$$

where $Q_\theta : \mathbb{R} \times \underline{\mathbb{R}} \cong \mathbb{R} \times \underline{\mathbb{R}}$ is the rotation matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$, and $\underline{e}$ denotes the unit length vector in $\underline{\mathbb{R}}$. Write $r := R\,|_{F \times \{1\}} : F \hookrightarrow \mathbb{R}^{N+1} \times \underline{\{1\}}$, and consider

$$S : F \times [1, +\infty) \hookrightarrow \mathbb{R}^{N+1} \times \underline{\mathbb{R}}, \quad (x, t) \longmapsto \begin{cases} r(x) + (1-t) \cdot e_{N+1} + (t-1) \cdot \underline{e}, & t \in [1, 2], \\ r(x) - e_{N+1} + (t-1) \cdot \underline{e}, & t \geq 2. \end{cases}$$

Finally consider the region $D \subset \mathbb{R} \times \underline{\mathbb{R}}$ given by tuples (u, v) with

$$u \geq 0, \quad u \leq 2 - v, \quad \text{and if } 0 \leq u \leq 1, \text{ then } u \leq (1 - (v-1)^2))^{1/2}.$$

Then we define a topological manifold $\widehat{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$, depicted in Figure 2, by

$$\widehat{a}(\rho) := M \times \{0\} \times (-\infty \times -1] \cup W \cup R(F \times [0, 1]) \cup S(F \times [1, +\infty)) \cup \partial M \times D.$$

Now if ρ is a collared partition, i.e. a point in $H^s_{\mathsf{col}}(M)$, one can use the collar of F to smooth out the corners of the topological manifold $\widehat{a}(\rho)$, and thus obtain a smooth manifold $\widetilde{a}(\rho) \subset \mathbb{R}^{N+1} \times \underline{\mathbb{R}}$ with the same boundary as $M \times \underline{\mathbb{R}}$, and which is boundedly diffeomorphic to $M \times \mathbb{R}$ relative to the boundary. This construction can be done simplex-wise in $H^s_{\mathsf{col}}(M)_\bullet \simeq H^s(M)_\bullet$, and so up to weak equivalence gives rise to a (simple) Alexander trick-like map

$$\text{(5.2)} \quad \mathrm{alex}^s : H^s(M) \xrightarrow{\simeq_0} \Theta(M)_0 = \mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})/\mathrm{Diff}_\partial(M) := \mathrm{hofib}(B\mathrm{Diff}_\partial(M) \to B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})),$$

$$\rho = (W, F, V) \longmapsto \left(i_\rho(F), \gamma_V : [-\infty, \infty] \ni t \mapsto \begin{cases} M \times \underline{\mathbb{R}}, & t = -\infty, \\ \widetilde{a}(\rho) - t \cdot \underline{e}, & -\infty < t < +\infty, \\ i_\rho(F) \times \underline{\mathbb{R}}, & t = +\infty. \end{cases} \right)$$

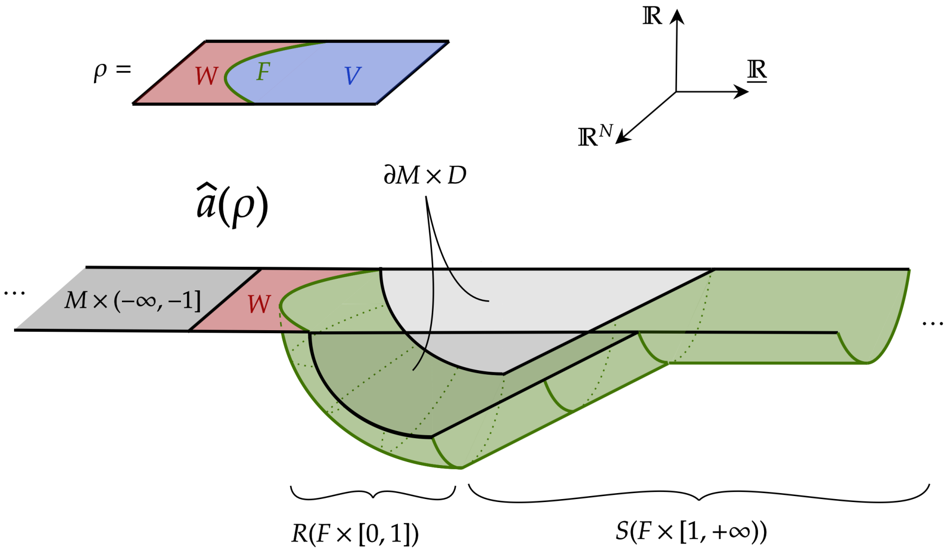

FIGURE 2. Depiction of the topological manifold $\widehat{a}(\rho)$ for $\rho \in H^s(M)$.

where we can regard the path γ_V as a 1-simplex in $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})_\bullet$ from $M \times \underline{\mathbb{R}}$ to $F \times \underline{\mathbb{R}}$. We adopt the convention that $\pm\infty + r \equiv \pm\infty$ for any real number $r \in \mathbb{R}$. Upon noting the well-known equivalence $H^s(M) \simeq BC(M)$ (cf. [Vog85, Prop. 2.1]), the map (5.2) is clearly a delooping of the Alexander trick-like equivalence $C(M) \simeq \Omega(\mathrm{Diff}^b_\partial(M \times \mathbb{R})/\mathrm{Diff}(M))$ of [WW88, Prop. 1.10] (i.e. (5.2) is a model for the map $B(\mathrm{alex})$ of Proposition 3.8 when $V = 0$), and therefore it is an equivalence onto the basepoint component. We keep track of the s-cobordism space $H^s(M)$ in the notation of alex^s as we will later extend this map to the whole of $H(M)$ whenever M^d is of the form $M' \times I$ with $d \geq 5$. Let us first prove the following.

Lemma 5.6. *The map (5.2) is equivariant up to homotopy for ι_H in the domain and τ_{WW} in the target.*

Proof. We will give an argument only in the topological setting; in the smooth setting one works with $H_{\mathsf{col}}(M)$ instead to smooth out corners, and uses smooth approximations of the continuous functions that will appear in proof below. We will however state the argument in the smooth setting to simplify notation. We will also assume at any point in the argument where it is necessary that a partition (W, F, V) is equipped with some bicollar of F in W and V.

The Weiss–Williams involution on $\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})/\mathrm{Diff}_\partial(M)$ is induced by the identity on $B\mathrm{Diff}_\partial(M)$ and the involution $U \mapsto U^* = (\mathrm{Id}_{\mathbb{R}^\infty} \times (-1)_{\underline{\mathbb{R}}})(U)$ on $B\mathrm{Diff}^b_\partial(M \times \underline{\mathbb{R}})$. Then for $\rho = (W, F, V) \in H^s(M)$,

$$\tau_{WW} \circ \mathrm{alex}^s(W, F, V) = (i_\rho(F), \gamma^*_W), \qquad \mathrm{alex}^s \circ \iota_H(W, F, V) = (i_{\rho^*}(F^*), \gamma_{V^*}),$$

where $\gamma^*_W(t) := (\gamma_W(t))^*$. We have depicted the paths γ^*_W and γ_{V^*} in Figure 3. We need to find a path $\eta : [-1, 1] \to B\mathrm{Diff}_\partial(M)$ from $i^*_\rho(F^*)$ to $i_\rho(F)$ and a homotopy $\{H_s(-)\}_{-1 \leq s \leq 1}$ from $\gamma_{V^*}(-)$ to $\gamma^*_W(-)$ such that $H_s(-\infty) = M \times \underline{\mathbb{R}}$ and $H_s(+\infty) = \eta(s) \times \underline{\mathbb{R}}$ for all $s \in [-1, 1]$.

For η, we use the last two coordinates in $\mathbb{R}^{N+2}$ to do a half rotation of that plane. More explictly, $\eta(s) := (\mathrm{Id}_{\mathbb{R}^N} \times Q_{\pi\cdot(s+1)/2})(i_\rho(F))$ where $Q_\theta : \mathbb{R}^2 \cong \mathbb{R}^2$ is as before. View $\mathbb{R}^\infty$ as $\mathbb{R}^\infty \times \underline{\{0\}} \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$ and write $N := \bigcup_{s \in [-1,1]} \eta(s) + s \cdot \underline{e}$. For $X \subset \mathbb{R}^\infty \times \underline{\mathbb{R}}$, write $X \mid_{[a,b]}$ for $X \cap (\mathbb{R}^\infty \times \underline{[a, b]})$. Then consider the compact manifold

$$U_\rho := \left(\widehat{a}(\rho^*) \mid_{[-1,2]} -3 \cdot \underline{e}\right) \cup N \cup \left((\widehat{a}(\rho) \mid_{[-1,2]})^* + 3 \cdot \underline{e}\right)$$

depicted in Figure 4. We can straighten $U_\rho - (V^* \cup W^*) \cong F^* \times (-3, 3)$ to a surface of revolution in $\mathbb{R}^N \times \mathbb{R}^2 \times \underline{\mathbb{R}}$ about a rotation in $\mathbb{R}^3 \cong \mathbb{R}^2 \times \underline{\mathbb{R}}$. Using the bicollar of F^* in W^* and V^* to gradually eat up this surface of revolution, we obtain a canonical path ψ from U_ρ to $M \times \underline{[-4, 4]} = M \times \underline{[-4, -1]} \cup V^* \cup W^* \cup M \times \underline{[1, 4]}$ in the moduli space of manifolds inside $\mathbb{R}^{N+2} \times \underline{[-4, 4]}$ which are diffeomorphic to $M \times \underline{[-4, 4]}$ relative to its boundary $M \times \underline{\{-4\}} \cup \partial M \times \underline{[-4, 4]} \cup M \times \underline{\{4\}}$.

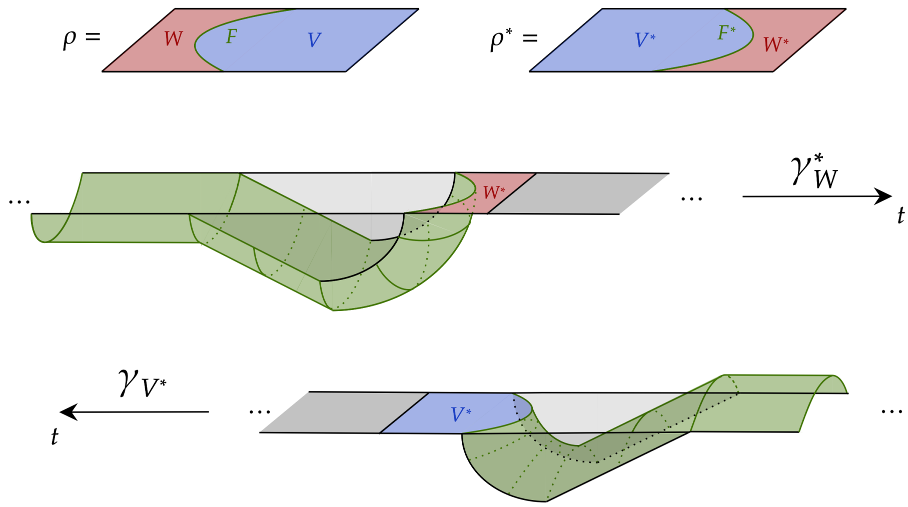

FIGURE 3. Paths γ_W^* and γ_{V^*} in $B\mathrm{Diff}_\partial^b(M \times \mathbb{R})$. The arrow indicates the direction of the path as time increases.

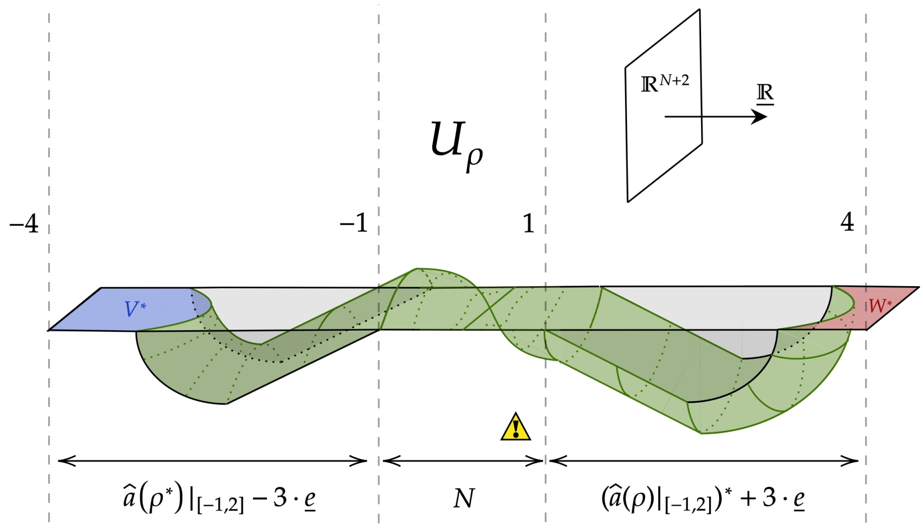

FIGURE 4. Depiction of the manifold U_ρ. Proceed with caution: the part of the picture corresponding to N takes place in an extra dimension that we are unable to depict accurately.

We now describe the homotopy $\{H_s(-)\}_{-1\leq s\leq 1}$. Fix some homeomorphism $l : [-1,1] \cong [-\infty,+\infty]$, and assume that the path ψ from $M \times \underline{[-4,4]}$ to U_ρ just described is parametrised by $[-\infty,+\infty]$. Then $H_s(-)$ is the concatenation of two paths $H_s^{(1)}(-)$ and $H_s^{(2)}(-)$ in $B\mathrm{Diff}_\partial^b(M \times \underline{\mathbb{R}})$: roughly speaking the path $H_s^{(1)}(-)$ performs $\psi(-)$ on $M \times \underline{[-4,4]} + l(s) \cdot \underline{e} \subset M \times \underline{\mathbb{R}}$ (if $s = \pm 1$, $H_s^{(1)}(-)$ is constant on $M \times \underline{\mathbb{R}}$). The path $H_s^{(2)}(-)$ starts at $H_s^{(1)}(+\infty)$, and sends $H_s^{(1)}(+\infty)\mid_{(-\infty,l(s)+s]}$ and $H_s^{(1)}(+\infty)\mid_{[l(s)+s,+\infty)}$ towards $\mathbb{R}^{N+2} \times \underline{\pm\infty}$, respectively, extending by $H_s^{(1)}(+\infty)\mid_{l(s)+s}$ times an interval of diverging length. The resulting paths $H_{\pm 1}(-) = H_{\pm 1}^{(1)}(-) \cdot H_{\pm 1}^{(2)}(-)$ are reparametrisations of γ_{V^*} and γ_{W^*} (the reparametrisations only depend on our choice of homeomorphism $l : [-1,1] \cong [-\infty,+\infty]$ and the parametrisation of the path ψ). Thus η and H give rise to the required homotopy $\tau_{WW} \circ \mathrm{alex}^s \simeq \mathrm{alex}^s \circ \iota_H$. □

Recall that $H^s(M) \simeq BC(M)$. Now if $P \subset M$ is a codimension zero embedding and $p \leq d-3$ (in the notation of Theorem A), then $C\mathrm{Emb}(P,M) \simeq \mathrm{hofib}(H^s(\overline{M-P}) \to H^s(M))$ by the isotopy extension sequence (3.3), and therefore $C\mathrm{Emb}(P,M)$ inherits an involution ι_H up to weak equivalence. The map (5.2) is functorial with respect to codimension zero embeddings, so the following diagram is commutative:

$$\begin{tikzcd} H^s(\overline{M-P}) \arrow[r, "\mathrm{alex}^s"] \arrow[d] & \Theta(\overline{M-P})_0 \arrow[r, hook] \arrow[d] & \Omega^\infty(\Theta(\overline{M-P})) \arrow[d] \\ H^s(M) \arrow[r, "\mathrm{alex}^s"] & \Theta(M)_0 \arrow[r, hook] & \Omega^\infty(\Theta(M)). \end{tikzcd} \tag{5.3}$$

Corollary 5.7. *If* $\dim P = d$*, the vertical homotopy fibre of the horizontal compositions in (5.3) gives a map*

$$\mathrm{alex} : C\mathrm{Emb}(P,M) \longrightarrow \Omega^\infty(\Sigma^{-1}\Theta(M, M-P))$$

which is $\phi_{C\mathrm{Emb}}(d,p)$*-connected and* C_2*-equivariant up to homotopy.*

Proof. The connectivity of this map is the content of Proposition 3.9. It is equivariant as both $\mathrm{alex}^s : H^s(M) \to \Omega^\infty(\Theta(M))$ and $\mathrm{alex}^s : H^s(\overline{M-P}) \to \Omega^\infty(\Theta(M-P))$ are by Lemma 5.6. □

Warning 5.8. There is a canonical involution in the concordance space $C(M)$ given by turning upside down a concordance and precomposing by the inverse of the top diffeomorphism (see e.g. [Vog85, p. 296]). The restriction map $C(M) \to C\mathrm{Emb}(P,M)$ is *not* C_2-equivariant with respect to ι_C and ι_H—rather, it is anti-equivariant. This may seem to contradict [Vog85, Prop. 2.2], but what Vogell really proves there is that there is a homotopy C_2-equivariant equivalence $C(M) \approx \Omega^\sigma H(M) := \mathrm{Map}_*(S^\sigma, H(M))$, where we recall S^σ stands for the representation sphere of the 1-dimensional sign representation σ. This is due to an extra flip in the loop component that he introduces at the end of the proof of the proposition.

Let us now extend the map (5.2) to the whole h-cobordism space in some cases. This will not be strictly necessary for the proof of Theorem B, so the reader may want to skip the rest of this section on first reading.

Construction 5.9. Let M^d be a smooth manifold with $d \geq 4$ and $\pi_1(M) =: \pi$, and let J denote a closed interval. The h-cobordism space $H(M \times J)$ is an $\mathbb{E}_1$-space with respect to stacking partitions in the J-direction, denoted simply by $+_J$. Remember that we imposed the technical assumption on a partition $\rho = (W, F, V) \in H(M \times J)$ to be such that the intersection of F with $\partial(M \times J) \times [-1,1]$ is standard and happens exactly at $\partial(M \times J) \times \{0\}$, so this makes $+_J$ well-defined. As $\dim(M \times J) \geq 5$, the $\mathbb{E}_1$-space $(H(M \times J), +_J)$ is in fact group-like by the h-cobordism theorem, with $\pi_0(H(M \times J)) \cong \mathrm{Wh}(\pi)$. This last isomorphism is in fact one of unital monoids by the additivity of Whitehead torsion (cf. [Coh73, Thm. 23.1]), where the unit in $\pi_0(H(M \times J))$ is the path-component of the standard partition $* = (M \times J \times [-1,0], M \times J \times \{0\}, M \times J \times [0,1])$. Moreover, if $[\rho] \in \mathrm{Wh}(\pi)$ denotes the Whitehead torsion of $\rho \in H(M \times J)$, i.e. the path-component of ρ in $\pi_0(H(M \times J))$, then

$$[\iota_H(\rho)] = (-1)^d \overline{[\rho]}, \tag{5.4}$$

where $\overline{(-)}$ is Milnor's involution on $\mathrm{Wh}(\pi)$ (see Warning 5.18). This follows by the *duality formula* of [Mil66, §10] and the fact that if $\rho = (W, F, V)$, then the Whitehead torsion of the h-cobordism W with respect to $M \times J$ is (roughly) *minus* that of V with respect to F (cf. [Mil66, Lem. 7.8]).

On the other hand, the space $\Theta(M \times J)_0 := \mathrm{Diff}^b_\partial(M \times J \times \mathbb{R})/\mathrm{Diff}_\partial(M \times J)$ is also an $\mathbb{E}_1$-group under $+_J$, with $\pi_0(\Theta(M \times J)_0) \cong \mathrm{Wh}(\pi)$ as groups. Explicitly, the identification is

$$\begin{tikzcd}[column sep=small] & \pi_0\left(\Omega \frac{\mathrm{Diff}^b_\partial(M\times\mathbb{R}^{1,1})}{\mathrm{Diff}^b_\partial(M\times\mathbb{R}^{1,0})}\right) \arrow[d, equal] & & \\ \pi_0\left(\frac{\mathrm{Diff}^b_\partial(M\times J\times\mathbb{R}^{0,1})}{\mathrm{Diff}_\partial(M\times J)}\right) \arrow[ur, "\cong", "{[\text{WW88, Prop 1.5}]}"'] & & \pi_0(C^b(M\times\mathbb{R}^{1,0})) \arrow[dl, "\cong"', "{[\text{WW88, Prop 1.8}]}"] \arrow[r, "\cong", "{[\text{WW88, Cor. 5.3}]}"'] & \mathrm{Wh}(\pi). \\ & \pi_0\left(\Omega^\sigma \frac{\mathrm{Diff}^b_\partial(M\times\mathbb{R}^{1,1})}{\mathrm{Diff}^b_\partial(M\times\mathbb{R}^{1,0})}\right) & & \end{tikzcd}$$

Here $\mathbb{R}^{a,b} = \mathbb{R}^a \oplus b \cdot \sigma$ as in Remark 2.1, and $C^b(-)$ is the space of bounded concordances, which is equipped with the concordance involution ι_C of Warning 5.8. The first and third map are C_2-equivariant, the second equality is so only after introducing a minus sign, and the last one after introducing a $(-1)^{d+1}$ (the homomorphism $\beta : \pi_0(C^b(M \times \mathbb{R}^{1,0})) \cong \pi_0(H(M \times I))$ introduced in the proof of [WW88, Cor. 5.3] identifies the concordance and h-cobordism directions, so the claim follows by the duality formula [Mil66, §10]). All in all, for $\phi \in \Theta(M \times J)_0$ we have

$$[\tau_{WW}(\phi)] = (-1)^d \overline{[\phi]}. \tag{5.5}$$

We construct an equivalence

$$\mathrm{alex} : H(M \times J) \xrightarrow{\sim} \Theta(M \times J)_0 = \mathrm{Diff}^b_\partial(M \times J \times \mathbb{R})/\mathrm{Diff}_\partial(M \times J) \tag{5.6}$$

as follows: for each $\kappa \in \mathrm{Wh}(\pi)$, fix elements $p_\kappa \in H(M \times J)$ and $\phi_\kappa \in \mathrm{Diff}^b_\partial(M \times J \times \mathbb{R})/\mathrm{Diff}_\partial(M \times J)$ with $[p_\sigma] = [\phi_\kappa] = \kappa$. Fix also paths in $H(M)$ between $\iota_H(p_\kappa)$ and $p_{[\iota_H(p_\kappa)]} = p_{(-1)^d\overline{\kappa}}$ (by (5.4)), and paths in $\Theta(M \times J)_0$ between $\tau_{WW}(\phi_\kappa)$ and $\phi_{[\tau_{WW}(\phi_\kappa)]} = \phi_{(-1)^d\overline{\kappa}}$ (by (5.5)). When $\kappa = 0 \in \mathrm{Wh}(\pi)$, we arrange p_0 and ϕ_0 to be the basepoints in their respective spaces, and the loops around them that we have just fixed to be constant. Then define

$$\mathrm{alex}(\rho) := \mathrm{alex}^s(\rho +_J p_{-[\rho]}) +_J \phi_{[\rho]}, \qquad \rho \in H(M \times J).$$

Lemma 5.10. *The map (5.6) is C_2-equivariant up to homotopy. Moreover, it is indeed an equivalence and the composition $H^s(M) \hookrightarrow H(M) \xrightarrow{\mathrm{alex}} \Theta(M \times J)_0$ is canonically homotopic to the map* alex^s *of (5.2).*

Proof. For $\phi, \psi \in \Theta(M \times J)_0$, let us write $\phi \sim \psi$ to indicate that there is a path between these two points in that space. The paths we have fixed above, together with the observation that the involutions ι_H and τ_{WW} respect the $\mathbb{E}_1$-structure $+_J$, give paths in $\Theta(M \times J)_0$

$$\begin{aligned}
\tau_{WW}(\mathrm{alex}(\rho)) &= \tau_{WW}(\mathrm{alex}^s(\rho +_J p_{-[\rho]})) +_J \tau_{WW}(\phi_{[\rho]}) \\
&\sim \mathrm{alex}^s(\iota_H(\rho) +_J \iota_H(p_{-[\rho]})) +_J \phi_{(-1)^d\overline{[\rho]}} \\
&\sim \mathrm{alex}^s(\iota_H(\rho) +_J p_{-[\iota_H(\rho)]}) +_J \phi_{[\iota_H(\rho)]} =: \mathrm{alex}(\iota_H(\rho))
\end{aligned}$$

which depend continuously in $\rho \in H(M \times J)$. This defines the required equivalence $\tau_{WW} \circ \mathrm{alex} \simeq \mathrm{alex} \circ \iota_H$.

The canonical homotopy between the map $(-) +_J *$ and the indentity (in both spaces) gives the homotopy between alex^s and the composition $H^s(M) \hookrightarrow H(M) \xrightarrow{\mathrm{alex}} \Theta(M \times J)_0$. Therefore $\pi_{>0}(\mathrm{alex}) = \pi_{>0}(\mathrm{alex}^s)$ is an isomorphism. Now by construction, under the identifications $\pi_0(H(M \times J)) \cong \mathrm{Wh}(\pi)$ and $\pi_0(\Theta(M \times J)_0) \cong \mathrm{Wh}(\pi)$, the map $\pi_0(\mathrm{alex})$ is the identity of $\mathrm{Wh}(\pi)$. It follows then that alex is indeed an equivalence. □

5.3. **From h-cobordism spaces back to A-theory.** Given a spherical fibration ξ over M equipped with a section, fibrewise smashing a retractive space over M with ξ gives rise to a functor $- \cdot \xi : \mathbf{A}(M) \to \mathbf{A}(M)$ which, by [Vog85, Prop. 2.5], makes the following diagram homotopy commutative:

$$\begin{tikzcd}
\mathbf{A}(M) \arrow[r, "\tau_\epsilon"] \arrow[dr, "\tau_\xi"'] & \mathbf{A}(M) \arrow[d, "-\cdot\xi"] \\
& \mathbf{A}(M),
\end{tikzcd} \tag{5.7}$$

where $\epsilon := \epsilon^0 = M \times S^0$ is the trivial 0-dimensional sphere bundle over M. When $\xi = \epsilon^d = M \times S^d$ is the trivial d-spherical fibration, the functor $- \cdot \xi$ corresponds to $\Sigma^d_M(-) : \mathbf{A}(M) \to \mathbf{A}(M)$, the d-fold *fibrewise suspension over M* (cf. [Vog85, p. 281]). By the *additivity theorem* of [Wal85, Prop. 1.6.2] applied to the Waldhausen category of retractive spaces over M, it follows that Σ^d_M acts (up to homotopy) as $(-1)^d$ on $\mathbf{A}(M)$, and thus by (5.7)

$$\xi = M \times S^d \implies \mathbf{A}(M; \xi) \approx S^{d\cdot(\sigma-1)} \wedge \mathbf{A}(M; \epsilon). \tag{5.8}$$

Vogell also defined [Vog85, p. 299] a homotopy involution[4] $\mathcal{T}$ on $A(M) = \Omega^\infty\mathbf{A}(M)$, compatible with the splitting $A(M) \simeq \mathrm{Wh}^{\mathrm{Diff}}(M) \times Q_+M$, such that $\Omega\mathcal{T}$ extends ι_H on $H(M) \subset \mathcal{H}(M) \simeq \Omega\mathrm{Wh}^{\mathrm{Diff}}(M)$. Here $\mathcal{H}(M) := \mathrm{hocolim}_k\ H(M \times I^k)$ is the *stable h-cobordism space* of M, which is equivalent to $\Omega\mathrm{Wh}(M)$ by the stable parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13, Thm. 0.1]. Vogell further showed in [Vog85, Cor. 2.10] that $\mathcal{T}$ and the involution τ_ξ agree up weak homotopy when ξ is the d-spherical fibration associated to the stable tangent bundle $TM \oplus \epsilon^1$ of M^d. The upshot then is that when M^d is stably paralellisable, the Weiss–Williams involution is compatible with $(-1)^d\tau_\epsilon$ in the following sense.

Theorem 5.11. *If M is stably paralellisable, then there is an equivalence away from two*

$$\Omega^\infty\left(\Theta(M)_{hC_2}\right) \simeq_{[\frac{1}{2}]} \Omega^\infty\left(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M; \epsilon)_{hC_2}\right).$$

[4] Vogell refers to $\mathcal{T}$ as a *weak* involution in the sense that it is a homotopy involution when restricted to "any compactum" (cf. [Vog85, Lem. 2.4]) or, in better words, to each stage of the colimit in [Vog85, p. 299] modelling $A(M)$.

Remark 5.12. The equivalence above is not stated to be one of infinite loop spaces (even though it most probably is). Moreover, the weaker statement that the equivalence in Theorem 5.11 holds only after taking Postnikov sections $\tau_{\geq N}(-)$ for any $N \geq 0$ (and maybe even replacing $\Theta(-)$ and $\mathbf{Wh}^{\mathrm{Diff}}(-)$ by $\Theta^s(-)$ and $\mathbf{Wh}^{\mathrm{Diff}}_s(-)$) requires a much simpler proof than the one we will give below; namely, one does not need Construction 5.9, Lemma 5.14 and Claim 2 anymore. Even though this weaker version will be enough for the proof of Theorem B, we prefer to state and proof Theorem 5.11 in this generality as the reader may find it of independent interest. We encourage the eager reader to figure out the details in the weaker case.

We will need the a few preliminary results for the proof of Theorem 5.11.

Lemma 5.13. *For each $k \geq 0$, there is a natural C_2-equivariant equivalence of spectra*

$$e_k : \Theta(M \times I^k) \simeq S^{k\cdot(\sigma-1)} \wedge \Theta(M). \tag{5.9}$$

such that the following square is homotopy commutative:

$$\begin{tikzcd} \Theta(M \times I^k) \arrow[r, "\sim", "e_1"'] \arrow[d, "\wr"', "e_k"] & S^{\sigma-1} \wedge \Theta(M \times I^{k-1}) \arrow[d, "\wr"', "S^{\sigma-1}\wedge e_{k-1}"] \\ S^{k\cdot(\sigma-1)} \wedge \Theta(M) \arrow[r, equal] & S^{k\cdot(\sigma-1)} \wedge \Theta(M). \end{tikzcd}$$

Proof. We may assume $k = 1$, so that the above square is homotopy commutative by construction. Recall $\mathbb{R}^{a,b} := \mathbb{R}^a \oplus b \cdot \sigma$, and for any orthogonal functor $F(-)$ let $C_2 = O(1)$ act on $F(\mathbb{R}^{a,b})$ by the induced action. Finally let $B(-) := B\mathrm{Diff}^b_\partial(M \times (-))$ and $BI(-) := B\mathrm{Diff}^b_\partial(M \times I \times (-))$. The Alexander trick-like map of [WW88, Prop. 1.5] is a C_2-equivariant map

$$\mathrm{alex} : \mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^{a,b}) \xrightarrow{\sim} \Omega\mathrm{Diff}^b_\partial(M \times \mathbb{R}^{a+1,b})$$

which, upon delooping, gives rise to a C_2-equivariant equivalence on basepoint components $B(\mathrm{alex.}) : BI(\mathbb{R}^{a,b}) \simeq_0 \Omega B(\mathbb{R}^{a+1,b})$. Writing Ξ for the C_2-spectrum whose n-th space is $B^{(1)}(\mathbb{R}^{1,n+1})$ and with stabilisation maps $s_{0,1} : S^1 \wedge B^{(1)}(\mathbb{R}^{1,n}) \to B^{(1)}(\mathbb{R}^{1,n+1})$, we obtain a C_2-equivariant equivalence of spectra

$$B(\mathrm{alex.}) : \Theta(M \times I) := \Theta(BI)^{(1)} \xrightarrow{\sim} \Xi := \{B^{(1)}(\mathbb{R}^{1,n+1})\}_{n\geq 0}. \tag{5.10}$$

But now the stabilisation map $s_{1,0} : S^\sigma \wedge B^{(1)}(\mathbb{R}^{0,n}) \to B^{(1)}(\mathbb{R}^{1,n}) = \Xi_{n-1}$ induces another C_2-equivariant equivalence of spectra

$$S^{\sigma-1} \wedge \Theta B^{(1)} \xrightarrow{\sim} \Xi.$$

Composing these two equivalences gives the one in the statement. □

Vogell introduced in [Vog85, p. 298] the *lower* and *upper stabilisation maps* $\Sigma_\ell, \Sigma_u : H(M) \to H(M \times I)$. Roughly, the former sends a partition $\rho = (W, F, V)$ to $(U(W), W \cup_F W, \overline{M \times I \times I \setminus U(W)})$, where $U(W)$ is obtained from W by bending $W \times I$ into a U-shape, whilst Σ_u does the same to V instead of W (see Figure 7 for a pictorial representation of Σ_ℓ). We will only be interested in the lower stabilisation Σ_ℓ, which we will denote by Σ for simplicity. Here's how it interacts with the h-cobordism involution ι_H.

Lemma 5.14. *Let $+_I$ stand for the "stacking in the I-direction" $\mathbb{E}_1$-structure in $H(M \times I)$. Then:*

$$(a)\ \iota_H\Sigma +_I \Sigma\iota_H \simeq * : H(M) \to H(M \times I), \qquad (b)\ \iota_H\Sigma^2 \simeq \Sigma^2\iota_H : H(M) \to H(M \times I^2).$$

Proof. We defer the proof of (a) to Lemma B.1 in Appendix B as it is a bit technical. Then (b) follows from

$$\iota_H\Sigma^2 \simeq \iota_H\Sigma^2 +_I \Sigma(\iota_H\Sigma +_I \Sigma\iota_H) \simeq (\iota_H\Sigma +_I \Sigma\iota_H)\Sigma +_I \Sigma^2\iota_H \simeq \Sigma^2\iota_H.$$

□

We are now ready to deal with Theorem 5.11. If the reader has decided to skip Construction 5.9, then they should replace in the argument below the symbols alex, $H(-)$, $\mathcal{H}(-)$, $\mathbf{Wh}^{\mathrm{Diff}}(-)$ and $\Theta(-)$ by alex^s, $H^s(-)$, $\mathcal{H}^s(-)$, $\mathbf{Wh}^{\mathrm{Diff}}_s(-)$ and $\Theta^s(-) := \tau_{\geq 1}\Theta(-)$, respectively.

Proof of Theorem 5.11. We may assume without loss of generality that M is of the form $M' \times J$ for some $(d-1)$-manifold M' with $d \geq 5$: indeed as there is a homotopy C_2-equivariant equivalence $S^2 \approx S^{2\sigma}$, it follows by Lemma 5.13 and Proposition 5.3 that there are equivalences of spectra

$$\Theta(M \times J^{2k})_{hC_2} \simeq_{[\frac{1}{2}]} \Theta(M)_{hC_2},$$
$$\left(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M \times J^{2k}; \epsilon)\right)_{hC_2} \simeq_{[\frac{1}{2}]} \left(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M; \epsilon)\right)_{hC_2}.$$

In the second equivalence we also use that $\mathbf{Wh}^{\mathrm{Diff}}(-)$ and τ_ϵ are homotopy invariant. We will not explicitly use the J-intervals nor that $d \geq 5$ anymore, which we only need for the map "alex" of Construction 5.9 to be well-defined. The intervals I below should not be confused with the J one (present only in disguise).

Let $k \geq 0$ and let ξ denote the $(d+k)$-spherical fibration corresponding to the stable tangent bundle $T(M \times I^k) \oplus \epsilon^1$. If M (and hence $M \times I^k$) is stably parallelisable, the involution $(-1)^{d+k}\tau_\epsilon$ is homotopic to τ_ξ by (5.8), and by [Vog85, Prop. Cor. 2.10] it agrees with $\mathcal{T}$ (and hence extends ι_H). All in all, we obtain a zig-zag of homotopy C_2-equivariant maps

$$\begin{tikzcd} \Omega^\infty\Theta(M \times I^k) \arrow[d, "(5.9)\ \wr"'] & H(M \times I^k) \arrow[l, "\mathrm{alex}"', "(5.6)"] \arrow[r, "(\dagger)"] & \Omega^\infty\left(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)\right) \arrow[d, "\wr"] \\ \Omega^\infty(S^{k\cdot(\sigma-1)} \wedge \Theta(M)) & & \Omega^\infty\left(S^{(d+k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)\right) \end{tikzcd} \tag{5.11}$$

Every space involved in (5.11) is an $\mathbb{E}_1$-group if $k \geq 1$: both $H^s(M \times I^k)$ and $\Omega^\infty(\Theta(M \times I^k))$ by stacking in the first coordinate of I^k, and the others by their own infinite loop structures.

Claim 1. *All of the maps in (5.11) are H-maps if $k \geq 1$.*

Proof. The only maps that are not obviously H-maps are (5.9) and (†). For the former, we may assume that $k = 1$, and we need only show that $\Omega^\infty(\Theta(M \times I)) \to \Omega^\infty\Xi$ of (5.10) is an H-map, where Ξ is as in the proof of Lemma 5.13. Non-equivariantly, such map is the colimit as n goes to infinity of

$$\mathrm{alex} : \Omega^n\left(\frac{\mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^{n+1})}{\mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^n)}\right) \xrightarrow{\sim} \Omega^{n+1}\left(\frac{\mathrm{Diff}^b_\partial(M \times \mathbb{R}^{n+2})}{\mathrm{Diff}^b_\partial(M \times I \times \mathbb{R}^{n+1})}\right).$$

One verifies that stacking in the I-direction of the domain of this map corresponds, up to homotopy in the codomain, to composing loops in the $(n+1)$-st coordinate. So (5.9) is indeed an H-map.

Non equivariantly, the map (†) is the composition $(\dagger) : H(M \times I^k) \hookrightarrow \mathcal{H}(M \times I^k) \simeq \Omega^{\infty+1}\mathbf{Wh}^{\mathrm{Diff}}(M \times I^k)$, where the last equivalence is the stable parametrised h-cobordism theorem of Waldhausen–Jahren–Rognes [WJR13, Thm. 0.1]. As communicated to us in private by Bjørn Jahren and John Rognes, such equivalence is only stated to hold in the category of topological spaces (and not of infinite loop spaces, though it should definitely also hold there). We now explain why this equivalence is one of H-groups (which is the general consensus, but we couldn't find it written down anywhere): again assume $k = 1$. One reduces to the PL-case as in [WJR13, pp. 15 & 16]. Then if X is a simplicial set such that $|X| \simeq M$, the equivalence in the PL-setting is induced by a zig-zag of equivalences of simplicial sets (cf. the left vertical column of [WJR13, Eq. (0.4)])

$$\mathcal{H}(M \times I)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} s\mathcal{C}^h(X \times I),$$

where $s\mathcal{C}^h(X \times I)$ is the category (seen as a simplicial set by taking its nerve) of acyclic cofibrations $X \times I \hookrightarrow Y$ together with simple maps over $X \times I$. One can verify that it makes sense to stack in the I-direction in each of the simplicial sets involved in the zig-zag, and that the maps between them respect this monoidal structure. Let $\mu_0 : s\mathcal{C}^h(X \times I) \times s\mathcal{C}^h(X \times I) \to s\mathcal{C}^h(X \times I)$ stand for this monoidal structure, given explicitly by $\mu_0(Y, Z) := Y_{X \times 1} \cup_{X \times 0} Z$. There is another monoidal structure μ_1 induced by the pushout along $X \times I$, i.e. $\mu_1(Y, Z) := Y \cup_{X \times I} Z$. Sliding gives a homotopy between μ_0 and μ_1: intuitively, the maps

$$\mu_t : s\mathcal{C}^h(X \times I) \times s\mathcal{C}^h(X \times I) \longrightarrow s\mathcal{C}^h(X \times I), \quad (Y, Z) \longmapsto Y_{X \times [1-t,1]} \cup_{X \times [0,t]} Z, \quad t \in [0, 1],$$

constitute the homotopy. More precisely, consider the simplicial category $s\widetilde{\mathcal{C}}^h_\bullet(X)$ [WJR13, Defn. 3.1.1] whose objects in simplicial degree q consist of commutative diagrams

$$\begin{tikzcd} X \times \Delta^q_\bullet \arrow[rr, hook, "i", "\sim"'] \arrow[dr, "\mathrm{pr}"'] & & Y \arrow[dl, "\pi"] \\ & \Delta^q_\bullet, & \end{tikzcd}$$

where i is an acyclic cofibration and π is a Serre fibration. The inclusion $s\mathcal{C}^h(X) \hookrightarrow s\widetilde{\mathcal{C}}^h_\bullet(X)$ as the 0-simplices is a homotopy equivalence by [WJR13, Cor. 3.5.2], where a simplicial category is seen as a bisimplicial set by taking its nerve, and a bisimplicial set as an ordinary simplicial set by taking its totalisation. The monoidal structures μ_t make perfectly good sense in $s\widetilde{\mathcal{C}}^h_\bullet(X \times I)$, and one can indeed define a simplicial homotopy between μ_0 and μ_1 in this setting resembling the idea above.

But by Proposition 3.1.1, and Theorems 3.1.7 and 3.3.1 of [Wal85] (see also [WJR13, p. 5]), for any simplicial set T, there is a zig-zag of equivalences connecting $|s\mathcal{C}^h(T)|$ and $\Omega\mathrm{Wh}^{PL}(T) := \Omega^{\infty+1}(\mathbf{Wh}^{PL}(T))$ which is monoidal up to homotopy with respect to μ_1 in the domain and the loop structure on the looped Whitehead space. It hence follows that (†) is indeed a zig-zag of H-maps. $\square$

Claim 2. *For each $k \geq 1$, the diagram*

$$\begin{tikzcd} \Omega^\infty(S^{2k\cdot(\sigma-1)} \wedge \Theta(M)) \arrow[d, "\wr"] & H(M \times I^{2k}) \arrow[l] \arrow[r] \arrow[d, "\Sigma^2"] & \Omega^\infty(S^{(d+2k)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \arrow[d, "\wr"] \\ \Omega^\infty(S^{(2k+2)\cdot(\sigma-1)} \wedge \Theta(M)) & H(M \times I^{2k+2}) \arrow[l] \arrow[r] & \Omega^\infty(S^{(d+2k+2)\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \end{tikzcd} \tag{5.12}$$

is homotopy commutative, where the rows are the zig-zags (5.11) and the vertical external maps are induced by the homotopy C_2-equivariant equivalence $\mathbb{S}^0 \approx \mathbb{S}^{2\cdot(\sigma-1)}$.

Proof. The right hand square is clearly commutative. The commutativity of the left one will be a consequence of the commutativity of the one in Lemma 5.13 and the commutativity of the (non-equivariant) square

$$\begin{tikzcd} H(M \times I^k) \arrow[r, "\mathrm{alex}"] \arrow[d, "\Sigma"] & \Theta(M \times I^k)_0 \arrow[r, "{[WW88, Prop. 1.5]}"] & \Omega^k\Theta(M)_k \arrow[d, "s^\vee"] \\ H(M \times I^{k+1}) \arrow[r, "\mathrm{alex}"] & \Theta(M \times I^{k+1})_0 \arrow[r, "{[WW88, Prop. 1.5]}"] & \Omega^{k+1}\Theta(M)_{k+1}. \end{tikzcd}$$

This last square is commutative by a delooped version of the proof of [WW88, Prop. 1.12] (compare with Proposition 3.8). $\square$

Clearly Σ^2 is an H-map and also homotopy C_2-equivariant (with respect to ι_H) by Lemma 5.14(*b*). Therefore by Claim 1, all of the maps involved in (5.12) are H-maps and homotopy C_2-equivariant. Taking the homotopy colimit as $k \to \infty$, we obtain a homotopy C_2-equivariant zig-zag

$$\Omega^\infty(\Theta(M)) \xleftarrow{\approx} \operatorname*{hocolim}_k H(M \times I^{2k}) \xrightarrow{\approx} \Omega^\infty(S^{d\cdot(\sigma-1)-1} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M)) \tag{5.13}$$

of H-maps. The connectivity of, say, the upper horizontal maps in (5.12) is $\phi(d+2k) \gtrsim (d+2k)/3$ by Igusa's theorem and, as this lower bound increases linearly with k, the horizontal maps in (5.13) are indeed equivalences. The equivalence in the statement now follows by the second part of Proposition 5.3 applied to (5.13), and because taking homotopy C_2-orbits commutes up to equivalence with $\Omega^\infty(-)$ if 2 is inverted (as in Corollary 5.4). The proof of Theorem 5.11 is now complete. $\square$

Corollary 5.15. *If M^d is stably paralellisable and $P \subset M^d$ is a codimension zero submanifold with $p \leq d-3$ (in the notation of Theorem A), then there is an equivalence away from two*

$$\Omega^\infty\left(\Sigma^{-1}\Theta(M, \overline{M-P})_{hC_2}\right) \simeq_{[\frac{1}{2}]} \Omega^\infty\left(\left(S^{d\cdot(\sigma-1)-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(M, M-P; \epsilon)\right)_{hC_2}\right).$$

Proof. Note that $\overline{M-P}$ is stably parallelisable because M is. The zig-zag (5.13) is functorial with respect to codimension zero embeddings of stably paralellisable manifolds, and hence taking the homotopy fibre of the map between (5.13) applied to $\overline{M-P}$ and (5.13) applied to M, we obtain another homotopy C_2-equivariant zig-zag of equivalences

$$\Omega^\infty(\Sigma^{-1}\Theta(M,\overline{M-P})) \overset{\approx}{\leftarrow} \operatorname{hocolim}_k C\mathrm{Emb}(P\times I^{2k}, M\times I^{2k}) \overset{\approx}{\rightarrow} \Omega^\infty(S^{d\cdot(\sigma-1)-2}\wedge \mathbf{Wh}^{\mathrm{Diff}}(M,M-P)).$$

Note that $M-P\to M$ is 2-connected because $p\le d-3$, and hence

$$\begin{array}{ccc} \Theta(\overline{M-P}) & \longrightarrow & \Theta^s(M) \\ \downarrow & & \downarrow \\ \Theta(M) & \longrightarrow & \Theta^s(M) \end{array}$$

is cartesian (and similarly for $H(-)$ and $\mathbf{Wh}^{\mathrm{Diff}}(-)$ in place of $\Theta(-)$). So, indeed as mentioned before, one gets the same zig-zag if dealing only with the simple version of Theorem 5.11 (so as to skip Construction 5.9). In any case, the claim now follows as before by applying Proposition 5.3 to the zig-zag above. ☐

5.4. **The canonical involution in algebraic K-theory.** We now define the canonical involution τ_ϵ on $A(X)$ and explore its connection to an involution in the model of A-theory via "spaces of matrices with values in the ring up to homotopy" $Q_+\Omega X$ [Wal85, §2.2]. All throughout, let $G := GX$ denote the topological monoid of Moore loops on X, and write $\mathbb{S}[G]$ for the $\mathbb{E}_1$-ring spectrum $\mathbb{S}\wedge G_+$.

We will work over the ∞-category $\mathsf{Mod}_{\mathbb{S}[G]}$ of right $\mathbb{S}[G]$-module spectra; we will also write ${}_{\mathbb{S}[G]}\mathsf{Mod}$ for the ∞-category of left $\mathbb{S}[G]$-modules. Then for $m\ge 1$, if $\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])$ denotes the homotopy invertible components of the mapping space $\mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G],\oplus^m\mathbb{S}[G])$, Waldhausen showed in [Wal85, Thm. 2.2.1] that for X connected, there is a natural equivalence

$$A(X)\simeq \mathbb{Z}\times \operatorname{hocolim}_m B\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])^+. \tag{5.14}$$

In order to define τ_ϵ, we will introduce compatible anti-involutions on $\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])$ defined in terms of Spanier–Whitehead duality. As $\mathbb{S}[G]$ is not commutative, this duality really arises as an instance of a *duality in the symmetric closed bicategory* $\mathsf{Bimod}_{\mathbb{S}}$ of bimodule spectra, in the sense of [MS06, §16.4]. This duality coincides with the one considered by Vogell in [Vog85, §1].

First observe that a right $\mathbb{S}[G]$-module M can always be regarded as a left $\mathbb{S}[G]$-module by

$$\mathbb{S}[G]\otimes M \xrightarrow{\text{swap}} M\otimes \mathbb{S}[G] \xrightarrow{\mathrm{Id}_M\otimes \mathrm{inv}} M\otimes \mathbb{S}[G^{\mathrm{op}}] = M\otimes\mathbb{S}[G] \xrightarrow{\text{act}} M,$$

where "inv" stands for inversion in the monoid G–write M_ℓ for this left $\mathbb{S}[G]$-module. Here $\otimes=\otimes_{\mathbb{S}}$ stands for the usual smash product of spectra. Note also that ${}_{\mathbb{S}[G]}\mathsf{Mod}(M_\ell,M_\ell)\simeq \mathsf{Mod}_{\mathbb{S}[G]}(M,M)$ as $\mathbb{E}_1$-algebras. If $\nu:\mathbb{S}\to\mathbb{S}[G]_\ell$ denotes the unit, consider the map of spectra

$$\eta_1:\mathbb{S}\simeq \mathbb{S}\otimes_{\mathbb{S}[G]}\mathbb{S}[G]\xrightarrow{\nu\otimes 1}\mathbb{S}[G]\otimes_{\mathbb{S}[G]}\mathbb{S}[G]_\ell$$

and the map of $(\mathbb{S}[G],\mathbb{S}[G])$-bimodules

$$I_1:\mathbb{S}[G]_\ell\otimes\mathbb{S}[G]\xrightarrow{\mathrm{inv}\otimes 1}\mathbb{S}[G]\otimes\mathbb{S}[G]\xrightarrow{\text{act}}\mathbb{S}[G].$$

Then (η_1,I_1) make $(\mathbb{S}[G],\mathbb{S}[G]_\ell)$ a dual pair [MS06, Defn. 16.4.1]. More generally, the map of spectra

$$\eta_m:\mathbb{S}\xrightarrow{\bigoplus_{i,j}\delta_{ij}\eta_1}\bigoplus_{i,j=1}^m\mathbb{S}[G]\otimes_{\mathbb{S}[G]}\mathbb{S}[G]_\ell\cong\bigoplus_{j=1}^m\mathbb{S}[G]\otimes_{\mathbb{S}[G]}\left(\bigoplus_{i=1}^m\mathbb{S}[G]\right)_\ell$$

together with the map of $(\mathbb{S}[G],\mathbb{S}[G])$-bimodules

$$I_m:\left(\bigoplus_{i=1}^m\mathbb{S}[G]\right)_\ell\otimes\bigoplus_{j=1}^m\mathbb{S}[G]\cong\bigoplus_{i,j=1}^m\mathbb{S}[G]_\ell\otimes\mathbb{S}[G]\xrightarrow{\bigoplus_{i,j}\delta_{ij}I_1}\mathbb{S}[G],$$

exhibit $(\oplus^m\mathbb{S}[G])_\ell$ as a *right* dual to $\oplus^m\mathbb{S}[G]$. Therefore as in [MS06, Prop. 16.4.9], I_m induces an equivalence of left $\mathbb{S}[G]$-modules $\widetilde{I}_m:(\oplus^m\mathbb{S}[G])_\ell\simeq D_r(\oplus^m\mathbb{S}[G]) := \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G],\mathbb{S}[G])$, where the right hand side stands for the right $\mathbb{S}[G]$-linear mapping spectrum.

With (5.14) and Lemma 5.5 in mind, the involution τ_ϵ on $A(X;\epsilon)$ is then induced by the map of $\mathbb{E}_1$-algebras

$$\mathrm{Aut}_G(\oplus^m\mathbb{S}[G]) \xrightarrow{D_r} {}_G\mathrm{Aut}(D_r(\oplus^m\mathbb{S}[G]))^{\mathrm{op}} \overset{\widetilde{I}_\#}{\simeq} {}_G\mathrm{Aut}((\oplus^m\mathbb{S}[G])_\ell)^{\mathrm{op}} \simeq \mathrm{Aut}_G(\oplus^m\mathbb{S}[G])^{\mathrm{op}}, \tag{5.15}$$

where $\widetilde{I}_\#$ stands for conjugation with the equivalence $\widetilde{I}_m$. It will be convenient to think of (5.15) in the following way: let $GL_m(Q_+G)$ denote the union of path components in $(Q_+G)^{m\times m}$ in the image of $\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])$ under the natural equivalence

$$u : \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G], \oplus^m\mathbb{S}[G]) \xrightarrow{\sim} \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G], \textstyle\prod^m \mathbb{S}[G]) \simeq \mathsf{Sp}(\mathbb{S}, \mathbb{S}[G])^{m\times m} \simeq (Q_+G)^{m\times m},$$

where $\mathsf{Sp} \simeq \mathsf{Mod}_{\mathbb{S}}$ stands for the ∞-category of spectra. So $u : \mathrm{Aut}_G(\oplus^m\mathbb{S}[G]) \simeq GL_m(Q_+G)$ and, just as in standard linear algebra, under this equivalence the anti-involution (5.15) corresponds to the rule that sends a matrix A to its conjugate transpose $A^\dagger$ (conjugate with respect to inversion of G in $\mathbb{S}[G]$). More precisely:

Proposition 5.16. *Write* $\mathrm{End}_G(\oplus^m\mathbb{S}[G]) := \mathsf{Mod}_{\mathbb{S}[G]}(\oplus^m\mathbb{S}[G], \oplus^m\mathbb{S}[G])$ *with the action of the cyclic group* C_m *by conjugation with the permutation automorphisms of* $\oplus^m\mathbb{S}[G]$*. Let* C_m *act similarly on* $(Q_+G)^{m\times m}$ *by conjugation. Then the following square is commutative in the homotopy category of* C_m*-spaces:*

$$\begin{tikzcd} \mathrm{End}_G(\oplus^m\mathbb{S}[G]) \arrow[r, "(5.15)", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{End}_G(\oplus^m\mathbb{S}[G]) \arrow[d, "u", "\wr"'] \\ (Q_+G)^{m\times m} \arrow[r, "\dagger", "\sim"'] & (Q_+G)^{m\times m}. \end{tikzcd} \tag{5.16}$$

Remark 5.17. Passing to the homotopy invertible components in (5.16), we obtain the following commutative square in the homotopy category of C_m-spaces

$$\begin{tikzcd} \mathrm{Aut}_G(\oplus^m\mathbb{S}[G]) \arrow[r, "(5.15)", "\sim"'] \arrow[d, "u", "\wr"'] & \mathrm{Aut}_G(\oplus^m\mathbb{S}[G]) \arrow[d, "u", "\wr"'] \\ GL_m(Q_+G) \arrow[r, "\dagger", "\sim"'] & GL_m(Q_+G). \end{tikzcd}$$

We have suppressed the $(-)^{\mathrm{op}}$ in the codomain of the map (5.15) in the previous squares to emphasise that such squares take place in the homotopy category of (C_m-)spaces, as opposed to that of $\mathbb{E}_1$-spaces.

Proof of Proposition 5.16. First note that all the maps involved in (5.16) are indeed C_m-maps: the only one that is not obviously so is (5.15), but this follows from the observation that I_m is C_m-equivariant for the diagonal action on the domain and the trivial action on the target. Note also that the C_m-action on $(Q_+G)^{m\times m}$ restricts to a cofree C_m-action on each of the right C_m-cosets of the diagonal subspace, and hence $(Q_+G)^{m\times m} = \prod^m \mathrm{coInd}_e^{C_m} Q_+G$ as a C_m-space. Thus, in order to show that (5.16) commutes in the homotopy category of C_m-spaces, it suffices to prove that it commutes in the homotopy category of spaces after postcomposing it with the map

$$(Q_+G)^{m\times m} = \prod^m \mathrm{coInd}_e^{C_m} Q_+G \longrightarrow \prod^m Q_+G$$

that records the first column of a matrix.

Now given an endomorphism h of $\oplus^m\mathbb{S}[G]$, the $(m\times m)$-matrix $u(h) = (h_{ij}) \in GL_m(Q_+G)$ has components

$$h_{ij} : \mathbb{S} \xrightarrow{\nu} \mathbb{S}[G] \xrightarrow{\mathrm{inc}_j} \textstyle\bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{h} \textstyle\bigoplus_{k=1}^m \mathbb{S}[G] \xrightarrow{\mathrm{pr}_i} \mathbb{S}[G].$$

Slightly abusing the notation, we will write τ_ϵ to mean (5.15). Then we must only check that $\tau_\epsilon(h)_{ij}$ is homotopic to $\overline{h}_{ji}$, coherently in h (and for $j = 1$, though it is still true for all j of course). Observe now that $(-)_\ell : \mathsf{Mod}_{\mathbb{S}[G]} \to {}_{\mathbb{S}[G]}\mathsf{Mod}$ is a functor over Sp, and hence the last equivalence in (5.15) happens over the automorphism space of $\oplus^m\mathbb{S}[G]$ as a regular spectrum. Consequently, $\tau_\epsilon(h)_{ij}$ is by definition the top horizontal composition in the diagram of spectra

$$\begin{tikzcd}
\mathbb{S} \arrow[r, "\nu"] \arrow[dr, "\nu"'] & \mathbb{S}[G]_\ell \arrow[r, "\mathrm{inc}_j"] \arrow[d, "\mathrm{inv}", "\wr"'] & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[r, "\tau_\epsilon(h)"] \arrow[dd, "\widetilde{I}_m", "\wr"'] & \bigoplus_{k=1}^m \mathbb{S}[G]_\ell \arrow[r, "\mathrm{pr}_i"] \arrow[dd, "\widetilde{I}_m", "\wr"'] & \mathbb{S}[G]_\ell \\
 & \mathbb{S}[G] \arrow[r, phantom, "(*_1)"] \arrow[d, phantom, "\wr\Vert"] & {} \arrow[r, phantom, "(*_2)"] & {} \arrow[r, phantom, "(*_3)"] & \mathbb{S}[G] \arrow[u, "\mathrm{inv}"', "\wr"] \arrow[d, phantom, "\wr\Vert"] \\
 & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G]) \arrow[r, "\mathrm{pr}_j^*"] & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \mathbb{S}[G]) \arrow[r, "h^*"] & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \mathbb{S}[G]) \arrow[r, "\mathrm{inc}_i^*"] & \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G]).
\end{tikzcd}$$

On the other hand, one recognises the composition that goes through the bottom row to be $\overline{h}_{ji}$. Note that the left triangle is commutative, and that the square $(*_2)$ is too by definition of $\tau_\epsilon(h)$. Moreover, $(*_1)$ and $(*_3)$ do not depend on h; we must then argue that $(*_1)$ and $(*_3)$ are commutative up to homotopy.

For the commutativity of $(*_1)$, first observe that the left vertical composition of $(*_1)$ coincides up to homotopy with the equivalence of left $\mathbb{S}[G]$-modules $\widetilde{I}_1 : \mathbb{S}[G]_\ell \simeq \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G])$. This is because, under the usual tensor-hom adjunction, both maps represent the same element in

$$\pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}\left(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\mathbb{S}[G], \mathbb{S}[G])\right)\right) \cong \pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}_{\mathbb{S}[G]}\left(\mathbb{S}[G]_\ell \otimes \mathbb{S}[G], \mathbb{S}[G]\right)\right)$$

by definition of I_1. But now $(*_1)$, with $\widetilde{I}_1$ in place of the left vertical composition, commutes up to homotopy as both compositions represent the same element in

$$\pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}\left(\mathbb{S}[G]_\ell, \underline{\mathrm{Hom}}_{\mathbb{S}[G]}(\oplus^m \mathbb{S}[G], \mathbb{S}[G])\right)\right) \cong \pi_0\left({}_{\mathbb{S}[G]}\mathsf{Mod}_{\mathbb{S}[G]}\left(\mathbb{S}[G]_\ell \otimes (\oplus^m \mathbb{S}[G]), \mathbb{S}[G]\right)\right)$$

simply because the following diagram commutes by definition of I_1 and I_m:

$$\begin{tikzcd}
\mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "\mathrm{inc}_j \otimes 1"] \arrow[d, "1 \otimes \mathrm{pr}_j"] & \bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[d, "I_m"] \\
\mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "I_1"] & \mathbb{S}[G].
\end{tikzcd}$$

Finally the commutativity of $(*_3)$ follows by a similar reasoning using that

$$\begin{tikzcd}
\bigoplus^m \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[r, "\mathrm{pr}_i \otimes 1"] \arrow[d, "1 \otimes \mathrm{inc}_i"] & \mathbb{S}[G]_\ell \otimes \mathbb{S}[G] \arrow[d, "I_1"] \\
\bigoplus^m \mathbb{S}[G]_\ell \otimes \bigoplus^m \mathbb{S}[G] \arrow[r, "I_m"] & \mathbb{S}[G]
\end{tikzcd}$$

is also commutative by definition. □

Warning 5.18. The canonical involution τ_ϵ induces an involution on the Whitehead group

$$\mathrm{Wh}(X) := \pi_1^s(\mathbf{Wh}^{\mathrm{Diff}}(X)) \cong GL(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}/(\pm\pi_1(X)).$$

In the foundational paper [Mil71], Milnor also defined an involution $\mathrm{Wh}(X) \ni \kappa \mapsto \overline{\kappa}$ induced by sending a matrix in $GL(\mathbb{Z}[\pi_1(X)])^{\mathrm{ab}}$ to its conjugate transpose (conjugate with respect to inversion in $\pi_1(X)$). This is an actual homomorphism because of the abelianisation present in the general linear group of $\mathbb{Z}[\pi_1(X)]$. It is worth being aware that these two involutions on $\mathrm{Wh}(X)$ are only the same after introducing a minus sign, i.e.

$$\tau_\epsilon(\kappa) = -\overline{\kappa}. \tag{5.17}$$

This does *not* contradict the commutativity of (5.16). On the contrary, this extra minus sign is the result of having to deloop the anti-involution (5.15) in the sense of Lemma 5.5 in order to obtain τ_ϵ.

5.5. *A*-**theory of a suspension.** In this section we focus our attention on the homotopy type of $A(X;\epsilon)$ when X is the suspension ΣY of a connected based space Y. By a theorem of Carlsson–Cohen–Goodwillie–Hsiang[5]

[5]As pointed out in [BCC+96, p. 543], the proof in [CCGH87] has a serious flaw around p. 71. This issue was fixed in [BCC+96, Cor. 4.15], and in particular the map θ of (5.18) constructed in [CCGH87, §1] is still an equivalence. We are indebted to Tom Goodwillie for his help in clearing out this matter and for carefully explaining to us another more general principle for which (5.18) holds—namely, it is the observation that if F is a functor (from based spaces to based spaces, say) whose m-th derivative spectrum is of the form $X \mapsto X^{\wedge m}_{hC_m}$ for every $m \geq 1$, then its Taylor tower must split globally. This is indeed the case for the functor $F(-) := \Omega \circ A \circ \Sigma(-)$.

[CCGH87, Thm. 3], in such cases there is an equivalence of infinite loop spaces

$$\theta : \prod_{m\geq 1} Q(Y^{\wedge m}_{hC_m}) \xrightarrow{\sim} \Omega\widetilde{A}(\Sigma Y), \tag{5.18}$$

where $\widetilde{A}(-) := \mathrm{hofib}(A(-) \to A(*))$ and C_m acts on $Y^{\wedge m}$ by cyclic permutation of the factors. In this section, we argue that (5.18) can be upgraded to be C_2-equivariant up to homotopy.

Proposition 5.19. *Let Y be a connected, based C_2-space. There is an equivalence of spectra*

$$\boldsymbol{\theta} : \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}\left((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\right) \xrightarrow{\sim} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon)$$

that is C_2-equivariant up to homotopy, and whose underlying (non-equivariant) equivalence induces (5.18). Here $\Sigma^\sigma Y := S^\sigma \wedge Y$ and $D_m \subset \Sigma_m$ acts on $Y^{\wedge m}$ by

$$g\cdot(y_1\wedge\cdots\wedge y_m) \longmapsto g\cdot y_{g(1)}\wedge\cdots\wedge g\cdot y_{g(m)}, \quad g\in D_m, \quad y_i\in Y,$$

where Y is now seen as a based D_m-space (on which C_m acts trivially). Finally $C_2 = D_m/C_m$ acts on $(ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ by its residual diagonal action.

Remark 5.20. The C_2-space $\Sigma^\sigma Y$ induces an involution on $\mathbf{A}(\Sigma^\sigma Y)$ which commutes with the canonical involution τ_ϵ described in the previous section, by naturality of its construction. Therefore its composite gives the involution on $\widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon)$ appearing in the statement of Proposition 5.19. Alternatively, we can allow X in the previous section to mean a C_2-space (e.g. $\Sigma^\sigma Y$), and agree that $\mathrm{inv} : G = GX \to G^{\mathrm{op}}$ there stands for inversion in the monoid G followed by the C_2-action on X.

In practice, we will apply Proposition 5.19 to the case when $Y = S^\sigma \wedge Z$ for some trivial C_2-space Z, as then $\Sigma^\sigma Y \simeq S^{2\sigma}\wedge Z \approx S^2 \wedge Z$ because of the homotopy C_2-equivariant equivalence $S^{2\sigma}\approx S^2$. In such case, as $\mathbf{A}(-;\epsilon)$ is a homotopy functor, Proposition 5.19 provides a simple description of the homotopy C_2-equivariant homotopy type of $\widetilde{\mathbf{A}}(\Sigma^2 Z;\epsilon)\approx \widetilde{\mathbf{A}}(\Sigma^{2\sigma}Z;\epsilon)$. This, together with Proposition 5.3, can then be used to analyse the homotopy type of $\widetilde{\mathbf{A}}(\Sigma^2 Z;\epsilon)_{hC_2}$ away from 2.

We will need the following observation for the proof of Proposition 5.19.

Lemma 5.21. *Let X be a based, connected C_m-space. The equivalence*

$$\iota : B((C_m \ltimes \Omega X)^{\mathrm{op}}) \simeq B(C_m\ltimes\Omega X)$$

of Lemma 5.5 coincides up to equivalence with the delooping of the inversion map

$$\mathrm{inv} : (C_m\ltimes\Omega X)^{\mathrm{op}} \longrightarrow C_m\ltimes\Omega X, \quad (s^i,\gamma)\mapsto(s^{-i}, s^i\cdot\overline{\gamma}),$$

where $\overline{\gamma}$ stands for the loop γ with the reversed orientation.

Proof. Given a topological monoid M equipped with a C_m-action, it is well-known (see e.g. [AM04, §II, Thm. 1.12]) that the classifying space of the semi-direct product $C_m \ltimes M$ is equivalent to $EC_m \times_{C_m} BM$. On the simplicial level, this equivalence is given by

$$\beta : B_\bullet(C_m\ltimes M) \xrightarrow{\sim} E_\bullet C_m \times_{C_m} B_\bullet M,$$
$$\left((s^{i_1},m_1),\ldots,(s^{i_q},m_q)\right) \longmapsto \left[(e,s^{i_1},\ldots,s^{i_q}),(s^{i_1}\cdot m_1, s^{i_1+i_2}\cdot m_2,\ldots,s^{i_1+\cdots+i_q}\cdot m_q)\right].$$

Now, for simplicity, we may assume that $\Omega(-)$ stands for the Moore loop space, so that $C_m\ltimes\Omega X$ is strictly associative (a *Moore loop* is a pair (γ,t) where $t\geq 0$ and $\gamma:[0,t]\to Y$ is a map with $\gamma(0)=\gamma(t)=*$; multiplication of Moore loops is given by concatenation of loops and addition of its lengths). Also recall that there is an identification $\Delta^q \cong \{\mathbf{v}=(v_1,\ldots,v_q) : 0\leq v_1\leq\cdots\leq v_q\leq 1\}$; under this identification, the self-isomorphism $\Phi_q : \Delta^q\cong\Delta^q$ in the proof of Lemma 5.5 becomes the rule that sends a partition $\mathbf{v}=(0\leq v_1\leq\cdots\leq v_q\leq 1)$ to $1-\mathbf{v} := (0\leq 1-v_q\leq\cdots\leq 1-v_1\leq 1)$. There is a C_m-equivariant map

$$\xi : B\Omega X \longrightarrow X, \quad \left[(\gamma_1,t_1),\ldots,(\gamma_q,t_q),\mathbf{v}\right] \longmapsto \gamma_q\cdot\gamma_{q-1}\cdot\ldots\cdot\gamma_1\left(\sum_{i=1}^q t_i v_i\right)$$

that is an equivalence if X is connected. This map satisfies the property that

$$\xi\left(\left[(\overline{\gamma}_1,t_1),\ldots,(\overline{\gamma}_q,t_q),\mathbf{v}\right]\right) = \xi\left(\left[(\gamma_q,t_q),\ldots,(\gamma_1,t_1),1-\mathbf{v}\right]\right).$$

With all of this in mind, one verifies that the following diagram commutes up to homotopy

$$\begin{tikzcd}
B((C_m \ltimes \Omega X)^{\mathrm{op}}) \arrow[r, "\iota", "\sim"'] \arrow[d, "B(\mathrm{inv})"] & B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[d, "EC_m \times_{C_m} \xi", "\wr"'] \\
B(C_m \ltimes \Omega X) \arrow[r, "\beta", "\sim"'] & EC_m \times_{C_m} B\Omega X \arrow[r, "EC_m \times_{C_m} \xi", "\sim"'] & EC_m \times_{C_m} X.
\end{tikzcd}$$

□

Proof of Proposition 5.19. In the notation of the previous section, we let $X = \Sigma^\sigma Y$ now, so that $G = GX = \Omega^\sigma \Sigma^\sigma Y$ as a monoid with anti-involution (i.e. "inv" now means inversion in the monoid G followed by the C_2-action on $X = \Sigma^\sigma Y$). For each $m \geq 1$, let us write $\varrho : GL_m(Q_+G) \simeq \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ for u^{-1} (meaning u as a wrong way equivalence); it should be thought of as given by the rule that sends a matrix $h = (h_{ij}) \in \mathrm{Map}(\mathbb{S}, \mathbb{S}[G])^{m\times m} \cong \mathrm{Map}_G(\mathbb{S}[G], \mathbb{S}[G])^{m\times m}$ to

$$\varrho(h) : \bigoplus_{j=1}^m \mathbb{S}[G] \xrightarrow{\bigoplus_j \left(\bigoplus_i h_{ij}\right)} \bigoplus_{i=1}^m \mathbb{S}[G].$$

The map θ of (5.18) is constructed in several steps in [CCGH87, §1], each of which we now upgrade to the homotopy C_2-equivariant setting. As these homotopy C_2-actions will get mixed up with strict C_m-actions, it will be more convenient and clear, at least throughout the first few steps, to avoid speaking about "homotopy equivariance" and rather regard a homotopy involution as what it is, i.e. a map whose square happens to be homotopic to the identity.

Step 1. For each $m \geq 2$, consider the D_m-equivariant map of spaces

$$\widetilde{\theta}_{m,1} : Y^{\times m} \longrightarrow GL_m(Q_+G), \quad (y_1, \dots, y_m) \longmapsto \begin{pmatrix} 1 & y_1 - 1 & & & & \\ & 1 & y_2 - 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & & \\ & & & & 1 & y_{m-1} - 1 \\ y_m - 1 & & & & & 1 \end{pmatrix}$$

where, if $D_m := \langle s, r \mid s^m = r^2 = rsrs = e \rangle$, the notation is as follows:

- A point $y \in Y$ is identified in G with the path $\eta(y) := (t \mapsto t \wedge y) \in G$, which is itself identified with a point in $\{1\} \times QG \subset QS^0 \times QG \simeq Q_+G$. Then $y - 1$ is the corresponding point in $\{0\} \times QG \subset Q_+G$. Here the n-th component of QS^0 has been fixed a basepoint $n \in QS^0$.
- The action of D_m on $(y_1, \dots, y_m) \in Y^{\times m}$ is given by

$$s \cdot (y_1, \dots, y_m) := (y_m, y_1, \dots, y_{m-1}), \quad r \cdot (y_1, \dots, y_m) := (y^*_{m-1}, y^*_{m-2}, \dots, y^*_1, y^*_m),$$

  where $y \mapsto y^*$ denotes the C_2-action on Y.
- Let $S, R \in GL_m(\mathbb{Z})$ be the permutation matrices that send the i-th unit vector e_i to e_{i+1} and e_{m+1-i} (with subindexes taken modulo m), respectively. Then D_m acts on $A \in GL_m(Q_+G)$ by

$$s \cdot A := SAS^{-1}, \qquad r \cdot A := RA^\dagger R.$$

  In other words, r acts by transposition along the "$x = y$"-axis together with conjugation on G.

We also define $\widetilde{\theta}_{1,1} : Y \to GL_1(Q_+G) = (Q_+G)^\times$ by sending $y \in Y$ to $y \in \{1\} \times QG \subset Q_+G$. We note that $\widetilde{\theta}_{m,1}(y_1, \dots, y_m)$ is homotopy invertible by choosing a path from each of the y_i's to the basepoint $* \in Y$. Then for $m \geq 1$, define $\theta_{m,1}$ as the composition

$$\theta_{m,1} : Y^{\times m} \xrightarrow{\widetilde{\theta}_{m,1}} GL_m(Q_+G) \xrightarrow[\sim]{\varrho} \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]).$$

By construction, $\theta_{m,1}$ is a C_m-map as both $\widetilde{\theta}_{m,1}$ and u are. Recall that $s \in C_m \subset D_m$ acts on $\mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$ by conjugation with $S \in \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])$, which we denote by $S_\#$.

Step 2. Recall that the free $\mathbb{E}_1$-algebra on a based, connected space X is naturally equivalent to $\Omega\Sigma X$. Therefore, we can extend $\theta_{m,1}$ to a C_m-equivariant $\mathbb{E}_1$-map

$$\theta_{m,2} : \Omega\Sigma(Y^{\times m}) \longrightarrow \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]).$$

For any based space X, let us write $\sigma : \Omega X \to (\Omega X)^{\mathrm{op}}$ for inversion in ΩX (i.e. reversing the loop direction). Given a C_m-space X, we will denote $X^{\mathrm{op}_{C_m}}$ for X with the opposite C_m-action (i.e. that in which s acts by s^{-1}, which is a valid left action as C_m is abelian). If X additionally has an $\mathbb{E}_1$-structure, we will write $X^{\mathrm{op},\mathrm{op}_{C_m}}$ for X with both the opposite $\mathbb{E}_1$-structure and the opposite C_m-action. Then the square of $\mathbb{E}_1$-maps

$$\begin{tikzcd}
\Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,2}"] \arrow[d, "\sigma\circ r"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_\#\circ\tau_\epsilon"] \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op},\mathrm{op}_{C_m}} \arrow[r, "\theta_{m,2}^{\mathrm{op},\mathrm{op}_{C_m}}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op},\mathrm{op}_{C_m}}
\end{tikzcd} \tag{5.19}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces with a C_m-action. Here $r : \Omega\Sigma(Y^{\times m}) \to \Omega\Sigma(Y^{\times m})^{\mathrm{op}_{C_m}}$ is induced by the action of $r \in D_m$ on $Y^{\times m}$ together with the flip of the suspension coordinate, and τ_ϵ really stands for (5.15). To see this, consider the diagram

$$\begin{tikzcd}
Y^{\times m} \arrow[rr, "\widetilde{\theta}_{m,1}"] \arrow[dr, "r"] \arrow[dd, "\sigma\circ r\circ\eta"'] & & GL_m(Q_+G) \arrow[r, "\varrho", "\sim"'] \arrow[d, "r=R_\#\circ\dagger"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[dd, "R_\#\circ\tau_\epsilon"] \\
& (Y^{\times m})^{\mathrm{op}_{C_m}} \arrow[r, "\widetilde{\theta}_{m,1}^{\mathrm{op}_{C_m}}"] \arrow[dl, "\eta^{\mathrm{op}_{C_m}}"] & GL_m(Q_+G)^{\mathrm{op}_{C_m}} \arrow[dr, "\varrho^{\mathrm{op}_{C_m}}", "\sim"'] & \\
\Omega\Sigma(Y^{\times m})^{\mathrm{op}_{C_m}} \arrow[rrr, "\theta_{m,2}^{\mathrm{op}_{C_m}}"] & & & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}_{C_m}}
\end{tikzcd} \tag{5.20}$$

of C_m-spaces. By definition, $\theta_{m,2}$ is the $\mathbb{E}_1$-map induced from the top horizontal composition in (5.20). Thus, in order to show that (5.19) homotopy commutes as C_m-equivariant $\mathbb{E}_1$-maps, it suffices to show that the outer square of (5.20) commutes in the homotopy category of C_m-spaces. But each of its subsquares/triangles commute in this category: indeed the lower subsquare does so by definition of $\theta_{m,2}$ (after applying $(-)^{\mathrm{op}_{C_m}}$), the left subtriangle and the upper subsquare too by an easy check, and the right subsquare by Proposition 5.16 and the observation that $R_\# \circ u = u^{\mathrm{op}_{C_m}} \circ R_\#$.

Step 3. The C_m-equivariant $\mathbb{E}_1$-map $\theta_{m,2}$ gives rise to an $\mathbb{E}_1$-map

$$\theta_{m,3} : C_m \ltimes \Omega\Sigma(Y^{\times m}) \xrightarrow{C_m\ltimes\theta_{m,2}} C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \xrightarrow{\ \mu\ } \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]),$$

where $\mu(s^i, h) := S^i h$. Observe that μ is indeed an $\mathbb{E}_1$-map as

$$\mu((s^i, h)\cdot(s^j, h')) = \mu(s^{i+j}, S^{-j} h S^j h') = S^i h S^j h' = \mu(s^i, h)\mu(s^j, h').$$

Now from the homotopy commutativity of (5.19), it immediately follows that the left subsquare in

$$\begin{tikzcd}
C_m \ltimes \Omega\Sigma(Y^{\times m}) \arrow[r, "C_m\ltimes\theta_{m,2}"] \arrow[rr, bend left, "\theta_{m,3}"] \arrow[d, "C_m\ltimes(\sigma\circ r)"] & C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[r, "\mu"] \arrow[d, "C_m\ltimes(R_\#\circ\tau_\epsilon)"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_\#\circ\tau_\epsilon"] \\
C_m \ltimes \Omega\Sigma(Y^{\times m})^{\mathrm{op},\mathrm{op}_{C_m}} \arrow[r, "C_m\ltimes\theta_{m,2}^{\mathrm{op},\mathrm{op}_{C_m}}"] & C_m \ltimes \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op},\mathrm{op}_{C_m}} \arrow[r, "\mu^{\mathrm{op}}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}}
\end{tikzcd} \tag{5.21}$$

commutes in the homotopy category of $\mathbb{E}_1$-spaces. Here $\mu^{\mathrm{op}}(s^i, h) := hS^i$, and since $\tau_\epsilon(S) = S^\dagger = S^{-1}$ and $RS = S^{-1}R$, it easily follows that the right subsquare also commutes as $\mathbb{E}_1$-maps. So the outer square of (5.21) commutes in the homotopy category of $\mathbb{E}_1$-spaces.

But given an $\mathbb{E}_1$-space X equipped with a C_m-action, there is an isomorphism of $\mathbb{E}_1$-spaces

$$\alpha : C_m \ltimes X^{\mathrm{op},\mathrm{op}_{C_m}} \xrightarrow{\ \cong\ } (C_m \ltimes X)^{\mathrm{op}}, \quad (s^i, x) \longmapsto (s^i, s^{-i}\cdot x).$$

Under this identification, the lower horizontal composition of (5.21) becomes $\theta^{\mathrm{op}}_{m,3}$, and hence

$$\begin{tikzcd} C_m \ltimes \Omega\Sigma(Y^{\times m}) \arrow[r, "\theta_{m,3}"] \arrow[d, "\alpha\circ(C_m\ltimes(\sigma\circ r))"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "R_\# \circ \tau_\epsilon"] \\ (C_m \ltimes \Omega\Sigma(Y^{\times m}))^{\mathrm{op}} \arrow[r, "\theta^{\mathrm{op}}_{m,3}"] & \mathrm{Aut}_G(\oplus^m \mathbb{S}[G])^{\mathrm{op}} \end{tikzcd} \tag{5.22}$$

is commutative in the homotopy category of $\mathbb{E}_1$-spaces.

Step 4. We wish to deloop (5.22), viewing the vertical maps as anti-involutions of their respective domains, and appealing to Lemma 5.5 to do so. But by Lemma 5.21, the delooping of the anti-involution $\alpha\circ(C_m \ltimes(\sigma\circ r))$ is homotopic to the delooping of the involution $\mathrm{inv}\circ\alpha\circ(C_m \ltimes (\sigma\circ r))$, where inv stands for inversion in the $\mathbb{E}_1$-space $C_m \ltimes \Omega\Sigma(Y^{\times m})$. It is given explictly by $\mathrm{inv}(s^i,\gamma) := (s^{-i}, s^i\cdot\sigma(\gamma))$, and hence we see that

$$\mathrm{inv}\circ\alpha\circ(C_m \ltimes (\sigma\circ r)) : (s^i,\gamma) \longmapsto (s^{-i}, r\cdot\gamma).$$

We denote this map simply by $\mathrm{inv}\ltimes r$. From now on we treat r as the action map on the D_m-space $\Sigma^\sigma(Y^{\times m})$, where σ is seen as a D_m-representation on which C_m acts trivially. Putting this together, the delooped version of (5.22) yields a homotopy commutative square of spaces

$$\begin{tikzcd} B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] \arrow[d, "B(\mathrm{inv}\ltimes r)"] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]) \arrow[d, "\overline{B}(R_\#\circ\tau_\epsilon)"] \\ B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \arrow[r, "B(\theta_{m,3})"] & B\mathrm{Aut}_G(\oplus^m \mathbb{S}[G]), \end{tikzcd}$$

where the notation $\overline{B}(-)$ stands for delooping in the sense of Lemma 5.5.

To simplify the terms in this last diagram, first observe that as $Y^{\times m}$ is connected, we have

$$B(C_m \ltimes \Omega\Sigma^\sigma(Y^{\times m})) \simeq ED_m \times_{C_m} B\Omega\Sigma^\sigma(Y^{\times m}) \simeq ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}).$$

The inversion on C_m coincides with the residual $C_2 = D_m/C_m$-action on C_m by conjugation, which explains why we chose to write ED_m instead of EC_m. As for the right hand side, note that $R_\#$ is an inner automorphism of $\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])$, and hence it induces a map homotopic to the identity on the classifying space level [AM04, §II, Thm. 1.9]. But delooping is functorial, so $\overline{B}(R_\#\circ\tau_\epsilon)$ and $\overline{B}(\tau_\epsilon) =: \tau_\epsilon$ are homotopic involutions on $B\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])$. All together, we obtain a homotopy C_2-equivariant map

$$\theta_{m,4} : ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}) \xrightarrow{B(\theta_{m,3})^+} B\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])^+ \subset \mathbb{Z}\times B\mathrm{Aut}_G(\oplus^m\mathbb{S}[G])^+ \to A(\Sigma^\sigma Y;\epsilon),$$

where the last map is the passage to the colimit as $m\to\infty$ (see (5.14)).

Step 5. The following diagram commutes up to homotopy:

$$\begin{tikzcd} \theta_{m,4} : ED_m/C_m \simeq BC_m \arrow[r, "Bj"] \arrow[d] & B\mathrm{Aut}_G(\oplus^m\mathbb{S}) \arrow[r] \arrow[d] & A(*;\epsilon) \arrow[d] \\ \theta_{m,4} : ED_m \times_{C_m} \Sigma^\sigma(Y^{\times m}) \arrow[r] & B\mathrm{Aut}_G(\oplus^m\mathbb{S}[G]) \arrow[r] & A(\Sigma^\sigma Y;\epsilon), \end{tikzcd}$$

where $j : C_m \to \mathrm{Aut}_G(\oplus^m\mathbb{S})$ is the inclusion of the permutation automorphisms. As $A(-;\epsilon) = \Omega^\infty\mathbf{A}(-;\epsilon)$, we can adjoin the $\Omega^\infty(-)$ to get a similar homotopy commutative diagram of spectra. Then passing to vertical cofibres and noting that C_m acts trivially on the suspension coordinate of $\Sigma^\sigma(Y^{\times m})$, we get a homotopy C_2-equivariant map of spectra

$$\theta_{m,5} : \Sigma^{\infty+\sigma}\big((ED_m)_+ \wedge_{C_m} Y^{\times m}\big) \longrightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

Step 6. Now by [CCGH87, Lem. 1.4] (see also [CC87, Lem. 2.4]), the obvious projection $(ED_m)_+\wedge_{C_m} Y^{\times m} \to (ED_m)_+ \wedge_{C_m} Y^{\wedge m}$ has a stable section $\Sigma^\infty\big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\big) \to \Sigma^\infty\big((ED_m)_+ \wedge_{C_m} Y^{\times m}\big)$ that is D_m/C_m-equivariant. This observation gives rise to a homotopy C_2-equivariant map of spectra

$$\boldsymbol{\theta}_{m,6} : \Sigma^{\infty+\sigma}\big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\big) \longrightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

Finally set $\boldsymbol{\theta}$ to be

$$\boldsymbol{\theta} : \bigvee_{m\geq 1} \Sigma^{\infty+\sigma}\big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\big) \xrightarrow{\bigvee_{m\geq 1}(\boldsymbol{\theta}_{m,6})} \bigvee_{m\geq 1} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon) \longrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon).$$

This map is homotopy C_2-equivariant by construction, and non-equivariantly yields (5.18) after applying $\Omega^{\infty+\sigma}(-)$. This latter map is an equivalence of infinite loop spaces by [CCGH87, Thm. 1.6], and as both the domain and codomain of $\boldsymbol{\theta}$ are 1-connective, it follows that $\boldsymbol{\theta}$ is itself an equivalence of spectra. This concludes the proof of Proposition 5.19. □

Corollary 5.22. *Let Y be a connected, based C_2-space. There is an equivalence of spectra*

$$\boldsymbol{\theta} : \bigvee_{m\geq 2} \Sigma^{\infty+\sigma}\big((ED_m)_+ \wedge_{C_m} Y^{\wedge m}\big) \xrightarrow{\sim} \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$$

that is C_2-equivariant up to homotopy. Here $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(-) := \mathrm{hofib}(\mathbf{Wh}^{\mathrm{Diff}}(-) \to \mathbf{Wh}^{\mathrm{Diff}}(*))$.

Proof. It is clear from the construction that the map

$$\Sigma^\infty(\Sigma^\sigma Y) \simeq \Sigma^{\infty+\sigma}\big((ED_1)_+ \wedge_{C_1} Y^{\wedge 1}\big) \xrightarrow{\boldsymbol{\theta}_{1,6}} \widetilde{\mathbf{A}}(\Sigma^\sigma Y;\epsilon)$$

is the (reduced version of the) usual inclusion of the stable homotopy into A-theory. Thus its cofibre is $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma Y;\epsilon)$, and the claim follows immediately. □

Remark 5.23. As the reader may have noticed by now, the last two sections are a tiny bit technical, and one may wonder if there could be alternative approaches to deal with them. Such an approach that may come to mind is to use trace methods to analyse $\widetilde{A}(\Sigma^\sigma Y;\epsilon)$, since it coincides (non-equivariantly) with the reduced TC of $\mathbb{S}[\Omega\Sigma Y]$ (as Y is connected). In fact, recent developments have been made towards a C_2-equivariant version of topological cyclic homology for ring spectra with anti-involutions, commonly known as *real topological cyclic homology* (cf. [Hø16, HM16, DMP21]). This approach has two caveats:

- A *real cyclotomic trace* map does not yet exist (at the time of writing). The construction of such a map was supposed to appear in [HM16], but it never saw the light in the end. This is, nevertheless, current work in progress by Harpaz–Nikolaus–Shah [HNS21, p. 24].
- Even though much is known about the p-complete homotopy type of the TC of spherical group rings (cf. [BCC+96] or [NS18, §4.3]), the analysis of its integral homotopy type does not seem to be present in the literature.

For these two reasons, we prefer to proceed as we have.

6. The homotopy type of spaces of long knots

This section is devoted to Theorem B, which describes the homotopy type of $\mathrm{Emb}_\partial(D^p, D^d)$ for $p \leq d-3$ and $d \geq 5$, localised at odd primes and up to the concordance embedding stable range $\phi_{\mathrm{CEmb}}(d,p)$. After its proof, which will not take too much effort given the results in the preceeding sections, we will draw some conclusions on the homotopy groups of spaces of long knots. For convenience let us recall the statement of Theorem B. Recall that ψ_m stands for the real m-dimensional representation of the dihedral group D_m and σ for the sign representation, regarded in this statement as a D_m-representation by restricting along the determinant $D_m \hookrightarrow O(2) \overset{\det}{\to} \{\pm 1\} = C_2$.

Theorem (Theorem B)**.** *For $p \leq d-3$ and $d \geq 5$, there exists a homotopy fibre sequence which, after localising away from 2 and taking $(\phi_{\mathrm{CEmb}}(d,p)-1)$-th Postnikov sections, takes the form*

$$\begin{array}{ccc} \prod_{m\geq 2} \Omega^\infty\Big(\big(\mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m})\big)_{hC_2}\Big) & \longrightarrow & \mathrm{Emb}_\partial(D^p, D^d) \\ & & \big\downarrow \\ & & \Omega^p\,\mathrm{hofib}\big(G(d-p)/O(d-p) \to G/O\big). \end{array}$$

It is split for $p \geq 2$, and splits after being looped once for $p = 1$.

Recall from Remark 1.4(*ii*) that the space $\mathrm{Emb}_\partial(D^p, D^d)$ (and $\widetilde{\mathrm{Emb}}_\partial(D^p, D^d)$ too) can be localised.

6.1. **Proof of Theorem B.** Recall from (1.4) that we write $\mathrm{Emb}_\partial^{(\sim)}(P,M) := \mathrm{hofib}_\iota(\mathrm{Emb}_\partial(P,M) \to \widetilde{\mathrm{Emb}}_\partial(P,M))$ when ι is clear from the context. We saw in Corollary 4.3 that the fibration sequence

$$\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \longrightarrow \mathrm{Emb}_\partial(D^p, D^d) \longrightarrow \widetilde{\mathrm{Emb}}_\partial(D^p, D^d), \tag{6.1}$$

upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, is split for $2 \le p \le d-3$, and splits for $p=1$ after looping once. So we need to describe the exterior terms of (6.1), after inverting 2.

For the block embeddings, the graphing map

$$\Gamma : \Omega^p \widetilde{\mathrm{Emb}}(*, D^{d-p}) \xrightarrow{\sim} \widetilde{\mathrm{Emb}}_\partial(D^p, D^{d-p} \times D^p) \cong \widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \tag{6.2}$$

of (4.1) is an equivalence by inspection. Then by [GKW01, Thm. 2.2.1] and the example right after it, when $d-p \ge 3$ and $d \ge 5$ (see Remark 6.1 below), it follows that

$$\widetilde{\mathrm{Emb}}_\partial(D^p, D^d) \simeq \Omega^p \,\mathrm{hofib}(O/O(d-p) \to G/G(d-p)) \simeq \Omega^p \,\mathrm{hofib}(G(d-p)/O(d-p) \to G/O), \tag{6.3}$$

yielding the base of (6.1).

Remark 6.1. The equivalence (6.3) is only valid if $d-p \ge 3$ and $d \ge 5$. As pointed out right after [GKW01, Thm. 2.2.1], the second condition is not that important. For instance in the case $p=1$ and $d=4$, it follows directly from (6.2) and (6.3) that

$$\Omega \widetilde{\mathrm{Emb}}_\partial(D^1, D^4) \simeq \widetilde{\mathrm{Emb}}_\partial(D^2, D^5) \simeq \Omega^2 \,\mathrm{hofib}(G(3)/O(3) \to G/O).$$

The codimension condition $d-p \ge 3$, however, is essential.

For the fibre of (6.1), we know by Theorem A that, for $N = \phi_{C\mathrm{Emb}}(d,p) - 1$, there is an equivalence

$$\tau_{\le N} \mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \simeq \tau_{\le N}\, \Omega^\infty \left(\Sigma^{-1}\Theta(D^d, S^{d-p-1} \times D^{p+1})_{hC_2}\right).$$

We now use Corollaries 5.15 and 5.22 to describe the right hand side of the equivalence above.

Proposition 6.2. *There is an equivalence*

$$\Omega^\infty \left(\Sigma^{-1}\Theta(D^d, S^{d-p-1} \times D^{p+1})_{hC_2}\right) \simeq_{[\frac12]} \prod_{m\ge 2} \Omega^\infty \Big(\big(\mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m})\big)_{hC_2}\Big).$$

Proof. First observe that there is a homotopy C_2-equivariant equivalence $S^2 \approx S^{2\sigma}$. So by Corollary 5.22, for each $n \ge 2$ there is a homotopy C_2-equivariant equivalence of spectra

$$\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^n;\epsilon) \approx \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(\Sigma^\sigma S^{n-2+\sigma};\epsilon) \approx \bigvee_{m\ge 2} \Sigma^{\infty+\sigma}\left(S_{hC_m}^{\psi_m\otimes(n-2+\sigma)}\right) = \bigvee_{m\ge 2} \mathbb{S}^\sigma \wedge \mathrm{Th}(\psi_m \otimes (n-2+\sigma)\,|_{C_m}).$$

Using this for $n = d-p-1 \ge 2$, we obtain a chain of equivalences

$$\begin{aligned}
\Omega^\infty\left(\Sigma^{-1}\Theta(D^d, S^{d-p-1}\times D^{p+1})_{hC_2}\right) &\simeq_{[\frac12]} \Omega^\infty \left((S^{d\cdot(\sigma-1)-1} \wedge \widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}))_{hC_2}\right)\\
&\simeq_{[\frac12]} \Omega^\infty \Big(\big(\bigvee_{m\ge 2} \mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m})\big)_{hC_2}\Big).\\
&= \prod_{m\ge 2} \Omega^\infty \Big(\big(\mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m})\big)_{hC_2}\Big).
\end{aligned}$$

The first equivalence follows from the observation that $\widetilde{\mathbf{Wh}}^{\mathrm{Diff}}(S^{d-p-1}) \simeq \Sigma^{-1}\mathbf{Wh}^{\mathrm{Diff}}(D^p, S^{d-p-1};\epsilon)$ because both τ_ϵ and $\mathbf{Wh}^{\mathrm{Diff}}(-)$ are homotopy invariants of $(-)$, together with Corollary 5.15. The second equivalence is a consequence of the previous argument and Proposition 5.3. This establishes the desired equivalence. □

All together, this completes the proof of Theorem B. □

Remark 6.3 (Topological version of Theorem B). The space $\mathrm{Emb}_\partial^{\mathrm{Top}}(D^p, D^d)$ of topological long knots is contractible (for all $p \le d$) by the Alexander trick. We could still be interested in the homotopy type of the space $\mathrm{Emb}_{\partial_0}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d)$ of thickened topological long knots with $p \le d-3$, and one can get a description of it localised away from 2 and up to the concordance embedding stable range, similar to the one in Theorem B—let us explain how. As before, we have a homotopy fibre sequence

$$\mathrm{Emb}_{\partial_0}^{\mathrm{Top},(\sim)}(D^p \times D^{d-p}, D^d) \longrightarrow \mathrm{Emb}_{\partial_0}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d) \longrightarrow \widetilde{\mathrm{Emb}}_{\partial_0}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d)$$

which, upon localising at odd primes and taking $(\phi_{C\mathrm{Emb}}(d,p)-1)$-th Postnikov sections, is split for $2 \le p \le d-3$, and splits for $p=1$ after looping once. So we should describe the side terms.

For the block embeddings, consider the space

$$B\widetilde{\mathrm{Top}}(q) := \operatorname{holim}\left(\begin{array}{ccc} & & B\mathrm{Top} \\ & & \downarrow \\ BG(q) & \longrightarrow & BG \end{array} \right)$$

which is responsible for the classification of topological block normal bundles if $q \ge 3$ (cf. [RS68b, §2], [RS68a] and [Wal99, §11]). As $\widetilde{\mathrm{Emb}}_\partial^{\mathrm{Top}}(D^p, D^d)$ is contractible by the Alexander trick, it then follows that

$$\widetilde{\mathrm{Emb}}_{\partial_0}^{\mathrm{Top}}(D^p \times D^{d-p}, D^d) \simeq \mathrm{Map}_\partial(D^p, \widetilde{\mathrm{Top}}(d-p)) = \Omega^p \widetilde{\mathrm{Top}}(d-p).$$

As for the pseudoisotopy embeddings, the topological version of Theorem A (see Remark 1.3) tells us that for $N = \phi_{C\mathrm{Emb}(d,p)} - 1$, there is an equivalence

$$\tau_{\le N} \mathrm{Emb}_{\partial_0}^{\mathrm{Top},(\sim)}(D^p \times D^{d-p}, D^d)) \simeq \tau_{\le N}\, \Omega^\infty \left(\Sigma^{-1}\Theta^{\mathrm{Top}}(D^d, S^{d-p-1} \times D^{p+1})_{hC_2}\right), \tag{6.4}$$

where $\Theta^{\mathrm{Top}}(M)$ stands for the first orthogonal derivative of the functor $B\mathrm{Homeo}_\partial^b(M \times -)$ (as for $\Theta(M) = \Theta^{\mathrm{Diff}}(M)$, this is a C_2-spectrum whose underlying spectrum is (equivalent to) $\Sigma^{-1}\mathbf{Wh}^{\mathrm{Top}}(M)$). Noting that there is fibre sequence of C_2-spectra $\mathbf{Wh}^{\mathrm{Diff}}(M;\epsilon) \to \mathbf{Wh}^{\mathrm{Top}}(M;\epsilon) \to \Sigma\mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon) \wedge M_+$, one verifies that the infinite loop space in the right hand side of (6.4) fits in a fibre sequence away from 2

$$\begin{array}{ccc} \prod_{m\ge 2} \Omega^\infty\left(\left(\mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m})\right)_{hC_2}\right) & \longrightarrow & \Omega^\infty\left(\Sigma^{-1}\Theta^{\mathrm{Top}}(D^d, S^{d-p-1} \times D^{p+1})_{hC_2}\right) \\ & & \downarrow \\ & & \Omega^\infty\left((S^{d\cdot\sigma-p-2} \wedge \mathbf{Wh}^{\mathrm{Diff}}(*;\epsilon))_{hC_2}\right). \end{array}$$

In particular, it is easy to check (e.g. rationally) that the left hand side of (6.4) is not contractible (at least if d is sufficiently large). This was claimed in Remark 3.4.

6.2. **On the homotopy groups of spaces of long knots.** We can get plenty of information about the homotopy groups of $\mathrm{Emb}_\partial(D^p, D^d)$ from Theorem B. First observe that by Morlet's lemma of disjunction [BLR06, Thm. 3.1] (and Proposition 3.3 to reduce to the codimension zero case), the pseudoisotopy embedding space $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ is at least $(2(d-p-2)-1)$-connected. So by (6.3), it follows that

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d)) \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)), \quad * < 2(d-p-2). \tag{6.5}$$

Remark 6.4. This should be compared to work of Budney [Bud08, Prop. 3.9]. The main result there is the computation of the first non-trivial homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d-p \ge 3$, which lies in degree $2d-3p-3$, together with a geometric interpretation of the generators. From our point of view, he shows that $\mathrm{hofib}(G(d-p)/O(d-p) \to G/O)$ is exactly $(2d-2p-4)$-connected, which follows by work of Haefliger (see Section 3, Equation 4.11 and Corollary 6.6 of [Hae66]), and computes $\pi_{2d-2p-3}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))$. Moreover, it is stated in [Bud08, Prop. 3.9(1)] that the graphing map

$$\Gamma : \pi_*(\Omega\,\mathrm{Emb}_\partial(D^{p-1}, D^{d-1})) \longrightarrow \pi_*(\mathrm{Emb}_\partial(D^p, D^d))$$

is surjective for $* \le 2d-2p-5$. From (6.3) and the fact that $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ is $(2d-2p-5)$-connected, we see that it is in fact an isomorphism.

Recall that $\phi_{\mathrm{CEmb}}(d,p) \geq 2d-p-5$ by work of Goodwillie–Krannich–Kupers [GKK23], and so the pseudoisotopy embedding space $\mathrm{Emb}_\partial^{(\sim)}(D^p, D^d)$ has interesting homotopy in degrees from $2d-2p-4$ up to that range that we can understand by Theorem B. Let E^m denote the $C_2 = D_m/C_m$-spectrum

$$E^m := \mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathrm{Th}(\psi_m \otimes (d-p-3+\sigma)\,|_{C_m}) \simeq \mathbb{S}^{(d+1)\cdot(\sigma-1)} \wedge \mathbb{S}_{hC_m}^{\psi_m \otimes (d-p-3+\sigma)},$$

for $m \geq 2$. Then by Theorem B, for any odd prime ℓ there are isomorphisms in degrees $* \leq \phi_{\mathrm{CEmb}}(d,p)-1$

$$\pi_*(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \cong \pi_{*+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)} \oplus \bigoplus_{m\geq 2} \pi_*^s(E^m_{hC_2})_{(\ell)},$$

and if $\phi = \phi_{\mathrm{CEmb}}(d,p)$, there is also an exact sequence of abelian groups

$$\textstyle\bigoplus_{m\geq 2} \pi_\phi^s(E^m_{hC_2})_{(\ell)} \longrightarrow \pi_\phi(\mathrm{Emb}_\partial(D^p, D^d))_{(\ell)} \longrightarrow\!\!\!\!\rightarrow \pi_{\phi+p}(\mathrm{hofib}(G(d-p)/O(d-p) \to G/O))_{(\ell)},$$

where $A_{(\ell)}$ denotes $A \otimes \mathbb{Z}_{(\ell)}$, for A an abelian group. It remains to understand the groups $\pi_*^s(E^m_{hC_2})_{(\ell)}$, which are easier to study when ℓ is coprime to m.

Proposition 6.5. *Let $m \geq 2$ and $d-p \geq 3$. For $\ell \nmid 2m$ a prime,*

- *if d is even and p is even, then*
$$\pi_*^s(E^m_{hC_2})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3,5,7,\ldots \\ 0, & \textit{otherwise}. \end{cases}$$
- *if d is odd and p is odd, then*
$$\pi_*^s(E^m_{hC_2})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 2,4,6,\ldots \\ 0, & \textit{otherwise}. \end{cases}$$
- *if d is even and p is odd, then*
$$\pi_*^s(E^m_{hC_2})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 5,9,13,\ldots \\ 0, & \textit{otherwise}. \end{cases}$$
- *if d is odd and p is even, then*
$$\pi_*^s(E^m_{hC_2})_{(\ell)} \cong \begin{cases} \pi^s_{*-m(d-p-2)} \otimes \mathbb{Z}_{(\ell)}, & m = 3,7,11,\ldots \\ 0, & \textit{otherwise}. \end{cases}$$

Remark 6.6 (Rational homotopy of spaces of long knots). The rational homology and homotopy of $\mathrm{Emb}_\partial(D^p, D^d)$ for $d-p \geq 3$ has been extensively studied in recent years (see e.g. [Tur10, AT14, AT15]) through the lens of embedding calculus and its relation to the little disks operads and their formality, finally culminating in the work of Fresse–Turchin–Willwacher [FTW17]. There they compute the rational homotopy groups of $\overline{\mathrm{Emb}}_\partial(D^p, D^d) := \mathrm{hofib}_\iota(\mathrm{Emb}_\partial(D^p, D^d) \to \mathrm{Imm}_\partial(D^p, D^d))$ as the homology of the *hairy graph complex* (shifted appropriately). Observationally, our results correspond to the 0- and 1-loop order parts of this graph complex up to degree $\phi_{\mathrm{CEmb}}(d,p) \geq 2d-p-5$, where higher loop orders are still not seen. More precisely, the 0-loop part corresponds to the rational homotopy of $G/G(d-p)$, the lowest summand (i.e. $m=1$ when $d-p$ is even) of the 1-loop part appears as that of $O/O(d-p)$, and the higher summands of the 1-loop part come from the rational homotopy of the spectra $E^m_{hC_2}$ for $m \geq 2$, which we just computed. It is worth noting that:

- The first non-trivial rational homotopy group of $\mathrm{Emb}_\partial(D^p, D^d)$ coming from the 2-loop part of the hairy graph complex lies in degree $2d-p-4$ when both d and p are odd (cf. [FTW17, Eq. 3]). Therefore, the lower bound $\phi_{\mathrm{CEmb}}(d,p) \geq 2d-p-5$ on the concordance stable range of Goodwillie–Krannich–Kupers is very much sharp.
- The 1-loop part of the hairy graph complex seems to be completely generated by the spectra $E^m_{hC_2}$ for $m \geq 2$ in all degrees outside of the concordance embedding stable range. In other words, the computations in [FTW17] give evidence for the existence of a rational left splitting of the Weiss–Williams map
$$\Phi^{\mathrm{Emb}} : \mathrm{Emb}_\partial^{(\sim)}(D^p, D^d) \longrightarrow \prod_{m\geq 2} \Omega^\infty E^m_{hC_2}.$$

To investigate this, one should first understand the attachment of the second orthogonal derivative $\Theta F^{(2)}$ of the orthogonal functor $F(U) := \mathrm{Emb}^b_\partial(D^p \times U, D^d \times U)$ to its Taylor tower (2.1). It would also be interesting to understand the integral picture.

Proof of Proposition 6.5. When ℓ is coprime to $2m$, we have that

$$\pi_*^s(E^m_{hC_2})_{(\ell)} \cong H_0(D_m; \pi_*^s(\mathbb{S}^{(d+1)(\sigma-1)+\psi_m\otimes(d-p-3+\sigma)}))_{(\ell)}$$

by the Bousfield–Kan spectral sequence, because the higher group homology of D_m is $2m$-torsion. Let t and r denote the generators of D_m with $t^m = e$ and $rtr = t^{-1}$ such that

$$\psi_m(t) : \mathbb{R}^m \ni (a_1, \dots, a_m) \longmapsto (a_m, a_1, \dots, a_{m-1}),$$
$$\psi_m(r) : \mathbb{R}^m \ni (a_1, \dots, a_m) \longmapsto (a_m, a_{m-1}, \dots, a_1).$$

Then t and r act on the group $\pi_*^s(\mathbb{S}^{(d+1)(\sigma-1)+\psi_m\otimes(d-p-3+\sigma)})$ by $(-1)^{\epsilon_t}$ and $(-1)^{\epsilon_r}$, respectively, where

$$\epsilon_t = (m-1)(d-p-2), \qquad \epsilon_r = \underbrace{d+1}_{(1)} + \underbrace{\frac{1}{2}m(m-1)(d-p-3)}_{(2)} + \underbrace{m + \frac{1}{2}m(m-1)}_{(3)}$$
$$\equiv d+1+m+\frac{1}{2}m(m-1)(d-p-2) \mod 2.$$

The terms (1), (2) and (3) are the contributions coming, respectively, from $(d+1)(\sigma-1)$, $\psi_m \otimes (d-p-3)$ and $\psi_m \otimes \sigma$. One then readily verifies that the groups $H_0(D_m; \pi_*^s(\mathbb{S}^{(d+1)(\sigma-1)+\psi_m\otimes(d-p-3+\sigma)}))$ are given by the fomuli in the statement. ☐

A bit more interesting are the homotopy groups $\pi_*^s(E^m_{hC_2})_{(\ell)}$ when ℓ is odd but divides m. We treat the case when $\ell = m = 3$, which hopefully serves as a sample computation for other cases.

Proposition 6.7. *The first few homotopy groups $\pi_*^s(E^3_{hC_2}) \otimes \mathbb{Z}_{(3)}$ of the spectrum $E^3_{hC_2}$, localised at 3 and when $d-p=3$, are given in Table 1. Equally coloured groups in this table correspond to the same case depending on whether certain differentials in Figure 5 vanish or not. Entries containing "?" correspond to potentially more complicated answers that do not conveniently fit in the table.*

TABLE 1. $\pi_*^s(E^3_{hC_2}) \otimes \mathbb{Z}_{(3)}$ for $d-p=3$ for low values of $* \geq 3$.

$*$	3	4	5	6	7	8	9	10	11	12	13
p even	$\mathbb{Z}_{(3)}$	0	0	$\mathbb{Z}/9$	0	0	0	$\mathbb{Z}/9$	0	0	$\mathbb{Z}/3$
p odd	0	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/9$	0

$*$	14	15	16	17	18	19	20	21	22	23	24
p even	$\mathbb{Z}/27$	0	0	$\mathbb{Z}/3$ $\mathbb{Z}/3 \oplus \mathbb{Z}/3$	$\mathbb{Z}/3^4$ $\mathbb{Z}/3^5$	0	?	?	$\mathbb{Z}/9$	$\mathbb{Z}/3$	0
p odd	$\mathbb{Z}/3$	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	0	$\mathbb{Z}/3$	$\mathbb{Z}/3$	0	0	$\mathbb{Z}/9 \oplus \mathbb{Z}/3$

To prove Proposition 6.7, we will need to understand the cohomology of E^3 as a module over the Steenrod algebra $\mathcal{A}_3$, which we recall is generated by the Steenrod powers P^k and the Bockstein operation β.

Lemma 6.8. *The spectrum cohomology of E^3 is given by*

$$H^*(E^3; \mathbb{F}_3) \cong \mathbb{F}_3\langle u\rangle \otimes_{\mathbb{F}_3} \mathbb{F}_3[\alpha, s]/(\alpha^2), \qquad |\alpha| = 1, \quad |s| = 2, \quad |u| = 3(d-p-2),$$

with

$$P^k(u\alpha^i s^j) = \left(\sum_{r=0}^{k} \binom{d-p-2}{r}\binom{j}{k-r}\right) u\alpha^i s^{j+2k}, \qquad \beta(u\alpha^i s^j) = \begin{cases} 0, & i=0, \\ -us^{j+1}, & i=1. \end{cases}$$

Moreover $C_2 = D_m/C_m$ acts on $H^*(E^3; \mathbb{F}_3)$ by $u\alpha^i s^j \mapsto (-1)^{p+i+j} u\alpha^i s^j$.

Proof. The key observation to carry out this calculation is that the C_3-representation $\psi_3 |_{C_3}$ decomposes as $1+\theta$, where θ is the 2-dimensional representation pulled back from the standard complex $U(1) \cong SO(2)$-representation on $\mathbb{C} \cong \mathbb{R}^2$. In particular, the associated vector bundle of the representation $\psi_m \otimes (d-p-3+\sigma) |_{C_m}$ is orientable; write $u \in H^*(E^3; \mathbb{F}_3)$ for the corresponding Thom class. The $\mathbb{F}_3$-cohomology of BC_3 is $\mathbb{F}_3[\alpha, s]/(\alpha^2)$ with $|\alpha| = 1$, $|s| = 2$ and

$$\beta(\alpha) = s, \qquad P^k(\alpha^i s^j) = \binom{j}{k} \alpha^i s^{j+2k}.$$

So it remains to understand the action of $\mathcal{A}_3$ on u. Clearly $\beta(u) = 0$, as if $\beta(u) = nu\alpha$ for some $n \in \mathbb{F}_3$, then $0 = \beta^2(u) = n^2 u\alpha^2 - nus$ and hence $n = 0$. For u_θ the Thom class of θ, $P^1(u_\theta) = u_\theta^3 = u_\theta c_1(\theta)^2 = u_\theta s^2$. It then easily follows that $P^k(u) = \binom{d-p-2}{k} us^{2k}$.

The residual D_m/C_m-action on the $\mathbb{F}_3$-cohomology of BC_3 sends s to $-s$, and hence α to $-\alpha$. This action also switches the orientation of the vector bundle associated to θ, and hence that of $\psi_3 = 1+\theta$. So that of $\psi_3 \otimes \sigma$ does not change then, as ψ_3 is odd-dimensional. All in all, this means that u is sent by the C_2-action to $(-1)^\epsilon u$ with $\epsilon = (d+1)+(d-p-3) \equiv p \mod 2$, where the first term is the contribution of $\mathbb{S}^{(d+1)(\sigma-1)}$. This establishes the claim. ☐

Proof of Proposition 6.7. Consider the $\mathcal{A}_3$-submodules of $H^*(E^3; \mathbb{F}_3)$ given by

$$J_0 := \langle u\alpha^i s^j : i+j \equiv 0 \mod 2\rangle, \qquad J_1 := \langle u\alpha^i s^j : i+j \equiv 1 \mod 2\rangle.$$

Then $H^*(E^3; \mathbb{F}_3) = J_0 \oplus J_1$ as $\mathcal{A}_3$-modules, and J_p, where p here is taken mod 2, is the $(+1)$-eigenspace of the residual $C_2 = D_m/C_m$-action. Therefore $H^*(E^3_{hC_2}; \mathbb{F}_3) = J_p$ as an $\mathcal{A}_3$-module, and the Adams spectral sequence of J_p then converges to the stable homotopy of $E^3_{hC_2}$. The software [CCBFY22] computes the E_2-page of this spectral sequence, from which we determine some of the first homotopy groups of $E^3_{hC_2}$. We illustrate the case $d-p=3$ in Figure 5.

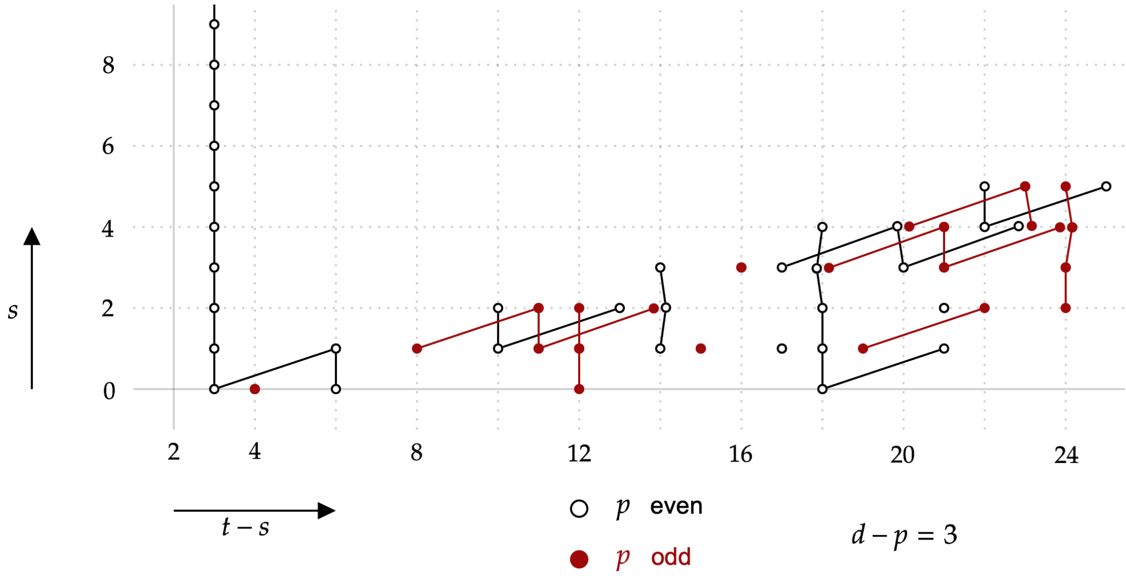

FIGURE 5. E_2-page of the Adams spectral sequence at the prime 3 for $E^3_{hC_2}$ when $d-p=3$. Here s denotes the degree in the Adams filtration and $t-s$ is the total degree.

Apart from standard arguments exploting the Leibniz rule and multiplicative structures of the differentials in Figure 5, we can appeal to some more refined tricks that allow us to solve the first few possible non-zero differentials in the case when p is even. Denote by $\widehat{\theta}$ the D_3-representation pulled back from the standard $O(2)$-representation on $\mathbb{R}^2$. Observe that $\widehat{\theta} |_{C_3} \equiv \theta$ in the notation of Lemma 6.8. Write $\underline{S}(\widehat{\theta} \otimes \sigma |_{C_3})$ for the unit sphere bundle of the associated vector bundle $ED_3 \times_{C_3} (\widehat{\theta} \otimes \sigma)$. As it is an S^1-bundle over BC_3, $\underline{S}(\widehat{\theta} \otimes \sigma |_{C_3})$ must itself be a $K(\pi, 1)$, and in fact it must be homotopy equivalent to S^1 because

$q : \underline{S}(\widehat{\theta} \otimes \sigma\ |_{C_3}) \to BC_3$ does not admit a section (its euler class is $s \in H^2(BC_3; \mathbb{F}_3) = \mathbb{F}[\alpha, s]/(\alpha^2)$). Moreover the homology class represented by q is the dual of α, and hence the residual $C_2 = D_m/C_m$-action on $\underline{S}(\widehat{\theta} \otimes \sigma\ |_{C_3}) \simeq S^1$ must have degree -1. So there is an equivalence of unbased spaces $\underline{S}(\widehat{\theta} \otimes \sigma\ |_{C_3}) \simeq S^\sigma$ which is C_2-equivariant up to homotopy. We thus get a cofibration

$$S^\sigma_+ \simeq \underline{S}(\widehat{\theta} \otimes \sigma\ |_{C_3})_+ \xrightarrow{q} (BC_3)_+ \twoheadrightarrow \mathrm{Th}(\widehat{\theta} \otimes \sigma\ |_{C_3}) \simeq S^{-\sigma} \wedge \mathrm{Th}(\psi_3 \otimes \sigma\ |_{C_3})$$

which is C_2-equivariant up to homotopy (see Notation 5.2(i)). By equipping both S^σ and BC_3 with distinguished basepoints which are fixed under the respective involutions and which match under q, we can get rid of the added basepoints and yield a homotopy cofibre sequence of C_2-spectra

$$\mathbb{S}^{(d+1)(\sigma-1)+2\sigma} \xrightarrow{q} S^{(d+1)(\sigma-1)+\sigma} \wedge \Sigma^\infty BC_3 \twoheadrightarrow E^3.$$

Then, upon inverting 2 and taking homotopy C_2-orbits in the sequence above, we obtain equivalences of spectra

$$E^3_{hC_2} \simeq_{[\frac{1}{2}]} \begin{cases} (\Sigma^{\infty+1} BC_3)_{hC_2} \simeq \Sigma^{\infty+1} BD_3, & d \text{ even (so } p \text{ odd)}, \\ \mathrm{hocofib}(q_{hC_2} : \mathbb{S}^1 \to (S^\sigma \wedge \Sigma^\infty BC_3)_{hC_2}), & d \text{ odd (so } p \text{ even)}. \end{cases} \tag{6.6}$$

Now by the Kahn–Priddy theorem [KP78] at the prime 3, the transfer-like map $\Sigma^{\infty+1} BD_3 \to \tau_{>1}\mathbb{S}^1$ is split surjective on homotopy groups localised at 3, and hence by (6.6), the group $\pi^s_*(E^3_{hC_2})_{(3)}$ split surjects onto $\pi^s_{*-1} \otimes \mathbb{Z}_{(3)}$ for $* > 1$ when p is odd. We will use this fact together with knowledge of π^s_* to determine the differentials of the red spectral sequence in Figure 5. For convenience, let us reillustrate a different portion of it in Figure 6.

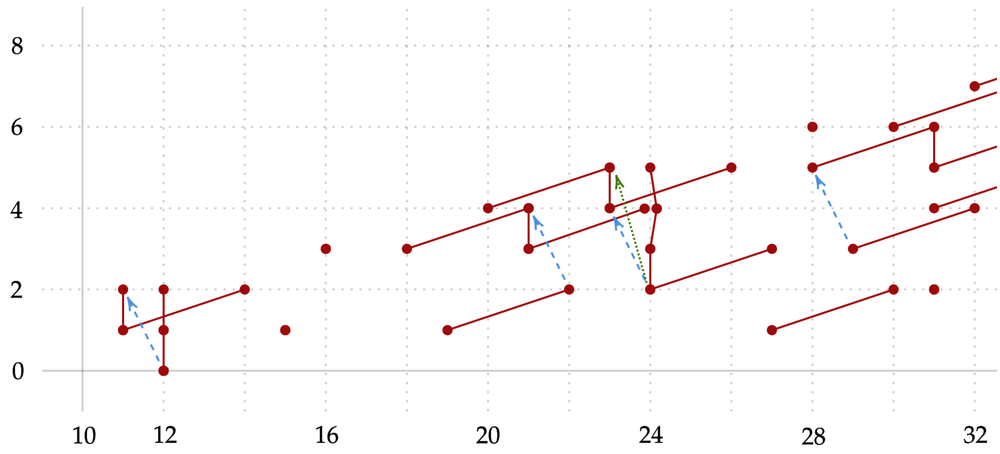

FIGURE 6. E_2-page of Adams spectral sequence at the prime 3 for $E^3_{hC_2}$ when $d - p = 3$ and p is odd. Some of the d_2 (dashed blue) and d_3 (dotted green) differentials that will be analysed are depicted.

The first possible non-zero differential goes from $(12, 0)$ to $(11, 2)$. Depending on its (non-)vanishing, we must have that either

$$(a)\ \pi^s_*(E^3_{hC_2})_{(3)} \cong \begin{cases} \mathbb{Z}/9, & * = 11, \\ \mathbb{Z}/27, & * = 12, \end{cases} \quad \text{or} \quad (b)\ \pi^s_*(E^3_{hC_2})_{(3)} \cong \begin{cases} \mathbb{Z}/3, & * = 11, \\ \mathbb{Z}/9, & * = 12. \end{cases}$$

Since $\pi^s_{10} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$, we must rule out possibility (a) as $\mathbb{Z}/9$ does not split surject onto $\mathbb{Z}/3$. In other words, the d_2 differential in Figure 5 from $(12, 0)$ to $(11, 2)$ is non-zero and (b) holds.

The d_2 differential from $(22, 2)$ to $(21, 4)$ must be non-zero, and hence so will that from $(19, 1)$ to $(18, 3)$ by the multiplicative structure. Indeed if such differential was zero, it would follow that $\pi^s_{21}(E^3_{hC_2})_{(3)} \cong \mathbb{Z}/9$, which does not split surject onto $\pi^s_{20} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3$.

The d_2 differential from $(24, 2)$ to $(23, 4)$ must also be non-zero (and hence that from $(24, 3)$ to $(23, 5)$) by a similar reason; indeed if it were trivial, then $\pi^s_{23}(E^3_{hC_2})_{(3)}$ would be isomorphic to $\mathbb{Z}/3 \oplus \mathbb{Z}/81$ or

$\mathbb{Z}/3 \oplus \mathbb{Z}/27$ (depending on whether the d_3 differential from $(24, 2)$ to $(24, 5)$ vanishes or not), either of which do not split surject onto $\pi^s_{23} \otimes \mathbb{Z}_{(3)} \cong \mathbb{Z}/3 \oplus \mathbb{Z}/9$.

The arguments we just made above, together with standard ones exploiting the Leibniz rule and the multiplicative structure of the differentials in the spectral sequence in Figure 5, establish the homotopy groups appearing in Table 1. □

One can keep up using the Kahn–Priddy theorem and go quite far up determining all possible non-zero differentials when p is odd; we leave it as a fun exercise to the eager reader. One would also hope that (6.6) could be used in the case when p is even, but this approach has somehow been inconclusive for us (at least for the first non-zero differentials that cannot be ruled out by elementary means).

Appendix A. Bounded geometry

All throughout, let M^d denote a smooth compact manifold (possibly with boundary) of dimension d.

A.1. **Models for bounded diffeomorphisms and embeddings.** Let N be a (possibly non-compact) smooth manifold and fix some smooth embedding $\iota : M \hookrightarrow N$. For $V \in \mathcal{J}_0$, we will write $\mathrm{Emb}_\partial(M \times V, N \times V)$ for the space of smooth embeddings of M into N that agree with $\iota \times \mathrm{Id}_V$ on some neighbourhood of the boundary $\partial M \times V$, endowed with the Whitney C^∞-topology. Following Definition 2.4, the *space of bounded embeddings* of $M \times V$ into $N \times V$ relative to $\partial M \times V$ is the subspace of $[0, +\infty) \times \mathrm{Emb}_\partial(M \times V, N \times V)$ given by

$$\mathrm{Emb}^b_\partial(M \times V, N \times V) := \{(t, \varphi) \in [0, +\infty) \times \mathrm{Emb}_\partial(M \times V, N \times V) : \varphi \text{ is } t\text{-bounded}\}.$$

Define similarly its simplicial version $\mathrm{Emb}^b_\partial(M \times V, N \times V)_\bullet$ as in Definition 2.7. In this section we prove

Proposition A.1. *There is a zig-zag of weak equivalences of simplicial group-like monoids*

$$\mathrm{Diff}^b_\partial(M \times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M \times V)).$$

Similarly, there is a zig-zag of weak equivalences of simplicial sets

$$\mathrm{Emb}^b_\partial(M \times V, N \times V)_\bullet \xleftarrow{\sim} \cdot \xrightarrow{\sim} \mathrm{Sing}_\bullet(\mathrm{Emb}^b_\partial(M \times V, N \times V)).$$

We will only deal with the first part of the statement, as the proof for the embedding case is completely analogous. Let us first introduce some notation. Given a topological space X, let $\mathrm{Sing}^{\mathrm{col}}_\bullet(X)$ be the sub-simplicial set of $\mathrm{Sing}_\bullet(X)$ consisting of those singular simplices that satisfy the ϵ-collaring condition of Section 2.2 for some $0 < \epsilon < 1/2$. Denote by $\mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ the sub-simplicial group of $\mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}_\partial(M \times V))$ consisting of those j-simplices which are adjoint to a bounded map $\Delta^j \times M \times V \to M \times V$. Then, there is a zig-zag of maps of simplicial group-like monoids

(A.1)

$$\mathrm{Diff}^b_\partial(M \times V)_\bullet \xhookrightarrow{①} \mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V)) \xleftarrow{②} \mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}^b_\partial(M \times V)) \xhookrightarrow{③} \mathrm{Sing}_\bullet(\mathrm{Diff}^b_\partial(M \times V)),$$

where the map ② forgets the explicit bounding constant of a simplex. We will show that all the maps in (A.1) are weak equivalences. We start with ③.

Lemma A.2. *The inclusion $i : \mathrm{Sing}^{\mathrm{col}}_\bullet(X) \hookrightarrow \mathrm{Sing}_\bullet(X)$ is a weak equivalence for every topological space X.*

Proof. We show that the relative homotopy groups $\pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$ vanish for all $j \geq 0$. Indeed, a homotopy class $x \in \pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$ corresponds, by the Yoneda Lemma, to a singular j-simplex $g : \Delta^j \to X$ which satisfies the ϵ-collaring condition for all faces $\sigma \subset \partial\Delta^j$ and some $\epsilon > 0$. Now fix some identification $\Delta^j \cong \Delta^j \cup_{\partial\Delta^j} (\partial\Delta^j \times [0, \epsilon])$, and consider the singular j-simplex

$$\overline{g} = g \cup (g\,|_{\partial_{\Delta^j}} \circ \mathrm{proj}_{\partial\Delta^j}) : \Delta^j \cong \Delta^j \cup_{\partial\Delta^j} (\partial\Delta^j \times [0, \epsilon]) \longrightarrow X.$$

By construction $\overline{g}$ now satisfies the δ-collaring conditions for all faces $\sigma \subset \Delta^j$ and some $0 < \delta \leq \epsilon$, so the corresponding relative homotopy class $\overline{x}$ is trivial. But clearly g and $\overline{g}$ are homotopic relative to the boundary by shrinking the added collar, and hence $x = \overline{x} = 0$ in $\pi_j(\mathrm{Sing}_\bullet(X), \mathrm{Sing}^{\mathrm{col}}_\bullet(X))$, as claimed. □

Remark A.3. The inclusion $i : \mathrm{Sing}^{\mathrm{col}}_\bullet(X) \hookrightarrow \mathrm{Sing}_\bullet(X)$ is in fact a simplicial homotopy equivalence; a homotopy inverse is constructed by induction on the skeleta of Δ^j. We will not need this though.

Lemma A.4. *The map* ② $: \mathrm{Sing}^{\mathrm{col}}_\bullet(\mathrm{Diff}^b_\partial(M \times V)) \to \mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ *of (A.1) is a weak equivalence.*

Proof. We again show that the relative homotopy groups π_j(②) vanish for all $j \geq 0$. Such a homotopy x class can be represented, for some $\epsilon > 0$, by an ϵ-collared singular j-simplex $g : \Delta^j \to \mathrm{Diff}_\partial(M \times V)$, adjoint to a map $g^\vee : \Delta^j \times M \times V \to M \times V$ bounded by some $K \geq 0$, together with a continuous ϵ-collared map $r : \partial\Delta^j \to [0, \infty)$ such that $g(s)$ is $r(s)$-bounded for all $s \in \partial\Delta^j$. To show that x is trivial, we need to extend r to a continuous δ-collared map $R : \Delta^j \to [0, \infty)$, for some $0 < \delta \leq \epsilon$, such that $g(s)$ is $R(s)$-bounded for all $s \in \Delta^j$. Fix some identification $\Delta^j \cong (\partial\Delta^j \times [0, \epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$; then $R\mid_{\partial\Delta^j \times [0,\epsilon/2]} \equiv r \circ \mathrm{proj}_{\partial\Delta^j}$ whilst $R\mid_{\partial\Delta^j \times [\epsilon/2,\epsilon]}$ is a linear interpolation along $[\epsilon/2, \epsilon]$ between r and the constant map $c_K : \partial\Delta^j \times \{\epsilon\} \to [0, \infty)$ with value $K \geq 0$. Finally set R to be constant of value K in the inner $\Delta^j \subset (\partial\Delta^j \times [0, \epsilon]) \cup_{\partial\Delta^j \times \{\epsilon\}} \Delta^j$. Then R is as required, and hence the relative homotopy class $x \in \pi_j$(②) is trivial. □

Remark A.5. For $CAT = \mathrm{Top}$, the map ① of (A.1) is an equality and thus, at this point, Proposition A.1 is established in the topological case.

Proof of Proposition A.1. It remains to show that ① is a weak equivalence, i.e., that the relative homotopy groups π_k(①) vanish for all $k \geq 0$. This is clear for $k = 0$ by definition. Such a homotopy class in π_k(①) is represented by a bounded homeomorphism

$$\overline{g} = (\mathrm{proj}_{\Delta^k}, g) : \Delta^k \times M \times V \longrightarrow \Delta^k \times M \times V$$

which is collared in the simplex direction and such that $g\mid_{\partial\Delta^k \times M \times V}$ is smooth. Therefore g is smooth on (a neighbourhood of) $\partial\Delta^k \times M \times V$. We need to smooth g outside of such neighbourhood in the Δ^k-direction and preserving boundedness. For $r \in \Delta^k$, we will write $g_r \in \mathrm{Diff}_\partial(M \times V)$ for $g\mid_{\{r\} \times M \times V}$.

Standard smoothing techniques [Mun66, §4] (see also [Kup19, Prop. 6.4.2] or [Lur09, Prop. 1]) can be used to prove the following: given nested compact subsets $L \subset K \subset \Delta^k \times M \times V$ with $L \subset \mathrm{int}\, K$ and any arbitrarily small $\epsilon > 0$, there exists a homotopy $H : I \times \Delta^k \times M \times V \to M \times V$ from g to some map $g' : \Delta^k \times M \times V \to M \times V$ satisfying that:

(*i*) H remains fixed on $\Delta^k \times M \times V - \mathrm{int}\, L$. In particular g and g' agree there.
(*ii*) g' is smooth on L. Moreover if g was already smooth on some (open neighbourhood of a) closed subset $\partial\Delta^k \times M \times V \subset F \subset \Delta^k \times M \times V$, the homotopy H remains fixed on F.
(*iii*) For each $t \in I$ and $r \in \Delta^k$, the map $H^t_r = H\mid_{\{t\} \times \{r\} \times M \times V} : M \times V \to M \times V$ is smooth.
(*iv*) H^t remains arbitrarily close to g for all $t \in I$. Consequently, if g is bounded by some $C \geq 0$, then for every $(r, t) \in I \times \Delta^k$ the map $H^t_r : M \times V \to M \times V$ is bounded by $C + \epsilon$, and is a diffeomorphism (as diffeomorphisms of compact manifolds are open in the space of smooth self-maps).

With this in mind, we construct a homotopy in $\mathrm{Sing}^{\mathrm{col},b}_\bullet(\mathrm{Diff}_\partial(M \times V))$ from the k-simplex $\overline{g} = (\mathrm{proj}_{\Delta^k}, g)$ to some $\overline{h} \in \mathrm{Diff}^b_\partial(M \times V)_k$ (relative to $\partial\Delta^k \times M \times V$), as follows: without loss of generality assume $V = \mathbb{R}^n$. Also for $v \in \mathbb{R}^n$ and $\delta > 0$, let $C_\delta(v) \subset \mathbb{R}^n$ denote the cube of side length 2δ and centered at v (i.e. $C_\delta(v) := v + [-\delta, \delta]^n$). Fix an $\epsilon > 0$ (e.g. $\epsilon = 1$). Then for each $v \in 3\mathbb{Z}^n \subset \mathbb{R}^n$, choose a homotopy as above starting from g with $(K, L) = (C_1(v), C_{2/3}(v))$, and perform all of these at the same time[6] to obtain some $g' : \Delta^k \times M \times V \to M \times V$. Now apply the same process to g' on $(K, L) = (C_1(v), C_{2/3}(v))$ for each $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ with $v_1 \equiv 1 \mod 3$ and $v_i \equiv 0 \mod 3$ for $2 \leq i \leq n$, keeping in mind that, by condition (*ii*) above, the homotopies keep fixed the parts that have been smoothed in the previous step. Continue this process in a similar fashion. After 3^n steps, we will obtain a smooth $(C + 3^n \cdot \epsilon)$-bounded map $h : \Delta^k \times M \times V \to M \times V$ such that $\overline{h} := (\mathrm{proj}_{\Delta^k}, h)$ represents the required k-simplex of $\mathrm{Diff}^b_\partial(M \times V)_\bullet$. This means that the relative homotopy class $[\overline{g}] \in \pi_k$(①) is trivial, as was to be shown. □

[6] This can be done as, by condition (*i*), the supports of such homotopies are disjoint by construction.

A.2. **A moduli space model for classifying spaces of bounded diffeomorphism groups.** Fix an embedding $\iota : M \hookrightarrow \mathbb{R}^m \subset \mathbb{R}^\infty$. Recall that the classifying space $B\mathrm{Diff}_\partial(M)$ of the diffeomorphism group of M admits a model as the moduli space of all d-manifolds $N^d \subset \mathbb{R}^\infty$ with $\partial N = \partial M$ which are diffeomorphic to M relative to the boundary. In this section we give an analogous description of the classifying space $B\mathrm{Diff}^b_\partial(M \times V)$, for any real finite-dimensional inner product vector space $V \in \mathcal{J}_0$.

Proposition A.6. *Set* $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V) := \mathrm{colim}_n\ \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^n \times V)$*, and let* $\mathrm{Diff}^b_\partial(M \times V)$ *act on it by precomposition. Then there is an equivalence*

$$B\mathrm{Diff}^b_\partial(M \times V) \simeq \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)/\mathrm{Diff}^b_\partial(M \times V).$$

In other words, $B\mathrm{Diff}^b_\partial(M \times V)$ is (equivalent to) the moduli space of all submanifolds in $\mathbb{R}^\infty \times V$ with boundary $\partial M \times V$ which are diffeomorphic to $M \times V$ boundedly in V and relative to $\partial M \times V$.

Proof. By Proposition A.1, we have that $B\mathrm{Diff}^b_\partial(M \times V) \simeq B|\mathrm{Diff}^b_\partial(M \times V)_\bullet| \simeq |B\mathrm{Diff}^b_\partial(M \times V)_\bullet|$ and

$$\frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)}{\mathrm{Diff}^b_\partial(M \times V)} \simeq \frac{|\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet|}{|\mathrm{Diff}^b_\partial(M \times V)_\bullet|} \simeq \left|\frac{\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet}{\mathrm{Diff}^b_\partial(M \times V)_\bullet}\right|.$$

As the simplicial action of $\mathrm{Diff}^b_\partial(M \times V)_\bullet$ on $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is visibly free, we only need to show that $\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ is weakly contractible by [GJ09, Cor. 2.6]. To that end, let

$$\varphi = (\mathrm{proj}_{\Delta^k}, \varphi_n, \varphi_V) : \Delta^k \times M \times V \hookrightarrow \Delta^k \times \mathbb{R}^n \times V, \quad n \geq m,$$

represent some homotopy class in $\pi_k(\mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet)$ for some $k \geq 0$. We will show that $[\varphi] = [\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V]$ by constructing a simplicial map $H : \Delta^1_\bullet \to \mathrm{Emb}^b_\partial(M \times V, \mathbb{R}^\infty \times V)_\bullet$ such that, under the Yoneda isomorphism, $\partial_0 H = \varphi$ and $\partial_1 H = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$. The map H will be given by (a modification of) the usual straight-line homotopy between φ and $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$.

Let us fix some notation. Pick some open collar $c : [0,1) \times \partial M \hookrightarrow M$ of the boundary of M. We can arrange the embedding $\iota : M \hookrightarrow \mathbb{R}^m$ to be such that

(*i*) $\iota \equiv (\mathrm{Id}_{[0,1)]} \times i) \circ c^{-1} |_{c([0,1) \times \partial M)}$ for some embedding $i : \partial M \hookrightarrow \mathbb{R}^{m-1}$, and
(*ii*) $\iota(M \setminus c((0,1] \times \partial M)) \subset [1, +\infty) \times \mathbb{R}^{m-1}$.

From now on we will supress ι and c from the notation, i.e., we canonically identify M (resp. $[0,1) \times \partial M$) with its image under ι (resp. c). Choose some increasing smooth function $\alpha : [0,1] \to [0,1]$ for which there exists some $0 < \delta$ with $\alpha |_{[0,\delta]} \equiv 0$, $\alpha |_{[1-\delta,1]} \equiv 1$ and $0 < \alpha(t) < 1$ for $\delta < t < 1 - \delta$ (this δ is required for the collaring condition right before Definition 2.7). Now by the collaring condition,

$$\text{there exists some } 0 < \epsilon < 1 \text{ such that } \varphi \equiv \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V \text{ on } \Delta^k \times [0,\epsilon) \times \partial M \times V. \tag{A.2}$$

Finally, fix some smooth function $\rho : M \to [0,1]$ such that

$$\rho |_{[0,\epsilon/2] \times \partial M} \equiv 0 \quad \text{and} \quad \rho |_{M \setminus [0,\epsilon) \times \partial M} \equiv 1.$$

Then for $t \in [0,1]$, consider the map

$$H_t : \Delta^k \times M \times V \longrightarrow \Delta^k \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{|V|} \times V \subset \Delta^k \times \mathbb{R}^\infty \times V,$$

$$(r, x, v) \longmapsto \begin{pmatrix} r \\ \alpha(t) \cdot x + (1 - \alpha(t)) \cdot \varphi_n(r, x, v) \\ \rho(x)\alpha(t)(1 - \alpha(t)) \cdot x \\ \rho(x)\alpha(t)(1 - \alpha(t)) \cdot v \\ \alpha(t) \cdot v + (1 - \alpha(t)) \cdot \varphi_V(t, x, v) \end{pmatrix} \tag{A.3}$$

Here $x \in M \subset \mathbb{R}^m \subset \mathbb{R}^n$ and $\mathbb{R}^{|V|}$ is a Euclidean space of the same dimension as V, treated as a copy of V.

Claim. *Let* $C \geq 0$ *be the bound of* φ *on the* V*-coordinate. Then the map* H_t *is a* C*-bounded embedding for* $t \in [0,1]$*. Moreover,* H_t *agrees with* $\mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ *on* $\Delta^k \times [0,\epsilon/2] \times \partial M \times V$*.*

Proof of Claim. Indeed H_t is bounded by $C \geq 0$ (in the V-coordinate) as

$$\|(\alpha(t) \cdot v + (1-\alpha(t)) \cdot \varphi_V(t,x,v)) - v\|_V = (1-\alpha(t)) \cdot \|\varphi_V(t,x,v) - v\|_V \leq (1-\alpha(t)) \cdot C \leq C.$$

To see that H_t is an embedding, suppose that $H_t(r,x,v) = H_t(r',x',v')$. Clearly then $r = r'$ by the first coordinate in (A.3). Note that $H_t = \varphi$ if $t \leq \delta$ and $H_t = \mathrm{Id}_{\Delta^k} \times \iota \times \mathrm{Id}_V$ if $t \geq 1-\delta$. As both are embeddings, we may assume that $\delta < t < 1-\delta$ so that $\alpha(t)(1-\alpha(t)) \neq 0$. To show that $x = x'$ we consider three cases:

- If $x, x' \in [0,\epsilon] \times \partial M$, then by (A.2), the equation on the second coordinate of (A.3) yields $x = x'$.
- If $x, x' \notin [0,\epsilon] \times \partial M$, then $\rho(x) = \rho(x') = 1$ and thus the third coordinate equation yields $x = x'$.
- If $x \in [0,\epsilon] \times \partial M$ but $x' \notin [0,\epsilon] \times \partial M$, then the third coordinate equation becomes $\rho(x) \cdot x = x' \in \mathbb{R}^n$. On the first coordinate of $[0,+\infty) \times \mathbb{R}^{n-1} \subset \mathbb{R}^n$, this implies, by items (i) and (ii) above, that $\rho(x) > 1$ which is a contradiction.

In all cases $x = x'$. Then the equation on the fourth coordinate of (A.3) implies that $v = v'$, as required.

Finally, the last part of the claim again follows from (A.2) and the nature of ρ. ☐

The family of C-bounded embeddings $\{H_t\}_{t\in[0,1]}$ gives rise to the required simplicial map H. This finishes the proof of the proposition. ☐

APPENDIX B. THE h-COBORDISM STABILISATION MAP

The (lower) stabilisation map $\Sigma : H(M) \to H(M \times I)$ of [Vog85, p. 298] is depicted in Figure 7 below.

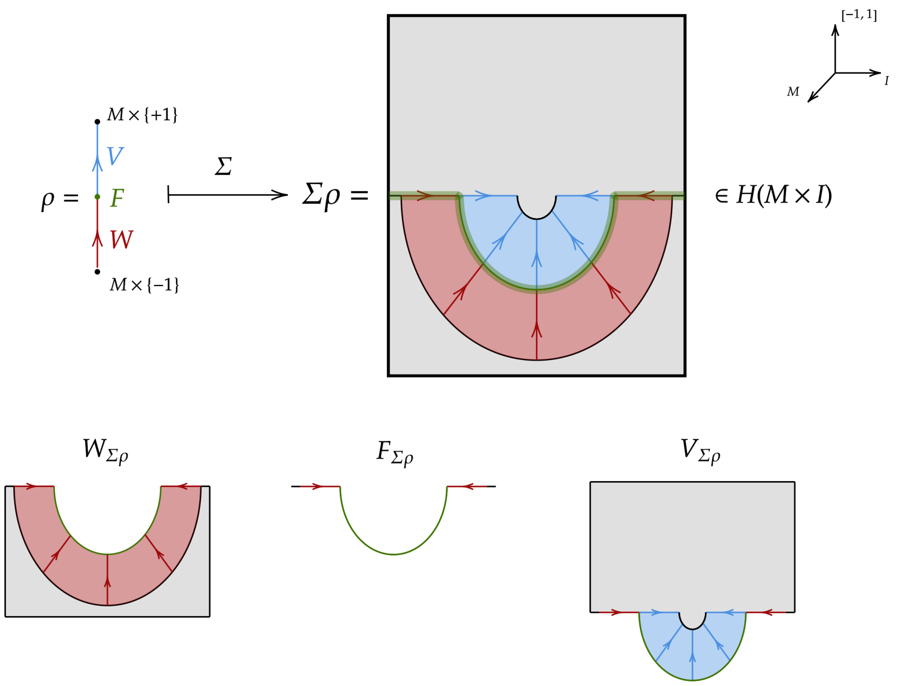

FIGURE 7. The h-cobordism (lower) stabilisation map $\Sigma_\ell = \Sigma : H(M) \to H(M \times I)$, sending $\rho = (W,F,V)$ to $\Sigma\rho := (W_{\Sigma\rho}, F_{\Sigma\rho}, V_{\Sigma\rho})$. A grey shaded region of shape S represents a manifold of the form $M \times S$.

Recall that the h-cobordism space $H(M \times I)$ is an $\mathbb{E}_1$-space under stacking in the I-direction, denoted $+_I$. In this section we argue that Σ anticommutes with the h-cobordism involution ι_H in the following sense:

Lemma B.1 (Lemma 5.14(a))**.** *The map $\iota_H\Sigma +_I \Sigma\iota_H : H(M) \to H(M \times I)$ is null-homotopic, i.e. it is homotopic to the constant map at the trivial partition $* \in H(M \times I)$.*

Proof. We describe the null-homotopy in the topological setting; the smooth case is very similar, but one has to be slightly careful with issues regarding corners (which can be overcome by working with the collared version $H_{\mathsf{col}}(M \times I)$ of the h-cobordism space). It will be convenient to work with yet another (upgraded) version of the h-cobordism space: let $\widehat{H}_{\mathsf{col}}(M)_\bullet$ denote the simplicial set in which a q-simplex consists of a pair (ρ, ϕ) with $\rho := (W, F, V) \in H_{\mathsf{col}}(M)_q$ and a diffeomorphism $\phi : V \cup_{M\times\Delta^q} W \cong F \times [-1,1] \times \Delta^q$ over Δ^q which fixes pointwise (a neighbourhood of) $\partial(F \times [-1,1]) \times \Delta^q$. There is a Kan fibration

$$\mathrm{Diff}_\partial(M \times [-1,1])_\bullet \xrightarrow{j} \widehat{H}_{\mathsf{col}}(M)_\bullet \xrightarrow{p} H_{\mathsf{col}}(M)_\bullet,$$

where $p(\rho, \phi) := \rho$. The inclusion j admits a (left) section up to homotopy $s : \widehat{H}_{\mathsf{col}}(M)_\bullet \to \mathrm{Diff}_\partial(M\times[-1,1])_\bullet$ given roughly by applying ϕ^{-1} on the collar of F in $M \times [-1,1] = W \cup_F V$ and then canonically identifying $W \cup_F V \cup_M W \cup_F V$ with $M \times [-1,1]$. This yields an equivalence

$$(p, s) : \widehat{H}_{\mathsf{col}}(M)_\bullet \xrightarrow{\sim} H_{\mathsf{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M \times [-1,1])_\bullet.$$

But now the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\widehat{H}_{\mathsf{col}}(M)_\bullet \arrow[d, "\wr"] \arrow[drr, "f_1"] & & \\
H_{\mathsf{col}}(M)_\bullet \times \mathrm{Diff}_\partial(M \times [-1,1])_\bullet \arrow[rr, "f_2"] \arrow[d, shift right, "\mathrm{pr}_1"'] & & H(M \times I)_\bullet \\
H_{\mathsf{col}}(M)_\bullet \arrow[u, shift right, "i"'] \arrow[urr, "f_3"] \arrow[d, "u", "\wr"'] & & \\
H(M)_\bullet \arrow[uurr, "f_4=\iota_H\Sigma+_I\Sigma\iota_H"'] & &
\end{tikzcd}$$

where u is the map that forgets the collaring data, and all the horizontal maps really factor through the bottom horizontal map $f_4 := \iota_H\Sigma +_I \Sigma\iota_H$ (on the nose). Therefore, in order to show that f_4 is null-homotopic, it suffices to show that f_1 is so: indeed this would imply that f_2 is null-homotopic. But $f_2 = f_3 \circ \mathrm{pr}_1$, so $f_3 \simeq f_3 \circ \mathrm{pr}_1 \circ i \simeq *$ too. This in turn would imply that f_4 is null-homotopic, as we aim to prove.

We therefore need to describe a null-homotopy of f_1. We will just describe a path (or rather, a 1-simplex) between $f_4(\rho, \phi) = (\iota_H\Sigma +_I \Sigma\iota_H)(\rho)$, for a fixed 0-simplex $(\rho = (W, F, V), \phi) \in \widehat{H}_{\mathsf{col}}(M)_0$, and the trivial partition $* \in H(M \times I)_0$—an exactly analogous argument yields an actual simplicial null-homotopy. This path is depicted in Figures 8, 9, 10 and 11. The green shaded regions in each picture represent the F-part of a partion, i.e., the intersection of the two h-cobordisms making up the partition, which is a $(d+1)$-manifold embedded in $M \times I \times [-1,1]$. The partition $\rho = (W, F, V) \in H(M)_0$ is as depicted in Figure 7.

Firstly, the path between the partition $P_0 = \iota_H\Sigma(\rho) +_I \Sigma\iota_H(\rho)$ and P_1 is obtained from rescaling (and slightly shifting inwards). But as $W \cup_F V$ is canonically[7] identified with $M \times [-1,1]$ as part of the data of ρ, the outer bent regions of the form $(W \cup_F V) \times I$ added to P_1 in order to obtain P_2 are canonically identified with $M \times [-1,1] \times I$, and therefore $P_1 = P_2$ in $H(M \times I)$.

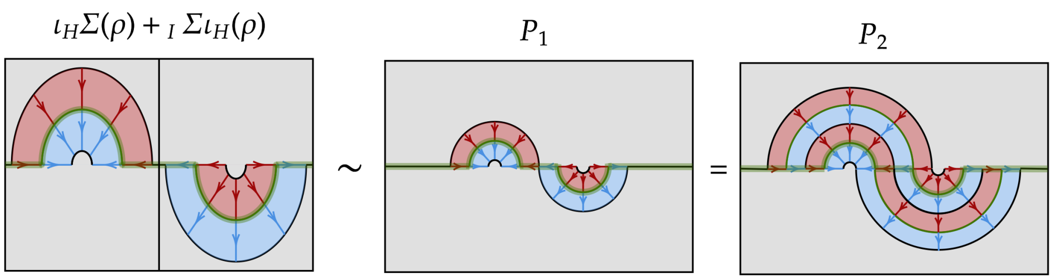

FIGURE 8. Path in $H(M \times I)$ between $\iota_H\Sigma(\rho) + \Sigma\iota_H(\rho)$ and P_2.

[7] *Canonically* in the sense that it does not depend in any other choice than the one of ρ.

Unbending and straightening the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I \equiv M \times [-1,1] \times I$ in P_2, we get the path to the partition P_3 of Figure 9 (this step is not strictly necessary, but convenient for depiction).

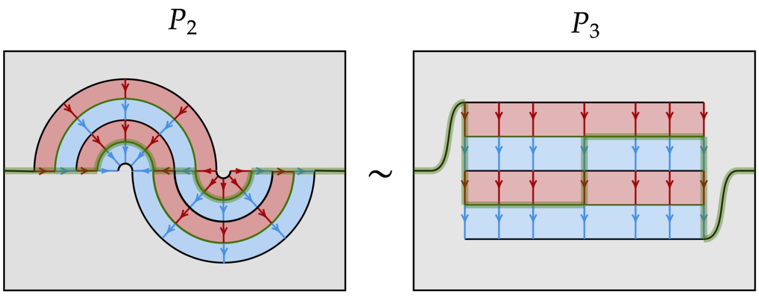

FIGURE 9. Path in $H(M \times I)$ between P_2 and P_3.

We now use the diffeomorphism $\phi : V \cup_M \cup W \cong F \times [-1,1]$ (rel. $\partial(F \times [-1,1])$) to carry out the path depicted in the lower part of Figure 10 locally in the circled sub-rectangle of P_3. This yields the path of Figure 10 between P_3 and P_4.

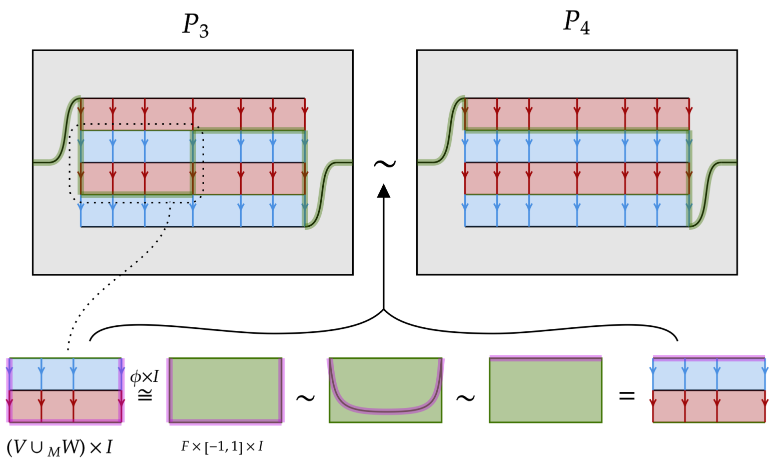

FIGURE 10. Path in $H(M \times I)$ between P_3 and P_4. The green shaded region in the lower part of the figure represents the manifold $F \times [-1,1] \times I$. The purple shaded region there represents the F-part of the partition (which used to be green, but is purple momentarily).

Retracting the region of the form $(W \cup_F V \cup_M W \cup_F V) \times I$ in P_4 onto its midpoint $(W \cup_F V \cup_M W \cup_F V) \times \{1/2\}$ yields the path of Figure 11 between P_4 and P_6 (passing through P_5). But now as $W \cup_F V \cup_M W \cup_F V \equiv M \times [-1,3]$, the F-part of the partition P_6 is of the form $M \times \gamma$, for the 1-dimensional submanifold $\gamma \subset [-1,1] \times I$ depicted in P_6. By straightening γ to the submanifold $\{0\} \times I \subset [-1,1] \times I$, we finally get the path of Figure 11 between P_6 and the trivial partition $P_7 = * \in H(M \times I)$.

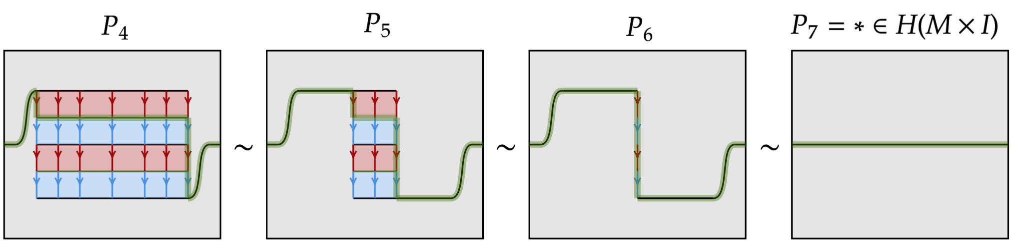

FIGURE 11. Path in $H(M \times I)$ between P_4 and P_7.

This process depends continuously on $(\rho, \phi) \in \widehat{H}_{\mathsf{col}}(M)$ (i.e. can be set up as a homotopy of simplicial maps $\widehat{H}_{\mathsf{col}}(M)_\bullet \to H(M \times I)_\bullet$), and therefore gives the required null-homotopy. □

Remark B.2. This result should be compared with [Hat78, Appendix I, Lem.] (or [BL82, Cor. A7]), where it is shown that the concordance stabilisation map $\Sigma : C(M) \to C(M \times I)$ anti-commutes (in the same sense of Lemma 5.14) with the concordance involution ι_C of Warning 5.8.

REFERENCES

[AM04] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Springer Berlin Heidelberg, 2004.

[AP83] D. R. Anderson and E. K. Pedersen, *Semifree topological actions of finite groups on spheres*, Mathematische Annalen **265** (1983), no. 1, 23–44.

[AT14] G. Arone and V. Turchin, *On the rational homology of high-dimensional analogues of spaces of long knots*, Geometry & Topology **18** (2014), no. 3, 1261–1322.

[AT15] G. Arone and V. Turchin, *Graph-complexes computing the rational homotopy of high dimensional analogues of spaces of long knots*, Annales de l'Institut Fourier **65** (2015), no. 1, 1–62.

[BC09] R. Budney and F. Cohen, *On the homology of the space of knots*, Geometry & Topology **13** (2009), no. 1, 99–139.

[BCC+96] M. Bökstedt, G. Carlsson, R. Cohen, T. Goodwillie, W. C. Hsiang, and I. Madsen, *On the algebraic K-theory of simply connected spaces*, Duke Mathematical Journal **84** (1996), no. 3, 1–1.

[BdBW18] P. Boavida de Brito and M. Weiss, *Spaces of smooth embeddings and configuration categories*, Journal of Topology **11** (2018), no. 1, 65–143.

[BF86] D. Burghelea and Z. Fiedorowicz, *Cyclic homology and algebraic K-theory of spaces|II*, Topology **25** (1986), no. 3, 303–317.

[BFJ20] M. Bustamante, F. T. Farrell, and Yi. Jiang, *Involution on pseudoisotopy spaces and the space of nonnegatively curved metrics*, Transactions of the American Mathematical Society **373** (2020), no. 10, 7225–7252.

[BL82] D. Burghelea and R. Lashof, *Geometric transfer and the homotopy type of the automorphism groups of a manifold*, Transactions of the American Mathematical Society **269** (1982), no. 1, 1–1.

[BLR06] D. Burghelea, R. Lashof, and M. Rothenberg, *Groups of automorphisms of manifolds*, 1975 ed., Lecture Notes in Mathematics, Springer, Berlin, Germany, November 2006.

[BM19] A. Blumberg and M. Mandell, *The homotopy groups of the algebraic K-theory of the sphere spectrum*, Geometry & Topology **23** (2019), no. 1, 101–134.

[Bud08] R. Budney, *A family of embedding spaces*, Geometry and Topology Monographs, Mathematical Sciences Publishers, February 2008.

[CC87] R. L. Cohen and G. E. Carlsson, *The cyclic groups and the free loop space*, Commentarii mathematici Helvetici **62** (1987), 423–449.

[CCBFY22] D. Chua, H. Chatham, J. Beauvais-Feisthauer, and M. Young, *The spectral sequences project (v0.0.4)*, `https://spectralsequences.github.io/sseq/`, April 2022.

[CCGH87] G. E. Carlsson, R. L. Cohen, T. Goodwillie, and W. C. Hsiang, *The free loop space and the algebraic K-theory of spaces*, K-Theory **1** (1987), no. 1, 53–82.

[Coh73] M. M. Cohen, *A course in simple-homotopy theory*, Graduate Texts in Mathematics, Springer, New York, NY, April 1973.

[DGM12] B. I. Dundas, T. G. Goodwillie, and R. McCarthy, *The local structure of algebraic K-theory*, Springer London, 2012.

[DH12] W. Dwyer and K. Hess, *Long knots and maps between operads*, Geometry & Topology **16** (2012), no. 2, 919 – 955.

[DM94] B. I. Dundas and R. McCarthy, *Stable K-theory and topological hochschild homology*, The Annals of Mathematics **140** (1994), no. 3, 685.

[DMP21] Emanuele Dotto, Kristian Moi, and Irakli Patchkoria, *On the geometric fixed-points of real topological cyclic homology*, 2021.

[Dun97] B. I. Dundas, *Relative K-theory and topological cyclic homology*, Acta Mathematica **179** (1997), no. 2, 223–242.

[EK71] R. D. Edwards and R. C. Kirby, *Deformations of spaces of imbeddings*, Annals of Mathematics **93** (1971), no. 1, 63–88.

[FTW17] B. Fresse, V. Turchin, and T. Willwacher, *The rational homotopy of mapping spaces of e_n operads*, 2017.

[GJ09] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Birkhäuser Basel, 2009.

[GKK23] T. Goodwillie, M. Krannich, and A. Kupers, *Stability of concordance embeddings*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics (2023), 1–36.

[GKW01] T. G. Goodwillie, J. R. Klein, and M. Weiss, *Spaces of smooth embeddings, disjunction and surgery*, pp. 221–284, Princeton University Press, 2001.

[Goo86] T. G. Goodwillie, *Relative algebraic K-theory and cyclic homology*, The Annals of Mathematics **124** (1986), no. 2, 347.

[Hae66] A. Haefliger, *Differentiable embeddings of S^n in S^{n+q} for $q > 2$*, The Annals of Mathematics **83** (1966), no. 3, 402.

[Hat78] A. Hatcher, *Algebraic and geometric topology: Concordance spaces, higher simple-homotopy theory, and applications*, American Mathematical Society, 1978.

[Hes09] L. Hesselholt, *On the whitehead spectrum of the circle*, pp. 131–184, Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[Hir66] M. W. Hirsch, *On normal microbundles*, Topology **5** (1966), no. 3, 229–240.

[Hir76] Morris W. Hirsch, *Differential topology*, Springer New York, 1976.

[HM16] L. Hesselholt and I. Madsen, *Real algebraic K-theory*, 2016, available at: `https://web.math.ku.dk/~larsh/papers/s05/book.pdf`.

[HNS21] Y. Harpaz, T. Nikolaus, and J. Shah, *Real topological cyclic homology of spherical group rings*, 2021, talk at the conference "Manifolds and K-theory: the legacy of Andrew Ranicki". Slides available at https://www.icms.org.uk/sites/default/files/documents/events/Manifolds%20and%20K-theory%20Y%20Harpaz%2022.06.21.pdf.

[Hud70] J. F. P. Hudson, *Concordance, isotopy, and diffeotopy*, The Annals of Mathematics **91** (1970), no. 3, 425.

[HW73] A. Hatcher and J. B. Wagoner, *Pseudo-isotopies of compact manifolds*, Société Mathématique de France (1973), no. 6, 1–275.

[Hø16] A. Høgenhaven, *Real topological cyclic homology of spherical group rings*, Ph.D. thesis, University of Copenhagen, 2016.

[Igu88] K. Igusa, *The stability theorem for smooth pseudoisotopies*, K-Theory **2** (1988), no. 1-2, 1–355.

[KP78] D. S. Kahn and S. B. Priddy, *The transfer and stable homotopy theory*, Mathematical Proceedings of the Cambridge Philosophical Society **83** (1978), no. 1, 103–111.

[Kup19] A. Kupers, *Lectures on diffeomorphisms groups of manifolds*, 2019, available at: https://people.math.harvard.edu/~kupers/teaching/272x/book.pdf.

[Lim64] E. L. Lima, *On the local triviality of the restriction map for embeddings*, Commentarii mathematici Helvetici **38** (1963/64), 163–164.

[Lur09] J. Lurie, *Diffeomorphisms and PL-homeomorphisms*, 2009, Lecture notes of "Topics in Geometric Topology", Lecture 6. Available at: https://www.math.ias.edu/~lurie/937notes/937Lecture6.pdf.

[Mil66] J. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), no. 3, 358–426.

[Mil71] ______, *Introduction to algebraic K-theory*, vol. 72, Ann. of Math. Studies, 1971.

[MS06] J. May and J. Sigurdsson, *Parametrized homotopy theory*, American Mathematical Society, December 2006.

[Mun66] J. R. Munkres, *Elementary differential topology*, Annals of Mathematics Studies, vol. 54, Princeton University Press, Princeton, NJ, December 1966.

[NS18] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Mathematica **221** (2018), no. 2, 203–409.

[Pal60] R. S. Palais, *Local triviality of the restriction map for embeddings*, Commentarii Mathematici Helvetici **34** (1960), no. 1, 305–312.

[Rog02] J. Rognes, *Two-primary algebraic K-theory of pointed spaces*, Topology **41** (2002), no. 5, 873–926.

[Rog03] ______, *The smooth Whitehead spectrum of a point at odd regular primes*, Geometry & Topology **7** (2003), no. 1, 155 – 184.

[RS67] C. P. Rourke and B. J. Sanderson, *An embedding without a normal microbundle*, Inventiones Mathematicae **3** (1967), no. 4, 293–299.

[RS68a] ______, *Block bundles: I*, The Annals of Mathematics **87** (1968), no. 1, 1.

[RS68b] ______, *Block bundles: III. homotopy theory*, The Annals of Mathematics **87** (1968), no. 3, 431.

[Sin09] D. P. Sinha, *The topology of spaces of knots: cosimplicial models*, American Journal of Mathematics **131** (2009), no. 4, 945–980.

[Tur10] V. Turchin, *Hodge-type decomposition in the homology of long knots*, Journal of Topology **3** (2010), no. 3, 487–534.

[Vog85] W. Vogell, *The involution in the algebraic K-theory of spaces*, Algebraic & Geometric Topology: Lecture Notes in Math., vol. 1126, Springer, 1985.

[Vol06] I. Volić, *Finite type knot invariants and the calculus of functors*, Compositio Mathematica **142** (2006), no. 01, 222–250.

[Wal85] F. Waldhausen, *Algebraic K-theory of spaces*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1985, pp. 318–419.

[Wal99] C. T. C. Wall, *Surgery on compact manifolds*, 2 ed., Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, March 1999.

[Wat07] T. Watanabe, *Configuration space integral for long n-knots and the alexander polynomial*, Algebraic & Geometric Topology **7** (2007), no. 1, 47–92.

[Wei95] M. Weiss, *Orthogonal calculus*, Trans. Amer. Math. Soc. (1995), no. 347, 3743–3796.

[WJR13] F. Waldhausen, B. Jahren, and J. Rognes, *Spaces of PL manifolds and categories of simple maps*, Annals of Mathematics Studies, no. 186, Princeton University Press, 2013.

[WW88] M. Weiss and B. Williams, *Automorphisms of manifolds and algebraic K-theory: I*, K-Theory **1** (1988), no. 6, 575–626.

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