**K₀ AND THE DIMENSION FILTRATION FOR p-TORSION IWASAWA MODULES**

KONSTANTIN ARDAKOV AND SIMON WADSLEY

**Abstract.** Let $G$ be a compact $p$-adic analytic group. We study $K$-theoretic questions related to the representation theory of the completed group algebra $kG$ of $G$ with coefficients in a finite field $k$ of characteristic $p$. We show that if $M$ is a finitely generated $kG$-module whose dimension is smaller than the dimension of the centralizer of any $p$-regular element of $G$, then the Euler characteristic of $M$ is trivial. Writing $\mathcal{F}_i$ for the abelian category consisting of all finitely generated $kG$-modules of dimension at most $i$, we provide an upper bound for the rank of the natural map from the Grothendieck group of $\mathcal{F}_i$ to that of $\mathcal{F}_d$, where $d$ denotes the dimension of $G$. We show that this upper bound is attained in some special cases, but is not attained in general.

1. Introduction

1.1. Iwasawa algebras. In this paper we study certain aspects of the representation theory of Iwasawa algebras. These are the completed group algebras

$$\Lambda_G := \lim_{\leftarrow} \mathbb{Z}_p[G/U],$$

where $\mathbb{Z}_p$ denotes the ring of $p$-adic integers, $G$ is a compact $p$-adic analytic group, and the inverse limit is taken over the open normal subgroups $U$ of $G$. Closely related is the epimorphic image $\Omega_G$ of $\Lambda_G$,

$$\Omega_G := \lim_{\leftarrow} \mathbb{F}_p[G/U],$$

where $\mathbb{F}_p$ is the field of $p$ elements.

This paper is a continuation of our earlier work [5], in which we investigated the relationship between the notion of characteristic element of $\Lambda_G$-modules defined in [10] and the Euler characteristic of $\Omega_G$-modules. We focus exclusively on the $p$-torsion $\Lambda_G$-modules, that is, those killed by a power of $p$. Because we are interested in $K$-theoretic questions, we only need to consider those modules actually killed by $p$, or equivalently, the $\Omega_G$-modules.

In this introduction we assume that the group $G$ has no elements of order $p$, although all of our results hold for arbitrary $G$ with slightly more involved formulations.

1.2. The dimension filtration. The category $\mathcal{M}(\Omega_G)$ of finitely generated $\Omega_G$-modules has a canonical dimension function $\dim(M) \mapsto d(M)$ defined on it which provides a filtration of $\mathcal{M}(\Omega_G)$ by admissible subcategories $\mathcal{F}_i$ whose objects are those modules of dimension at most $i$. If $d$ denotes the dimension of $G$ then $\mathcal{F}_d$ is just $\mathcal{M}(\Omega_G)$ and $\mathcal{F}_{d-1}$ is the full subcategory of torsion modules.

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Perhaps the central result of [5] was to classify those $p$-adic analytic groups $G$ for which the Euler characteristic of every finitely generated torsion $\Omega_G$-module is trivial. Following the proof of [5, Theorem 8.2] we see that the Euler characteristic of every module $M \in \mathcal{F}_i$ is trivial if and only if the natural map from $K_0(\mathcal{F}_i)$ to $K_0(\mathcal{F}_d)$ is the zero map. This raises the following

**Question.** When is the natural map $\alpha_i: K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d)$ the zero map?

Let $\Delta^+$ denote the largest finite normal subgroup of $G$. In [5] we answered the above question in the case $i = d - 1$: the map $\alpha_{d-1}$ is zero precisely when $G$ is $p$-nilpotent, that is, when $G/\Delta^+$ is a pro-$p$ group. There is a way of rephrasing this condition in terms of $G_{\text{reg}}$, the set of elements of $G$ of finite order: $G$ is $p$-nilpotent if and only if the centralizer of every element $g \in G_{\text{reg}}$ is an open subgroup of $G$.

1.3. Serre’s work. This question was answered by Serre in [23] in the case $i = 0$, although he did not use our language. In this case $\mathcal{F}_i$ consists of modules that are finite dimensional as $k$-vector spaces. He produced a formula which relates the Euler characteristic $\chi(G, M)$ of a module $M \in \mathcal{F}_0$ with its Brauer character $\varphi_M$:

$$\log_p \chi(G, M) = \int_G \varphi_M(g) \det(1 - \text{Ad}(g^{-1}))dg.$$ 

Here $\text{Ad}: G \to \text{GL}(L(G))$ is the adjoint representation of $G$. As a consequence, Serre proved that the Euler characteristic of every module $M \in \mathcal{F}_0$ is trivial precisely when the centralizer of every element $g \in G_{\text{reg}}$ is infinite.

1.4. Trivial Euler characteristics. The results of Serre and [5] mentioned above suggest that there might be a connection between the answer to Question 1.2 and the dimensions of the centralizers of the elements of $G_{\text{reg}}$. Indeed, one might wonder whether $\alpha_i$ is zero if and only if the $\dim C_G(g) > i$ for all elements $g \in G_{\text{reg}}$. Whilst this latter statement turns out to be not quite right — see (12.3) — such a connection indeed exists.

**Theorem A.** Let $M$ be a finitely generated $\Omega_G$-module such that $d(M) < \dim C_G(g)$ for all $g \in G_{\text{reg}}$. Then $\chi(G, M) = 1$.

The proof is given in (8.5).

1.5. A related question. In the remainder of the paper, we address the following

**Question.** What is the rank of the natural map $\alpha_i: K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d)$?

The group $G$ acts on $G_{\text{reg}}$ by conjugation and this action commutes with the action of a certain Galois group $G$ on $G_{\text{reg}}$, which essentially acts by raising elements to powers of $p$; see (3.1) for details. Let $S_i = \{ g \in G_{\text{reg}} : \dim C_G(g) \leq i \}$ — this is a union of $G \times G$-orbits. For example, $S_d = G_{\text{reg}}$ and $G_{\text{reg}} - S_{d-1}$ is the set of all elements of $G$ which have finite order and lie in a finite conjugacy class; in fact, $G_{\text{reg}} - S_{d-1}$ is just the largest finite normal subgroup $\Delta^+$ of $G$ mentioned in (1.2).

Our next result provides an upper bound for the rank of $\alpha_i$:

**Theorem B.** $\text{rk } \alpha_i$ is bounded above by the number of $G \times G$-orbits on $S_i$.

The proof is given in (9.3).
1.6. Some special cases. We next show that the rank of \( \alpha_i \) attains the upper bound given in Theorem B in some special cases.

Theorem C. The rank of \( \alpha_i \) equals the number of \( G \times G \)-orbits on \( S_i \) if either

(a) \( i = d \), or \( i = d - 1 \) or \( i = 0 \), or if

(b) \( G \) is virtually abelian.

This follows by combining Propositions 10.1, 10.6, 10.7 and Theorem 11.3.

Finally, in (12.3) we give an example to show that the upper bound of Theorem B is not always attained. Questions 1.2 and 1.5 remain open in general.

2. Generalities

Throughout this paper, \( k \) will denote a fixed finite field of characteristic \( p \) and order \( q \). Modules will be right modules, unless explicitly stated otherwise. We will conform with the notation of [5], with the exception of [5, §12].

2.1. Grothendieck groups. Let \( \mathcal{A} \) be a small abelian category. A full additive subcategory \( \mathcal{B} \) of \( \mathcal{A} \) is admissible if whenever \( 0 \to M' \to M \to M'' \to 0 \) is a short exact sequence in \( \mathcal{A} \) such that \( M \) and \( M'' \) belong to \( \mathcal{B} \), then \( M' \) also belongs to \( \mathcal{B} \) [17, 12.4.2].

The Grothendieck group \( K_0(\mathcal{B}) \) of \( \mathcal{B} \) is the abelian group with generators \([M]\) where \( M \) runs over all the objects of \( \mathcal{B} \) and relations \([M] = [M'] + [M'']\) for any short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{A} \) [17, 12.4.3].

We will frequently be dealing with vector space versions of these groups. To simplify the notation later on, whenever \( F \) is a field we will write \( FK_0(\mathcal{B}) := F \otimes \mathbb{Z} K_0(\mathcal{B}) \).

If \( A \) is a ring, then \( \mathcal{P}(A) \), the category of all finitely generated projective modules is an admissible subcategory of \( \mathcal{M}(A) \), the category of all finitely generated \( A \)-modules. The Grothendieck groups of \( A \) are defined as follows:

- \( K_0(A) := K_0(\mathcal{P}(A)) \), and
- \( G_0(A) := K_0(\mathcal{M}(A)) \).

We also set \( FK_0(A) := FK_0(\mathcal{P}(A)) \) and \( FG_0(A) := FK_0(\mathcal{M}(A)) \).

2.2. Homology and Euler characteristics. Let \( \mathcal{A} \) be an abelian category, let \( \Gamma \) be an abelian group and let \( \psi \) be an additive function from the objects of \( \mathcal{A} \) to \( \Gamma \). This means that for every object \( A \in \mathcal{A} \) there exists an element \( \psi(A) \in \Gamma \) such that

- \( \psi(B) = \psi(A) + \psi(C) \) whenever \( 0 \to A \to B \to C \to 0 \) is a short exact sequence in \( \mathcal{A} \).

In this context, if \( C_* = \cdots \to C_n \to \cdots \to C_0 \to \cdots \) is a bounded complex in \( \mathcal{A} \) (that is, \( C_i = 0 \) for sufficiently large \( |i| \)), we define the Euler characteristic of \( C_* \) (with respect to \( \psi \)) to be

\[
\psi(C_*) := \sum_{i \in \mathbb{Z}} (-1)^i \psi(C_i) \in \Gamma.
\]

Note that if we think of any object \( A \in \mathcal{A} \) as a complex \( A_* \) concentrated in degree zero, then \( \psi(A_*) = \psi(A) \). The following is well-known.
Lemma. Let $C_\ast$ be a bounded complex in $\mathcal{A}$. Then the Euler characteristic of $C_\ast$ equals the Euler characteristic of the homology complex $H_*(C_\ast)$ with zero differentials:

$$\psi(C_\ast) = \psi(H_*(C_\ast)).$$

Thus $\psi$ extends to a well-defined function from the objects of the derived category $\mathbf{D}(\mathcal{A})$ of $\mathcal{A}$ (if this exists) to $\Gamma$.

2.3. Spectral sequences and Euler characteristics. We will require a version of Lemma 2.2 for spectral sequences. Let $E = E_{ij}$ be a homology spectral sequence in $\mathcal{A}$ starting with $E^2$ [27, §5.2]. We say that $E$ is totally bounded if $E^r_{ij}$ is zero for all sufficiently large $|i|$ or $|j|$. This is equivalent to insisting that $E$ is bounded in the sense of [27, 5.2.5] and that there are only finitely many nonzero diagonals on each page. It is clear that $E$ converges.

We define the $r^{th}$-total complex $\text{Tot}(E^r)_\ast$ of $E$ to be the complex in $\mathcal{A}$ whose $n^{th}$ term is

$$\text{Tot}(E^r)_n = \bigoplus_{i+j=n} E^r_{ij}$$

and whose $n^{th}$ differential is $\bigoplus_{i\in\mathbb{Z}} d^r_{i,n-i}$. Because $E$ is totally bounded, $\text{Tot}(E^r)_n \in \mathcal{A}$ for all $r \geq a$ and $n \in \mathbb{Z}$ and each complex $\text{Tot}(E^r)$ is bounded.

The next result is folklore — see for example [18, Example 6, p.15] for a cohomological version and the proof of [16, Theorem 9.8] for a similar formulation — but we give a proof for the convenience of the reader.

Proposition. Let $E$ be a totally bounded homology spectral sequence in $\mathcal{A}$ starting with $E^2$. Then $\psi(\text{Tot}(E^\infty)_\ast) = \psi(\text{Tot}(E^\infty)_\ast)$.

Proof. From the definition of a spectral sequence we see that

$$H_n(\text{Tot}(E^r)_\ast) = \text{Tot}(E^{r+1})_n$$

for all $r \geq a$ and $n \in \mathbb{Z}$. Applying Lemma 2.2 repeatedly gives

$$\psi(\text{Tot}(E^\infty)_\ast) = \psi(\text{Tot}(E^{a+1})_\ast) = \cdots = \psi(\text{Tot}(E^\infty)_\ast)$$

as required.

2.4. Completed group algebras. Let $G$ be a profinite group. We will write $U_G$ (respectively, $U_{G,p}$) for the set of all open normal (respectively, open normal pro-$p$) subgroups $U$ of $G$.

Define the completed group algebra of $G$ by the formula

$$kG := k[[G]] := \lim_{U\in U_G} k[G/U].$$

As each group algebra $k[G/U]$ is finite, this is a compact topological $k$-algebra which “controls” the continuous $k$-representations of $G$ in the following sense. Whenever $V$ is a compact topological $k$-vector space and $\rho : G \to \text{Aut}_{cts}(V)$ is a continuous representation of $G$ then $\rho$ extends to a unique continuous homomorphism of topological $k$-algebras $\rho : kG \to \text{End}_{cts}(V)$, and $V$ becomes a compact topological (left) $kG$-module. Conversely, any compact topological $kG$-module $V$ gives rise to a continuous $k$-representation of $G$.

When $G$ is a compact $p$-adic analytic group, $kG$ is sometimes called an Iwasawa algebra — we refer the reader to [4] for more details. We note that in this case $kG$
3.3. Brauer characters.

G denotes the set of \( p \)-regular elements of \( G \) by construction. This action commutes with any automorphism of \( G \), so this definition extends the one given in \( (1.2) \).

Let \( m \) denote the \( p \)-part of \( |G| \) in the profinite sense; equivalently, \( m \) is the index of a Sylow pro-\( p \) subgroup of \( G \). As we are assuming that \( G \) is virtually pro-\( p \), \( m \) is finite. Let \( k' = k(\omega) \), where \( \omega \) is a primitive \( m \)-th root of unity over \( k \) and let \( G_k \) be the Galois group \( \text{Gal}(k(\omega)/k) \).

If \( \sigma \in G_k \), then \( \sigma(\omega) = \omega^{t_\sigma} \) for some \( t_\sigma \in (\mathbb{Z}/m\mathbb{Z})^\times \). This gives an injection \( \sigma \mapsto t_\sigma \) of \( G_k \) into \( (\mathbb{Z}/m\mathbb{Z})^\times \).

We can now define a left permutation action of \( G_k \) on \( G_{\text{reg}} \) by setting \( \sigma.g = g^{t_\sigma} \); this makes sense because \( t_\sigma \) is coprime to the order of any \( g \) in \( G_{\text{reg}} \) by construction. This action commutes with any automorphism of \( G \), so \( G_k \) permutes the \( p \)-regular conjugacy classes of \( G \). These constructions give a continuous action \( G \times G_k \) on \( G_{\text{reg}} \); note that \( G \times G_k \) is also virtually pro-\( p \) because \( G_k \) is a finite group.

3.2. Locally constant functions. Let \( X \) be a compact totally disconnected space. For any commutative ring \( A \) we let \( C(X; A) \) denote the ring of all locally constant functions from \( X \) to \( A \). If \( G \) acts on \( X \) continuously on the left then it acts on \( C(X; A) \) on the right as follows:

\[
(f.g)(x) = f(g.x) \quad \text{for all} \quad f \in C(X; A), g \in G, x \in X.
\]

We will identify the subring of invariants \( C(X; A)^G \) with \( C(G\backslash X; A) \), where \( G\backslash X \) denotes the set of \( G \)-orbits in \( X \).

3.3. Brauer characters. Our treatment is closely follows Serre [23, §2.1, §3.3].

Fix a finite unramified extension \( K \) of \( \mathbb{Q}_p \) with residue field \( k \). Let \( F = K(\bar{\omega}) \), where \( \bar{\omega} \) is a primitive \( m \)-th root of unity. Then the ring of integers of \( F \) is \( \mathcal{O}' = \mathcal{O}[\bar{\omega}] \) where \( \mathcal{O} \) is the ring of integers of \( K \). Reduction modulo \( p \) gives an isomorphism of the residue field of \( F \) with \( k' \) and we may assume that \( \bar{\omega} \) maps to \( \omega \) under this isomorphism. For each \( m \)-th root of unity \( \xi \in k' \) there is a unique \( m \)-th root of unity \( \bar{\xi} \in F \) such that \( \xi \) maps to \( \bar{\xi} \) modulo \( p \). This gives us an isomorphism \( \tilde{\cdot} : (\omega) \rightarrow (\bar{\omega}) \) between the two cyclic groups.

Let \( \mathcal{F}_0 = \mathcal{F}_0(G) \) be the abelian category of all topological \( kG \)-modules which are finite dimensional over \( k \). If \( A \in \mathcal{F}_0 \) and \( g \in G_{\text{reg}} \), the eigenvalues of the action of \( g \) on \( A \) are powers of \( \omega \) — say \( \xi_1, \ldots, \xi_d \) (always counted with multiplicity so that \( \dim A = d \)). Define

\[
\varphi_A(g) = \sum_{i=1}^d \bar{\xi}_i \in F.
\]

The function \( \varphi_A : G_{\text{reg}} \rightarrow F \) is called the Brauer character of \( A \). It has the following properties:

Lemma. Let \( A, B, C \in \mathcal{F}_0 \).
(i) $\varphi_A$ is a locally constant $G \times \mathcal{G}_k$-invariant $F$-valued function on $G_{\text{reg}}$:

$\varphi_A \in C(G_{\text{reg}}; F)^{G \times \mathcal{G}_k}$.

(ii) $\varphi_B = \varphi_A + \varphi_C$ whenever $0 \to A \to B \to C \to 0$ is a short exact sequence.

(iii) $\varphi_{A \otimes_k B} = \varphi_A \cdot \varphi_B$, where $G$ acts diagonally on $A \otimes_k B$.

(iv) Let $A' = A \otimes_k k'$ be the $k'$-$H$-module obtained from $A$ by extension of scalars. Then $\varphi_{A'} = \varphi_A$.

Proof. As $G$ acts continuously on the finite dimensional vector space $A$, some $U \in \mathcal{U}_G$ acts trivially. It follows that for any $g \in G$, $\varphi_A$ is constant on the open neighbourhood $gU$ of $g$, so $\varphi_A$ is a locally constant function. For the remaining assertions, we may assume that $G$ is finite. In this case, the result is well-known — see for example [12, Volume I, §17A, §21B] or [22, §18]. $\Box$

3.4. Berman–Witt Theorem. Lemma 3.3 shows that there is an $F$-linear map

$\varphi : FK_0(\mathcal{F}_0) \to C(G_{\text{reg}}; F)^{G \times \mathcal{G}_k}$

given by $\varphi(\lambda \otimes [A]) = \lambda \varphi_A$ for all $\lambda \in F$ and $A \in \mathcal{F}_0$ — see (2.1) for the notation.

Theorem. $\varphi$ is an isomorphism.

This is a generalization of the well-known Berman–Witt Theorem [12, Volume I, Theorem 21.25] in the case when $G$ is finite. When $k = \mathbb{F}_p$ and $G$ is finite, a short proof can also be found in [23, §2.3]. Our proof is given below in (3.8).

Until the end of §3, we fix $U \in \mathcal{U}_{G,p}$ and write $\overline{G}$ for the quotient group $G/U$.

3.5. Lemma. Let $\pi_U : G \to \overline{G}$ be the natural surjection. Then

$\pi_U(G_{\text{reg}}) = \overline{G}_{\text{reg}}$.

Proof. It is enough to show that $\pi_U(G_{\text{reg}}) \supseteq \overline{G}_{\text{reg}}$. First suppose that $G$ is a finite group so that $U$ is a normal $p$-subgroup. For any $x \in G$ we can find unique commuting $s \in G_{\text{reg}}$ and $u \in G$ such that $x = su$ and $u$ has order a power of $p$. Now if $xU \in \overline{G}$ is $p$-regular then raising $x$ to an appropriate sufficiently large power of $p$ doesn’t change $xU$ and has the effect of making $x = s$ $p$-regular.

Now suppose that $G$ is arbitrary. Serre [23, §1.1] has observed that $G_{\text{reg}}$ is a compact subset of $G$ which can be identified with the inverse limit of the various $(G/W)_{\text{reg}}$ as $W$ runs over $\mathcal{U}_G$. Because $U$ is open in $G$, we may assume that all the $W$’s are contained in $U$. Now the result follows from the first part. $\Box$

3.6. Proposition. The map $\pi_U$ induces a bijection

$\pi_U : (G \times \mathcal{G}_k) \backslash G_{\text{reg}} \to (\overline{G} \times \mathcal{G}_k) \backslash \overline{G}_{\text{reg}}$.

Proof. In view of Lemma 3.5, it is sufficient to prove that this map is injective. So let $x, y \in G_{\text{reg}}$ be such that $xU$ and $yU$ lie in the same $\overline{G} \times \mathcal{G}_k$-orbit. By replacing $y$ by a $G \times \mathcal{G}_k$-conjugate, we may assume that actually $xU = yU$. As $G$ is virtually pro-$p$, it will now be sufficient to show that $xW$ and $yW$ are conjugate in $G/W$ for any $W \in \mathcal{U}_{G,p}$ contained in $U$. Without loss of generality, we can assume that $G$ is finite and that $W = 1$.

Now as $x$ is $p$-regular and $U$ is a $p$-group, $\langle x \rangle \cap U = 1$. Similarly $\langle y \rangle \cap U = 1$, so $\langle x \rangle \cong \langle xU \rangle = \langle yU \rangle \cong \langle y \rangle$. 

Consider the finite solvable group $H := \langle x \rangle U$. As $xU = yU$, $y$ lies in $H$ and $U$ is the unique Sylow $p$-subgroup of $H$. It follows that $\langle x \rangle$ and $\langle y \rangle$ are Hall $p'$-subgroups of $H$ and as such are conjugate in $H$ [15]. Hence there exists $h \in H$ such that $x^h := h^{-1}xh = y^a$ for some $a \geq 1$. Now as $H/U$ is abelian, $xU = x^hU$. Hence $yU = y^aU$, but $y' \mapsto y'U$ is an isomorphism so $y = y^a = x^h$ as required. □

We would like to thank Jan Saxl for providing this proof.

Corollary. The $G \times G_k$-orbits in $G_{\text{reg}}$ are closed and open in $G_{\text{reg}}$.

Proof. Because $G \times G_k$ is a profinite group acting continuously on the Hausdorff space $G_{\text{reg}}$, the orbits are closed. But they are disjoint and finite in number by the Proposition, so they must also be open. □

3.7. Dévissage. Any finitely generated $k\mathcal{G}$-module is finite dimensional over $k$ since $\mathcal{G}$ is finite, so we have a natural inclusion $M(k\mathcal{G}) \subset F_0$ of abelian categories. Let $\lambda_U : G_0(\mathcal{G}) \to K_0(F_0)$ be the map induced on Grothendieck groups.

Lemma. $\lambda_U$ is an isomorphism.

Proof. Let $w_U := (U - 1)k\mathcal{G}$ be the kernel of the natural map $k\mathcal{G} \to k\mathcal{G}$ and let $A \in F_0$. As in the proof of Lemma 3.3 we can find $W \in U_G$ which acts trivially on $A$, which we may assume to be contained in $U$. Now $A$ is a $k[G/W]$-module and the image of $w_U$ in $k[G/W]$ is nilpotent because $U/W$ is a normal $p$-subgroup of the finite group $G/W$ [19, Lemma 3.1.6]. Thus $Aw_U^t = 0$ for some $t \geq 0$, so $A$ has a finite filtration 

$$0 = Aw_U^t \subset Aw_U^{t-1} \subset \cdots \subset Aw_U \subset A$$

where each factor is a $k\mathcal{G}$-module. Hence $\lambda_U$ is an isomorphism by dévissage — see, for example, [17, Theorem 12.4.7]. □

3.8. Proof of Theorem 3.4. Consider the commutative diagram

$$
\begin{array}{ccc}
F_0G_0(\mathcal{G}) & \xrightarrow{\varphi} & C(G_{\text{reg}}; F)^{G \times G_k} \\
\lambda_U \downarrow & & \downarrow \pi_U \\
FK_0(F_0) & \xrightarrow{\varphi} & C(G_{\text{reg}}; F)^{G \times G_k},
\end{array}
$$

where the map $\pi_U$ is defined by the formula $\pi_U(f)(g) = f(gU)$. The top horizontal map $\varphi$ is an isomorphism by the usual Berman–Witt Theorem [12, Volume I, Theorem 21.25]. $\pi_U$ is an isomorphism as a consequence of Proposition 3.6 and $\lambda_U$ is an isomorphism by Lemma 3.7. Hence the bottom horizontal map $\varphi$ is also an isomorphism, as required. □

4. Modules over Iwasawa algebras

4.1. Compact $p$-adic analytic groups. From now on, $G$ will denote an arbitrary compact $p$-adic analytic group. We will write $d := \dim G$ for the dimension of $G$. By the celebrated result of Lazard [13, Corollary 8.34], $G$ has an open normal uniform pro-$p$ subgroup, so $G$ is in particular virtually pro-$p$ and we can apply the theory developed in §3. We fix such a subgroup $N$ in what follows, and write $\overline{G}$ for the quotient group $G/N$. 

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4.2. **The base change map.** We begin by studying $G_0(kG)$ in detail. First, a preliminary result.

**Lemma.** The projective dimension of the left $kG$-module $kG$ is at most $d$.

**Proof.** It is well known that $kN$ has global dimension $\dim N = d$ [9, Theorem 4.1]. Now $kG \cong kG \otimes_{kN} k$ as a left $kG$-module and $kG$ is a free right $kN$-module of finite rank. The result follows. □

Hence the right $kG$-modules $\text{Tor}_j^{kG}(M, kG)$ are zero for all $j > d$ and all $M \in \mathcal{M}(kG)$, so we can define an element $\theta_N[M] \in G_0(kG)$ by the formula

$$\theta_N[M] := \sum_{j=0}^{\infty} (-1)^j [\text{Tor}_j^{kG}(M, kG)].$$

The long exact sequence for Tor shows that $M \mapsto \theta_N[M]$ is an additive function on the objects of $\mathcal{M}(kG)$, so we have a base change map [8, p. 454] on $G$-theory

$$\theta_N : G_0(kG) \rightarrow G_0(kG)$$

that will be one of our tools for studying $G_0(kG)$.

4.3. **Graded Brauer characters.** Let $\text{grmod}(kG)$ denote the category of all $kG$-modules $M$ which admit a direct sum decomposition

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

into a $kG$-submodules such that $M_n$ is finite dimensional for all $n \in \mathbb{Z}$ and zero for all sufficiently small $n$, thought of as a full subcategory of the abelian category of all $\mathbb{Z}$-graded $kG$-modules and graded maps of degree zero.

**Definition.** The graded Brauer character of $M \in \text{grmod}(kG)$ is the function

$$\zeta_M : G_{\text{reg}} \rightarrow \mathbb{C}[[t, t^{-1}]]$$

defined by the formula

$$\zeta_M(g) = \sum_{n \in \mathbb{Z}} \varphi_{M_n}(g) t^n.$$

This definition extends the notion of Brauer character presented in (3.3) if we think of any finite dimensional $kG$-module $M$ as a graded module concentrated degree zero.

Now let $M \in \mathcal{M}(kG)$ and let $w_N := (N-1)kG$ be the kernel of the natural map $kG \rightarrow kG$. As $kG$ is Noetherian, $w_N$ is a finitely generated right ideal in $kG$ for all $n \geq 0$, so the modules $Mw_N/Mw_{n+1}$ are finite dimensional over $k$ for all $n$. Hence the associated graded module

$$\text{gr} M := \bigoplus_{n=0}^{\infty} \frac{Mw_n}{Mw_{n+1}}$$

lies in $\text{grmod}(kG)$, and as such has a graded Brauer character $\zeta_{\text{gr} M}$.

We will see in (5.4) that $\zeta_{\text{gr} M}(g)$ is actually a rational function in $t$ for each $g \in G_{\text{reg}}$. 
The adjoint representation. Recall [13, §4.3] that there is an additive structure \((N, +)\) on our fixed uniform subgroup \(N\). In this way \(N\) becomes a free \(\mathbb{Z}_p\)-module of rank \(d\) so \(L(G) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} N\) is a \(\mathbb{Q}_p\)-vector space of dimension \(d\). There is a way of turning \(L(G)\) into a Lie algebra over \(\mathbb{Q}_p\) [13, §9.5], but we will not need this.

The conjugation action of \(G\) on \(N\) respects the additive structure on \(N\) and gives rise to the adjoint representation\(\text{Ad} : G \rightarrow \text{GL}(L(G))\) given by \(\text{Ad}(g)(n) = gng^{-1}\) for all \(g \in G\) and \(n \in N\). We define a function \(\Psi : G \rightarrow \mathbb{Q}_p[t]\) by setting \(\Psi(g) := \det(1 - \text{Ad}(g^{-1})t)\) for all \(g \in G\). As \(\Psi(g) \cdot \det \text{Ad}(g)\) is the characteristic polynomial of \(\text{Ad}(g)\), we can think of \(\Psi(g)\) as a polynomial in \(F[t]\) of degree \(d\).

The key result. By Theorem 3.4 and Lemma 3.7, the composite map \(\phi \circ \lambda_N : FG_0(kG) \rightarrow C(G_{\text{reg}}; F)^{G \times G}\) is an isomorphism. We therefore do not lose any information when studying \(\theta_N\) by postcomposing it with this isomorphism. Our main technical result reads as follows:

**Theorem.** Let \(\rho_N\) be the composite map
\[
\rho_N := \varphi \circ \lambda_N \circ \theta_N : FG_0(kG) \rightarrow C(G_{\text{reg}}; F)^{G \times G}.
\]
Then for any \(g \in G_{\text{reg}}\) and \(M \in F_d\), the number
\[
\rho_N[M](g) = \sum_{j=0}^{d} (-1)^j \varphi_{\text{Tor}^j_{\text{reg}}(M, kG)}(g) \in F
\]
equals the value at \(t = 1\) of the rational function
\[
\zeta_{G, M}(g) \cdot \Psi(g) \in F(t).
\]

We now begin preparing for the proof, which is given in §7.

5. **Graded modules for \(\text{Sym}(V) \# H\)**

5.1. **Notation.** Let \(V\) be a finite dimensional \(k\)-vector space and let \(H\) be a finite group acting on \(V\) by \(k\)-linear automorphisms on the right. We will write \(v^h\) for the image of \(v \in V\) under the action of \(h \in H\). This action extends naturally to an action of \(H\) on the symmetric algebra \(\text{Sym}(V)\) by \(k\)-algebra automorphisms. Let
\[
R := \text{Sym}(V) \# H
\]
denote the skew group ring [17, 1.5.4]: by definition, \(R\) is a free right \(\text{Sym}(V)\)-module with basis \(H\), with multiplication given by the formula
\[
(hr)(gs) = (hg)(r^g s)
\]
for all \(g, h \in H\) and \(r, s \in \text{Sym}(V)\). Thus \(R\) is isomorphic to \(kH \otimes_k \text{Sym}(V)\) as a \(k\)-vector space and setting \(R_n := kH \otimes_k \text{Sym}^n V\) turns \(R = \bigoplus_{n=0}^{\infty} R_n\) into a graded \(k\)-algebra.
5.2. **Dimensions.** If $S$ is a positively graded $k$-algebra, let $\mathcal{M}_\text{gr}(S)$ denote the category of all finitely generated $\mathbb{Z}$-graded right $S$-modules and graded maps of degree zero. Since $R = \text{Sym}(V) \# H$ is a finitely generated $\text{Sym}(V)$-module, we have an inclusion $\mathcal{M}_\text{gr}(R) \subset \mathcal{M}_\text{gr}(\text{Sym}(V))$ of abelian categories.

Following [7, §11] we define the dimension $d(M)$ of a module $M \in \mathcal{M}_\text{gr}(\text{Sym}(V))$ to be the order of the pole of the Poincaré series $P_M(t)$ of $M$ at $t = 1$, where

$$P_M(t) = \sum_{n \in \mathbb{Z}} (\dim_k M_n)t^n \in \mathbb{Z}[[t, t^{-1}]].$$

In fact [7, Theorem 11.1], there exists a polynomial $u(t) \in \mathbb{Z}[t, t^{-1}]$ such that

$$u(1) \neq 0$$

and

$$P_M(t) = \frac{u(t)}{(1 - t)^{d(M)}}.$$

Note that we have to allow Laurent polynomials and power series because our modules are $\mathbb{Z}$-graded. It is well-known that $d(M)$ also equals the Krull dimension $K(M)$ of $M$ in the sense of [17, §6.2] and the Gelfand-Kirillov dimension $\text{GK}(M)$ of $M$ [17, §8].

5.3. **Properties of graded Brauer characters.** Recall the definition of graded Brauer characters given in (4.3). As the finite group $H$ is in particular compact $p$-adic analytic, we may speak of the category $\text{grmod}(kH)$. The following result is a straightforward application of Lemma 3.3 and shows that graded Brauer characters behave well with respect to basic algebraic constructions.

**Lemma.** Let $A, B, C \in \text{grmod}(kH)$.

(i) If $0 \to A \to B \to C \to 0$ is an exact sequence in $\text{grmod}(kH)$ then

$$\zeta_B = \zeta_A + \zeta_C.$$

(ii) Define $A \otimes_k B \in \text{grmod}(kH)$ by letting $H$ act diagonally and giving $A \otimes_k B$ the tensor product gradation

$$(A \otimes_k B)_n = \bigoplus_{i+j=n} A_i \otimes_k B_j.$$

Then $\zeta_{A \otimes_k B} = \zeta_A \cdot \zeta_B$.

(iii) For each $m \in \mathbb{Z}$ define the shifted module $A[m] \in \text{grmod}(kH)$ by setting $A[m]_n = A_{m+n}$ for all $n \in \mathbb{Z}$. Then

$$t^m \zeta_{A[m]} = \zeta_A.$$

(iv) If $k'$ is the finite field extension of $k$ defined in (3.1) then $A' := A \otimes_k k'$ lies in $\text{grmod}(k'H)$ and

$$\zeta_{A'} = \zeta_A.$$

5.4. **Rationality of graded Brauer characters.** By picking a homogeneous generating set we see that any $M \in \mathcal{M}_\text{gr}(R)$ actually lies in $\text{grmod}(kH)$ and as such has a graded Brauer character $\zeta_M$. It is easy to see that $\zeta_M(1)$ is just the Poincaré series $P_M$ and is as such a rational function (5.2). The following result shows that $\zeta_M(h)$ is also a rational function for any $h \in H_{\text{reg}}$. Let $m$ denote the $p'$-part of $\lvert H \rvert$. 


Theorem. For any $M \in \mathcal{M}_{\text{gr}}(R)$ and any $h \in H_{\text{reg}}$ there exists a Laurent polynomial $u_h(t) \in F[t, t^{-1}]$ such that

$$\zeta_M(h) = \frac{u_h(t)}{(1 - t^m)^{d(M)}}.$$ 

Proof. Without loss of generality we may assume that $H = \langle h \rangle$. Moreover, since extension of scalars doesn’t change the graded Brauer character by Lemma 5.3(iv), we may also assume that $k = k'$. Hence $h$ acts diagonalizably on $V$.

We now prove the result by induction on $d(M) + \dim_k V$. Suppose first of all that $d(M) = 0$. It follows that $M_n = 0$ for sufficiently large $|n|$ and so $\zeta_M(h)$ is a Laurent polynomial as required. Since $d(M) < \dim_k V$ this also deals with the case when $\dim_k V = 0$, so we assume $\dim_k V > 0$ and $d(M) > 0$.

Choose an $h$-eigenvector $v \in V$ with eigenvalue $\lambda$. As $\lambda^m = 1$ we see that $h^{-1}v^m = (v^h)^m = (\lambda v)^m = v^m$, so $z := v^m$ is central in $R$.

Consider the graded submodule $T = \{\alpha \in M : \alpha z^r = 0 \text{ for some } r \geq 0\}$ of $M$. Because $R$ is Noetherian and $M$ is finitely generated, $T$ is finitely generated as an $R$-module and as such is killed by some power of the central element $z$.

Choose an $h$-invariant complement $W$ for $kv$ in $V$ so that $\text{Sym}(V) \cong \text{Sym}(W)[v]$. It is now easy to see that $T$ is a finitely generated over $\text{Sym}(W)$ and in fact $T \in \mathcal{M}_{\text{gr}}(\text{Sym}(W)\#H)$. Note that the dimension of $T$ viewed as a $\text{Sym}(V)$-module is the same as the dimension of $T$ viewed as an $\text{Sym}(W)$-module as both depend only on the Poincaré series of $T$.

Since $d(T) < d(M)$ and $\dim_k W < \dim_k V$, we know by induction that

$$\zeta_T(h) \cdot (1 - t^m)^{d(M)} \in F[t, t^{-1}].$$

By Lemma 5.3(i), $\zeta_M = \zeta_T + \zeta_{M/T}$, so it now suffices to prove the result for the graded module $M/T$. So, by replacing $M$ by $M/T$ we may assume $M$ is $z$-torsion-free.

Now as $z$ is a central element of $R$ of degree $m$, multiplication by $z$ induces a short exact sequence of graded $R$-modules

$$0 \to M \to M[z] \to L[z] \to 0,$$

where $L := M/Mz$. It follows from Lemma 5.3 that

$$t^m \zeta_M(h) = \zeta_M(h) + t^m \zeta_L(h).$$

But $L$ is a finitely generated graded $\text{Sym}(W)\#H$-module and $d(L) < d(M) - 1$ by [7, Proposition 11.3], so by induction

$$\zeta_L(h) \cdot (1 - t^m)^{d(M) - 1} \in F[t, t^{-1}].$$

The result follows. \qed

Inspecting the proof shows that the Theorem is still valid with $m$ replaced by the order of $h$.

6. Koszul resolutions

We continue with the notation established in §5. Let $d = \dim_k V$. 
6.1. The Koszul complex for graded \( R \)-modules. With any finitely generated graded right \( R \)-module \( M \) we associate the Koszul complex

\[
\mathbb{K}(M)_* := 0 \rightarrow M \otimes_k (\Lambda^d V)[-d] \xrightarrow{\phi_d} \cdots \xrightarrow{\phi_1} M \otimes_k (\Lambda^1 V)[-1] \xrightarrow{\phi_1} M \rightarrow 0
\]

whose maps \( \phi_j : M \otimes_k (\Lambda^j V)[-j] \rightarrow M \otimes_k (\Lambda^{j-1} V)[-j+1] \) are given by the usual formula

\[
\phi_j(m \otimes v_1 \wedge \cdots \wedge v_j) = \sum_{i=1}^j (-1)^{i+1} m. v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_j,
\]

for any \( m \in M \) and \( v_1, \ldots, v_j \in V \). The square brackets indicate that we are thinking of \( \Lambda^j V[-j] \) as a graded \( k \)-vector space concentrated in degree \( j \).

**Lemma.** \( \mathbb{K}(M)_* \) is a complex inside the abelian category \( \text{grmod}(kH) \).

**Proof.** It is well-known (and easily verified) that \( \mathbb{K}(M)_* \) is a complex of \( k \)-vector spaces. Letting \( H \) act diagonally on \( M \otimes_k \Lambda^j V \) makes each \( \phi_j \) into a map of right \( kH \)-modules as \( h \cdot v_i^h = v_i \cdot h \) inside the ring \( R \). Because of the shifts, each \( \phi_j \) is also a map of graded modules of degree zero, as required. \( \square \)

6.2. Homology of \( \mathbb{K}(M)_* \). The projection map \( \epsilon : R \twoheadrightarrow R_0 \) with kernel \( \bigoplus_{n=1}^{\infty} R_n \) is an algebra homomorphism from \( R \) to \( kH \) which gives \( kH \) an \( R\text{-}R \)-bimodule structure. The Koszul complex is useful because it allows us to compute \( \text{Tor}^R_j(M, kH) \) for any \( M \in \mathcal{M}_{gr}(R) \).

**Proposition.**

(i) \( \mathbb{K}(R)_* \) is a complex of \( R\text{-}kH \)-bimodules which is exact everywhere except in degree zero.

(ii) \( H_j(\mathbb{K}(R)_*) \cong kH \) as \( R\text{-}kH \)-bimodules.

(iii) For all \( M \in \mathcal{M}_{gr}(R) \) and all \( j \geq 0 \) there is an isomorphism of \( kH \)-modules

\[
H_j(\mathbb{K}(M)_*) \cong \text{Tor}^R_j(M, kH).
\]

**Proof.** (i) Consider first the special case when \( H = 1 \). Thinking of \( V \) as an abelian \( k \)-Lie algebra, \( R = \text{Sym}(V) \) becomes the enveloping algebra of \( V \) and

\[
\mathbb{K}(R)_* = \text{Sym}(V) \otimes_k \Lambda^* V
\]

is just the Chevalley-Eilenberg complex of left \( \text{Sym}(V) \)-modules [27, §7.7]. By [27, Theorem 7.7.2] \( \mathbb{K}(R)_* \) is exact everywhere except in degree zero.

Returning to the general case, there is a natural isomorphism of complexes of \( k \)-vector spaces

\[
R \otimes_{\text{Sym}(V)} (\text{Sym}(V) \otimes_k \Lambda^* V) \xrightarrow{\epsilon} \mathbb{K}(R)_*.
\]

Because \( R \) is free of finite rank as a right \( \text{Sym}(V) \)-module, it follows that \( \mathbb{K}(R)_* \) is also exact everywhere except in degree zero. We use this isomorphism to give \( \mathbb{K}(R)_* \) the structure of a complex of left \( R \)-modules.

On the other hand, \( \mathbb{K}(R)_* \) is a complex of right \( kH \)-modules by Lemma 6.1. It can be checked that the two structures are compatible, so \( \mathbb{K}(R)_* \) is a complex of \( R\text{-}kH \)-bimodules. Explicitly, the bimodule structure is given by the following formula:

\[
s.(r \otimes v_1 \wedge \cdots \wedge v_j).h = srh \otimes v_1^h \wedge \cdots \wedge v_j^h
\]

for all \( s, r \in R, v_1, \ldots, v_j \in V \) and \( h \in H \).
(ii) The map $\epsilon : R \to kH$ which gives $kH$ its $R$-$kH$-bimodule structure is the cokernel of the first map $\phi_1 : R \otimes_k V \to R$ of the complex $\mathcal{K}(R)_*$. Hence

$$H_0(\mathcal{K}(R)_*) = R/\text{Im} \phi_1 \cong kH$$

as $R$-$kH$-bimodules.

(iii) Each term in $\mathcal{K}(R)_*$ is free of finite rank as a left $R$-module, so by parts (i) and (ii) $\mathcal{K}(R)_*$ is a free resolution of the left $R$-module $kH$. We can therefore use it to compute $\text{Tor}^R_j(M,kH)$. Finally, the natural map $M \otimes_R \mathcal{K}(R)_* \to \mathcal{K}(M)_*$ is actually an isomorphism of complexes of right $kH$-modules, so

$$H_j(\mathcal{K}(M)_*) \cong H_j(M \otimes_R \mathcal{K}(R)_*) \cong \text{Tor}^R_j(M,kH)$$

as required. \hfill \Box

6.3. A formula involving graded Brauer characters. Let $M \in \mathcal{M}_{\gr}(R)$. By Lemma 6.1 and Proposition 6.2(iii), $\text{Tor}^R_j(M,kH)$ is an object in $\text{grmod}(kH)$ and as such has a graded Brauer character $\zeta_{\text{Tor}^R_j(M,kH)}$. On the other hand, since $R$ is Noetherian $M$ has a projective resolution consisting of finitely generated $R$-modules. Computing $\text{Tor}^R_j(M,kH)$ using this resolution shows that these $kH$-modules are finite dimensional over $kH$, so $\zeta_{\text{Tor}^R_j(M,kH)}(h)$ is actually a Laurent polynomial in $F[[t,t^{-1}]]$ for all $h \in H_{\text{reg}}$.

**Proposition.** For any $h \in H_{\text{reg}}$ and $M \in \mathcal{M}_{\gr}(R),$

$$\sum_{j=0}^{d} (-1)^j \zeta_{\text{Tor}^R_j(M,kH)}(h) = \zeta_M(h) \cdot \sum_{j=0}^{d} (-t)^j \varphi_{\Lambda^j V}(h).$$

**Proof.** In the notation of (2.2), Lemma 5.3(i) says that $\psi_h : A \mapsto \zeta_A(h)$ is an additive function from the abelian category $\text{grmod}(kH)$ to $F[[t,t^{-1}]]$ thought of as an abelian group. Applying Proposition 6.2(iii) and Lemma 2.2 we obtain

$$\sum_{j=0}^{d} (-1)^j \zeta_{\text{Tor}^R_j(M,kH)}(h) = \psi_h(\text{Tor}^R_*(M,kH)) = \psi_h(\mathcal{K}(M)_*).$$

Now $\mathcal{K}(M)_j = M \otimes_k \Lambda^j V[-j]$ so the result follows from Lemma 5.3(ii) and (iii). \hfill \Box

7. Proof of Theorem 4.5

7.1. Another expression for $\Psi(g)$. Set $V := k \otimes_{\mathbb{F}_p} (N/N^p)$. Because $N$ is uniform, $N/N^p$ is an $\mathbb{F}_p$-$p$-vector space of dimension $d$ and the right conjugation action of $G$ on $N$ induces a right action of $G$ on $N/N^p$ by linear automorphisms. In this way $V$ becomes a right $kG$-module with dim$_k V = d$.

**Lemma.** For any $g \in G_{\text{reg}}$, $\Psi(g) = \sum_{j=0}^{d} (-t)^j \varphi_{\Lambda^j V}(g)$.

**Proof.** Let $\beta = \text{Ad}(g^{-1}) \in \text{GL}(\mathcal{L}(G))$ and let $j \geq 0$. Recall from (4.4) that $\Psi(g) = \det(1 - t\beta)$. Now, $\Lambda^j N$ is a $\Lambda^j \beta$-stable lattice inside $\Lambda^j \mathcal{L}(G)$ whose reduction modulo $p$ is isomorphic to $\Lambda^j (N/N^p)$. Moreover, the endomorphism of $\Lambda^j V \cong \Lambda^j (N/N^p) \otimes_{\mathbb{F}_p} k$ induced by $\Lambda^j \beta$ is equal to the right action of $g$ on $\Lambda^j V$. The result now follows from the well-known formula

$$\det(1 - t\beta) = \sum_{j=0}^{d} (-t)^j \text{Tr}(\Lambda^j \beta).$$
See, for example, [23, p.487] or [14, p. 77, (6.2)].

**Corollary.** The restriction of $\Psi$ to $G_{\text{reg}}$ is locally constant.

**Proof.** Note that $V$ is a $k\mathcal{G}$-module because $[N, N] \subseteq N^p$ as $N$ is uniform, so the Brauer characters $\varphi_{\Lambda^j V}$ are constant on the cosets of $N$. Now apply the Lemma. □

7.2. The associated graded ring. We now make the connection with the theory developed on the preceding two sections. Let $H := \mathcal{G} = G/N$; then $V$ is a $kH$-module and we may form the skew group ring $\text{Sym}(V)\#H$. Recall from (4.3) that $w_N$ denotes the augmentation ideal $(N - 1)kG$.

**Lemma.** The associated graded ring of $kG$ with respect to the $w_N$-adic filtration is isomorphic to $R = \text{Sym}(V)\#H$.

**Proof.** When $N = G$ this is follows from [13, Theorem 7.24]; see also [1, Lemma 3.11]. Letting $m$ denote the augmentation ideal of $kN$ we see that $\text{gr}_{w_N} kG \cong kG \otimes_{kN} kN \cong kH \otimes_{k} \text{Sym}(V)$ as a right $\text{Sym}(V)$-module, because $kN$ acts trivially on its graded ring $\text{Sym}(V)$. Moreover, the zero-th graded part of $\text{gr}_{w_N} kG$ is isomorphic to $kH$ as a $k$-algebra, so $H$ embeds into the group of units of $\text{gr}_{w_N} kG$. It is now easy to verify that the multiplication works as needed. □

We will identify $R$ with $\text{gr}_{w_N} kG$ in what follows.

7.3. A spectral sequence. The last step in the proof involves relating Tor groups over $kG$ with Tor groups over the associated graded ring $R$. There is a standard spectral sequence originally due to Serre which does the job.

**Proposition.** For any finitely generated $kG$-module $M$ there exists a homological spectral sequence in $\mathcal{M}(kH)$

$$E^1_{ij} = \text{Tor}^{kG}_{i+j}(\text{gr} M, kH)_{\text{degree } -i} \Rightarrow \text{Tor}^{kG}_{i+j}(M, kH).$$

**Proof.** As in (4.3), we only consider the deduced filtration on $M$, given by $M^n = Mw_N^n$ for $n \geq 0$. As $M$ is finitely generated over $kG$, the associated graded module $\text{gr} M$ is finitely generated over the Noetherian ring $\text{gr} kG \cong R$. By the proof of [3, Proposition 3.4], the $w_N$-adic filtration on $kG$ is complete.

We claim that $M$ is complete with respect to the deduced filtration. Because $kG$ is Noetherian we can find an exact sequence $(kG)^n \xrightarrow{\alpha} (kG)^b \xrightarrow{\beta} M \rightarrow 0$ in $\mathcal{M}(kG)$. Giving all the modules involved deduced filtrations, $(kG)^n$ and $(kG)^b$ are compact and the maps $\alpha, \beta$ are continuous. Hence $\text{Im} \alpha$ and $M$ are compact, so $\text{Im} \alpha$ is closed in $(kG)^b$ and $M$ is complete as claimed.

We can now apply [25, Proposition 8.1] to the modules $A = M$ and $B = kH$ over the complete filtered $k$-algebra $kG$, where we equip $B$ with the trivial filtration $B^0 = B$ and $B^1 = 0$. This gives us the required spectral sequence of $k$-vector spaces. Examining the construction shows that it is actually a spectral sequence in $\mathcal{M}(kH)$. □
7.4. Proof of Theorem 4.5. The spectral sequence $E$ of Proposition 7.3 gives us suitable filtrations on the $kH$-modules $\text{Tor}^k(M, kH)$. In the notation of (2.3) we can rewrite the information we gain from the spectral sequence as follows:

\[ \text{Tot}(E^1)_n = \text{Tor}^R_n(\text{gr} M, kH) \quad \text{and} \quad \text{Tot}(E^\infty)_n = \text{gr} \text{Tor}^k(M, kH) \]

for each $n \in \mathbb{Z}$. We have already observed in (6.3) that $\text{Tor}^R_n(\text{gr} M, kH)$ is a finite dimensional $kH$-module, which is moreover zero whenever $n > d$ by Proposition 6.2(iii). Thus $E$ is totally bounded. Now, Lemma 3.3(ii) shows that $A \mapsto \varphi_A(g)$ is an additive $F$-valued function on the objects of $\mathcal{M}(kH)$. By Proposition 2.3,

\[ \rho_N[M](g) = \sum_{j=0}^{d} (-1)^j \varphi_{\text{Tor}^j}(M, kH)(g) = \sum_{j=0}^{d} (-1)^j \varphi_{\text{Tor}^j(\text{gr} M, kH)}(g). \]

We may now apply Proposition 6.3 and Lemma 7.1 to obtain

\[ \sum_{j=0}^{d} (-1)^j \varphi_{\text{Tor}^j(\text{gr} M, kH)}(g) = \sum_{j=0}^{d} (-1)^j \zeta_{\text{Tor}^j(\text{gr} M, kH)}(g)|_{t=1} = (\zeta_{\text{gr} M}(g) \cdot \Psi(g))|_{t=1}, \]

as required. □

8. Euler characteristics

8.1. Twisted $\mu$-invariants. Because we only deal with Iwasawa modules which are killed by $p$ in this paper, it is easy to see that the definition of the Euler characteristic [5, §1.5] of a finitely generated $kG$-module $M$ of finite projective dimension can be given as follows:

\[ \chi(G, M) := \prod_{j \geq 0} |\text{Tor}^j_k(M, k)|(-1)^j. \]

Let $\{V_1, \ldots, V_s\}$ be a complete list of representatives for the isomorphism classes of simple $kG$-modules. The $i$-th twisted $\mu$-invariant of $M$ [5, §1.5] for $i = 1, \ldots, s$ is defined by the formula

\[ \mu_i(M) = \frac{\log_q \chi(G, M \otimes_k V_i^*)}{\dim_k \text{End}_{kG}(V_i)}, \]

where $V_i^*$ is the dual module to $V_i$. We assume that $V_1$ is the trivial $kG$-module $k$, so that

\[ \mu_1(M) = \log_q \chi(G, M). \]

We proved in [5] that these twisted $\mu$-invariants completely determine the characteristic element of $M$ viewed as an $OG$-module [5, Theorem 1.5]. This adds to the motivation of the problem of computing the Euler characteristic $\chi(G, M)$.

8.2. The base change map. Before we can proceed, we need to record some information about the base change map $\theta_N : \mathcal{G}_0(kG) \to \mathcal{G}_0(k\mathcal{O})$.

We say that a map $f$ between two abelian groups is an $\mathbb{Q}$-isomorphism if it becomes an isomorphism after tensoring with $\mathbb{Q}$. Equivalently, $f$ has torsion kernel and cokernel.
Proposition. There is a commutative diagram of Grothendieck groups

\[
\begin{array}{ccc}
K_0(kG) & \overset{\pi_N}{\longrightarrow} & K_0(kG) \\
\downarrow c & & \downarrow c_N \\
\mathcal{G}_0(kG) & \overset{\theta_N}{\longrightarrow} & \mathcal{G}_0(kG)
\end{array}
\]

The map \(\pi_N\) is an isomorphism and the other maps are \(\mathbb{Q}\)-isomorphisms.

Proof. The vertical maps \(c\) and \(c_N\) are called Cartan maps and are defined by inclusions between admissible subcategories. The base change map \(\pi_N\) is defined by \(\pi_N[P] = [P \otimes_{kG} k\mathcal{G}]\) for all \(P \in \mathcal{P}(kG)\), and we have already discussed \(\theta_N\) in (4.2). This well-known diagram appears in \([8, p. 454]\) and expresses the fact that the Cartan maps form a natural transformation from \(K\)-theory to \(G\)-theory.

Now, \(kG\) is a crossed product of \(kN\) with the finite group \(G\):

\[kG \cong kN \ast \mathcal{G},\]

see, for example [4, §2.3] for more details. Because \(kN\) is a Noetherian \(k\)-algebra of finite global dimension, the map \(c\) is a \(\mathbb{Q}\)-isomorphism by a general result on the \(K\)-theory of crossed products [6]. Considering the case when \(G\) is finite shows that \(c_N\) is a \(\mathbb{Q}\)-isomorphism as well – this also follows from a well-known result of Brauer: see, for example, [22, Corollary 1 to Theorem 35].

Finally \(\pi_N\) is an isomorphism because \(kG\) is a complete semilocal ring – see [5, Lemma 2.6 and Proposition 3.3(a)]. It follows that \(\theta_N\) must also be a \(\mathbb{Q}\)-isomorphism, as required. \(\square\)

Recalling Theorem 3.4 and Lemma 3.7, we obtain

Corollary. The map \(\rho_N\) featuring in Theorem 4.5 is a \(\mathbb{Q}\)-isomorphism.

If the group \(G\) has no elements of order \(p\), then \(kG\) has finite global dimension and the Cartan map \(c\) is actually an isomorphism by Quillen’s Resolution Theorem [17, Theorem 12.4.8]. In this case, therefore, we do not have to rely on [6].

8.3. Computing Euler characteristics using \(\theta_N\). Let \(P \in \mathcal{P}(k\mathcal{G})\) and let \(V \in \mathcal{M}(k\mathcal{G})\). The rule

\[(P, V) \mapsto \dim_k \text{Hom}_{k\mathcal{G}}(P, V)\]

defines an additive function from \(\mathcal{P}(k\mathcal{G}) \times \mathcal{M}(k\mathcal{G})\) to \(\mathbb{Z}\) and hence a pairing

\[\langle - , - \rangle_N : K_0(k\mathcal{G}) \times \mathcal{G}_0(k\mathcal{G}) \to \mathbb{Z}.
\]

This pairing appears in [22, p. 121]. Now, by extending scalars, we can define a bilinear form

\[\langle - , - \rangle_N : \mathbb{Q}K_0(k\mathcal{G}) \times \mathbb{Q}\mathcal{G}_0(k\mathcal{G}) \to \mathbb{Q}\]

which is in fact non-degenerate. We saw in Proposition 8.2 that the Cartan map \(c_N : \mathbb{Q}K_0(k\mathcal{G}) \to \mathbb{Q}\mathcal{G}_0(k\mathcal{G})\) is an isomorphism. This allows us to define an non-degenerate bilinear form

\[\langle - , - \rangle_N : \mathbb{Q}\mathcal{G}_0(k\mathcal{G}) \times \mathbb{Q}\mathcal{G}_0(k\mathcal{G}) \to \mathbb{Q}\]

by setting \((x, y)_N = \langle c_N^{-1}(x), y \rangle_N\) for \(x, y \in \mathbb{Q}\mathcal{G}_0(k\mathcal{G})\).
Proposition. For any finitely generated $kG$-module $M$ of finite projective dimension, the Euler characteristic of $M$ can be computed as follows:

$$\log_q \chi(G, M) = (\theta_N[M], [k])_N.$$ 

Proof. Suppose first that $M$ is projective. The usual adjunction between $\otimes$ and $\text{Hom}$ gives isomorphisms

$$\text{Hom}_k(M \otimes kG, k) \cong \text{Hom}_{kG}(M, k) \cong \text{Hom}_k(M \otimes kG, kG).$$

As these Hom spaces are finite dimensional over $k$, we obtain

$$\dim_k(M \otimes kG) = (\pi_N[M], [k])_N.$$

Now because $M$ is projective, $\text{Tor}_i^{kG}(M, k) = 0 = \text{Tor}_i^{kG}(M, kG)$ for $i > 0$, and $\theta_N[M] = c_N(\pi_N[M])$ by Proposition 8.2. Hence

$$\log_q \chi(G, M) = \dim_k(M \otimes kG) = (\pi_N[M], [k])_N$$

as required. Returning to the general case, if $0 \to P_n \to \cdots \to P_0 \to M \to 0$ is a resolution of $M$ in $\mathcal{P}(kG)$ then Lemma 2.2 gives

$$\theta_N[M] = \sum_{i=0}^n (-1)^i \theta_N[P_i] \quad \text{and} \quad \log_q \chi(G, M) = \sum_{i=0}^n (-1)^i \log_q \chi(G, P_i).$$

The result now follows from the first part. □

8.4. Euler characteristics for modules of infinite projective dimension.

The definition of $\chi(G, M)$ given in (8.1) only makes sense when the module $M$ has finite projective dimension. However, the expression $(\theta_N[M], [k])_N$ makes sense for arbitrary $M \in \mathcal{M}(kG)$. We include the subscripts, because $a\text{ priori}$ this depends on the choice of the open normal uniform subgroup $N$.

Lemma. Let $M$ be a finitely generated $kG$-module. Then

(i) $\psi_N : M \mapsto (\theta_N[M], [k])_N$ is an additive function on the objects of $\mathcal{M}(kG)$,

(ii) $\psi_N(M)$ does not depend on the choice of $N$.

Proof. It suffices to prove part (ii). Now if $U \in U_{G, p}$ is another uniform subgroup of $G$, then $\psi_U(M) = \log_q \chi(G, M) = \psi_N(M)$ whenever $M$ is projective, by Proposition 8.3. Rephrasing this in the language of Grothendieck groups,

$$\mathbb{Q}K_0(kG) \xrightarrow{\psi_U \psi_N} \mathbb{Q}G_0(kG) \xrightarrow{\psi_U} \mathbb{Q}$$

is a complex of $\mathbb{Q}$-vector spaces. By Proposition 8.2, the Cartan map $c$ is an isomorphism, so $\psi_U(M) = \psi_N(M)$ for any finitely generated $kG$-module $M$, as required. □

In view of this result, we propose to extend the definition given in (8.1) as follows.

Definition. The Euler characteristic of a finitely generated $kG$-module $M$ is defined to be

$$\chi(G, M) := q^{(\theta_N[M], [k])_N} \in q^\mathbb{Q}$$

for any choice of open normal uniform subgroup $N$ of $G$. 

8.5. Trivial Euler characteristics. First, a preliminary

Lemma. For any \( g \in G_{\text{reg}} \), the multiplicity of 1 as a root of the polynomial \( \Psi(g) \) equals \( \dim C_G(g) \).

Proof. As \( \Psi(g) \cdot \det \text{Ad}(g) \) is the characteristic polynomial of \( \text{Ad}(g) \), the first number equals the dimension of the space \( C := \{ x \in \mathcal{L}(G) : \text{Ad}(g)(x) = x \} \). The definition of \( \text{Ad}(g) \) shows that \( C \cap N \) is just the centralizer \( C_N(g) \) of \( g \) in \( N \). Because \( C_N(g) = C_G(g) \cap N \) is open in \( C_G(g) \), the dimension of \( C_G(g) \) as a compact \( p \)-adic analytic group equals the dimension of \( C \) as a \( \mathbb{Q}_p \)-vector space. The result follows.

Recall [4, §5.4] that as \( kG \) is an Auslander-Gorenstein ring, every finitely generated \( kG \)-module \( M \) has a canonical dimension which we will denote by \( d(M) \). By [4, §5.4(3)] this is a non-negative integer which equals the dimension \( d(\text{gr} \ M) \) of the associated graded module \( \text{gr} \ M \), defined in (5.2).

Proposition. Let \( M \) be a finitely generated \( kG \)-module and let \( g \in G_{\text{reg}} \) be such that \( d(M) < \dim C_G(g) \). Then \( \rho_N[M](g) = 0 \).

Proof. By Theorem 4.5 and Theorem 5.4 there exists a Laurent polynomial \( u_p(t) \in F[t, t^{-1}] \) such that \( \rho_N[M](g) = 1 \) of the rational function

\[
\frac{u_p(t) \cdot \Psi(g)}{(1 - t^m)^{d(\text{gr} \ M)}}.
\]

Because we are assuming that \( \dim C_G(g) > d(M) = d(\text{gr} \ M) \), this rational function has a zero at \( t = 1 \) in view of the Lemma.

We can now give our first application of Theorem 4.5.

Proof of Theorem A. By the Proposition, \( \rho_N[M] = (\varphi \circ \lambda_N)(\theta_N[M]) = 0 \). As \( \varphi \) and \( \lambda_N \) are isomorphisms by Theorem 3.4 and Lemma 3.7, \( \theta_N[M] = 0 \). The result now follows from the new definition of \( \chi(G, M) \) given in (8.4).

\[ \square \]

9. K-theory

9.1. The dimension filtration. Recall from (3.3) that \( \mathcal{F}_0 \) denotes the category of all \( kG \)-modules which are finite dimensional over \( k \).

Now, a finitely generated \( kG \)-module \( M \) is finite dimensional over \( k \) if and only if \( d(M) = d(\text{gr} \ M) = 0 \), because both conditions are equivalent to the Poincaré series (5.2) of \( \text{gr} \ M \) being a polynomial in \( t \). We can therefore unambiguously define \( \mathcal{F}_i = \mathcal{F}_i(G) \) to be full subcategory of \( \mathcal{M}(kG) \) consisting of all modules \( M \) with \( d(M) \leq i \), for each \( i = 0, \ldots, d \). Thus we have an ascending chain of subcategories

\[
\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{d-1} \subseteq \mathcal{F}_d = \mathcal{M}(kG).
\]

Using [4, §5.3] we see that each \( \mathcal{F}_i \) is an admissible (in fact, Serre) subcategory of \( \mathcal{M}(kG) \), so we can form the Grothendieck groups \( K_0(\mathcal{F}_i) \). The inclusions \( \mathcal{F}_i \subseteq \mathcal{F}_d \) induce maps

\[
\alpha_i : K_0(\mathcal{F}_i) \to K_0(\mathcal{F}_d) \quad \text{and} \quad \alpha_i : F K_0(\mathcal{F}_i) \to F K_0(\mathcal{F}_d)
\]

which we would like to understand. Because \( \rho_N \) is a \( \mathbb{Q} \)-isomorphism by Corollary 8.2, we focus on the image of \( \rho_N \circ \alpha_i \).
9.2. Module structures. Note that $K_0(F_0)$ is a commutative ring with multiplication induced by the tensor product. By adapting the argument used in [5, Proposition 7.3] and using [4, §5A(5)], we see that the twist $M \otimes_k V$ of any $M \in F_d$ and any $V \in F_0$ satisfies $d(M \otimes_k V) = d(M)$. In this way $K_0(F_i)$ becomes a $K_0(F_0)$-module and it is clear that the maps $\alpha_i : K_0(F_i) \rightarrow K_0(F_d)$ respect this module structure.

Lemma. The map $\lambda_N \circ \theta_N : K_0(F_d) \rightarrow K_0(F_0)$ is a map of $K_0(F_0)$-modules.

Proof. Let $V \in M(kG)$. For any $M \in F_d$ we can define a function

$$\beta_M : (M \otimes_k V) \otimes_{kG} kG \rightarrow (M \otimes_{kG} kG) \otimes_k V$$

by the formula $\beta_M((m \otimes v) \otimes h) = (m \otimes h) \otimes vh$ for $m \in M$, $v \in V$ and $h \in G$. A straightforward check shows that $\beta_M$ is a homomorphism of right $kG$-modules with inverse $\gamma_M$, defined by the formula $\gamma_M((m \otimes h) \otimes v) = (m \otimes vh^{-1}) \otimes h$. Hence the functors $M \mapsto (M \otimes V) \otimes_{kG} kG$ and $M \mapsto (M \otimes_{kG} kG) \otimes_k V$ are isomorphic, so

$$\text{Tor}_j^{kG}(M \otimes_k V, kG) \cong \text{Tor}_j^{kG}(M, kG) \otimes_k V$$

for all $j \geq 0$. It follows that $\theta_N[M \otimes_k V] = \theta_N[M];[V]$ for all $M \in F_d$ and $V \in M(kG)$. Because $\lambda_N : G_0(kG) \rightarrow K_0(F_0)$ is an isomorphism by Lemma 3.7,

$$\lambda_N(\theta_N[M \otimes_k V]) = \lambda_N(\theta_N[M]);[V]$$

for all $M \in F_d$ and $V \in F_0$, as required. □

Hence the image of $\lambda_N \circ \theta_N \circ \alpha_i$ is always an ideal of $K_0(F_0)$. On the other hand, $C(G_{\text{reg}}; F)^{G_k \times \mathfrak{g}_k}$ is a commutative $F$-algebra via pointwise multiplication of functions, and Lemma 3.3(iii) shows that the map

$$\varphi : FK_0(F_0) \rightarrow C(G_{\text{reg}}; F)^{G_k \times \mathfrak{g}_k}$$

appearing in Theorem 3.4 is an $F$-algebra isomorphism, so $\text{Im}(\rho_N \circ \alpha_i)$ is always an ideal of $C(G_{\text{reg}}; F)^{G_k \times \mathfrak{g}_k}$.

It is easy to see that the ideals of this algebra are in bijection with the subsets of the orbit space $(G \times G_k)\backslash G_{\text{reg}}$, which subset does $\text{Im}(\rho_N \circ \alpha_i)$ correspond to?

9.3. An upper bound for $\text{rk} \alpha_i$. Define a subset $S_i$ of $G_{\text{reg}}$ by the formula

$$S_i := \{g \in G_{\text{reg}} : \dim C_G(g) \leq i\}.$$

We record some basic facts about these subsets of $G_{\text{reg}}$.

Lemma. (i) $S_i$ is a union of conjugacy classes in $G$.
(ii) $S_i$ is stable under the action of $G_k$ on $G_{\text{reg}}$.
(iii) $S_i$ is a clopen subset of $G_{\text{reg}}$ and hence a closed subset of $G$.

Proof. (i) This is clear.
(ii) If $g$ is a power of $h$ then $C_G(h) \subseteq C_G(g)$, therefore if $g$ and $h$ lie in the same $G_k$-orbit inside $G_{\text{reg}}$ then their centralizers are equal.
(iii) This follows from Corollary 3.6. □

We can now give our second application of Theorem 4.5.

Proof of Theorem B. By Proposition 8.5, $\rho_N[M]$ is zero on $G_{\text{reg}} - S_i$ for any $M \in F_i$. Hence $\text{Im}(\rho_N \circ \alpha_i) \subseteq C(S_i; F)^{G_k \times \mathfrak{g}_k}$ and the result follows from Corollary 8.2. □
10. Some special cases

10.1. The rank of $\alpha_d$. We begin by recording the rank of $\alpha_d$, or equivalently the rank of $K_0(F_d) = G_0(kG)$.

**Proposition.** The rank of $\alpha_d$ equals the number of $G \times G_k$-orbits on $G_{\text{reg}}$:

$$\text{rk} \alpha_d = |G \times G_k\backslash G_{\text{reg}}|.$$ 

**Proof.** This follows from Corollary 8.2. □

10.2. A localisation sequence. Consider the localisation sequence of $K$-theory [20, Theorem 5.5] for the Serre subcategory $F_i$ of the abelian category $F_d$:

$$K_0(F_i) \xrightarrow{\alpha_i} K_0(F_d) \xrightarrow{} K_0(F_d/F_i) \xrightarrow{} 0.$$ 

Because we already know the rank of $K_0(F_d)$, this sequence shows the problem of computing the rank of $\alpha_i$ is equivalent to the problem of computing the rank of the Grothendieck group of the quotient category $F_d/F_i$. At present, the only non-trivial case when we understand $F_d/F_i$ is the case $i = d - 1$ — see (10.6).

First, we require some results from noncommutative algebra.

10.3. Artinian rings and minimal primes. Recall that if $R$ is a (not necessarily commutative) ring, then an ideal $I$ of $R$ is said to be prime if whenever $A, B$ are ideals of $R$ strictly containing $I$, their product $AB$ also strictly contains $I$. A minimal prime of $R$ is a prime ideal which is minimal with respect to inclusion — equivalently, it has height zero. The following result is well-known.

**Lemma.** Let $R$ be an Artinian ring. Then

(i) $G_0(R)$ is a free abelian group on the isomorphism classes of simple $R$-modules.

(ii) There is a natural bijection between the isomorphism classes of simple $R$-modules and the minimal primes of $R$, given by

$$[M] \mapsto \text{Ann}_R(M).$$ 

10.4. The finite radical. Recall [3, 1.3] the important characteristic subgroup $\Delta^+$ of $G$, defined by

$$\Delta^+ = \{x \in G : |G : C_G(x)| < \infty \text{ and } o(x) < \infty\}.$$ 

This group is sometimes called the finite radical of $G$ and consists of all elements of finite order in $G$ whose conjugacy class is finite. It is also the largest finite normal subgroup of $G$. In our notation, $\Delta^+_{\text{reg}} = \Delta^+ \cap G_{\text{reg}}$ is just the complement of $S_{d-1}$ in $G_{\text{reg}}$:

$$\Delta^+_{\text{reg}} = G_{\text{reg}} \setminus S_{d-1}$$ 

and as such is a union of $G \times G_k$-orbits in $G_{\text{reg}}$.

10.5. The classical ring of quotients $Q(kG)$. By [5, Proposition 7.2] $kG$ has an Artinian classical ring of quotients $Q(kG)$. We have already computed the rank of $G_0(Q(kG))$ under the assumption that the order of the finite group $\Delta^+$ is coprime to $p$ [5, Theorem 12.7(b)]. We can now present a generalization of this result, valid without any restrictions on $G$.

**Theorem.** The rank of $G_0(Q(kG))$ equals the number of $G \times G_k$-orbits on $\Delta^+_{\text{reg}}$:

$$\text{rk} G_0(Q(kG)) = |(G \times G_k)\backslash \Delta^+_{\text{reg}}|.$$
Proof. We will show that both numbers in question are equal to the number of
minimal primes of \( kG \), \( r \) say. Let \( J \) denote the prime radical of \( kG \), defined
as the intersection of all prime ideals of \( kG \). By passing to \( kG/J \) and applying
[17, Proposition 3.2.2(i)], we see that minimal primes of \( kG \) are in bijection with
the minimal primes of \( Q(kG) \). As \( Q(kG) \) is Artinian, Lemma 10.3 implies that
\( \text{rk} \mathcal{G}_0(Q(kG)) = r \).

Next, the group \( G \) acts on \( \Delta^+ \) by conjugation and therefore permutes the the
minimal primes of \( k\Delta^+ \). It was shown in [2, Theorem 5.7] that there is a natural
bijection between the minimal primes of \( kG \) and \( G \)-orbits on the minimal primes
of \( k\Delta^+ \). The group \( G \) also permutes the simple \( k\Delta^+ \)-modules and respects the
correspondence between these and the minimal primes of \( k\Delta^+ \) given in Lemma
10.3(ii). Hence \( r \) is also the number of \( G \)-orbits on the simple \( k\Delta^+ \)-modules. Finally,
the \( G \)-equivariant version of the Berman–Witt Theorem [5, Corollary 12.6] shows
that the latter is just \( \left| (G \times \mathcal{G}_k)\backslash \Delta^+_{\text{reg}} \right| \) and the result follows.

\( \square \)

10.6. The rank of \( \alpha_{d-1} \). Recall that a finitely generated \( kG \)-module is torsion if and
only if \( M \otimes_{kG} Q(kG) = 0 \). By [11, Lemma 1.4], \( M \) is torsion if and only if
\( d(M) < d(kG) = d \), so \( \mathcal{F}_{d-1} \) is just the category of all finitely generated torsion
\( kG \)-modules, as mentioned in the introduction.

Lemma. The quotient category \( \mathcal{F}_d/\mathcal{F}_{d-1} \) is equivalent to \( \mathcal{M}(Q(kG)) \).

Proof. This follows from [24, Propositions XI.3.4(a) and XI.6.4], with appropriate
modifications to handle the finitely generated case.

We can now use the localisation sequence of (10.2) to show that the upper bound
for \( \text{rk} \alpha_i \) given in Theorem B is attained in the case when \( i = d - 1 \).

Proposition. The rank of \( \alpha_{d-1} \) equals the number of \( G \times \mathcal{G}_k \)-orbits on \( S_{d-1} \):
\[
\text{rk} \alpha_{d-1} = \left| (G \times \mathcal{G}_k)\backslash S_{d-1} \right|.
\]

Proof. In view of the Lemma, the localisation sequence becomes
\[
K_0(\mathcal{F}_{d-1}) \xrightarrow{\alpha_{d-1}} K_0(\mathcal{F}_d) \longrightarrow \mathcal{G}_0(Q(kG)) \longrightarrow 0.
\]

Hence \( \text{rk}(\alpha_{d-1}) = \text{rk} K_0(\mathcal{F}_d) - \text{rk} \mathcal{G}_0(Q(kG)) \). Now apply Proposition 10.1 and
Theorem 10.5, bearing in mind that \( S_{d-1} = \mathcal{G}_{\text{reg}} - \Delta_{\text{reg}}^+ \).

\( \square \)

10.7. The rank of \( \alpha_0 \). We will see in (12.3) that the rank of \( \alpha_i \) does not always
attain the upper bound of Theorem B. Here is another special case when \( \text{rk} \alpha_i \) is
well-behaved.

Proposition. The rank of \( \alpha_0 \) equals the number of \( G \times \mathcal{G}_k \)-orbits on \( S_0 \):
\[
\text{rk} \alpha_0 = \left| (G \times \mathcal{G}_k)\backslash S_0 \right|.
\]

Proof. Let \( M \in \mathcal{F}_0 \). Because \( M \) is finite dimensional over \( k \), the graded Brauer
character \( \zeta_{\text{gr}} \cdot M \) is a polynomial, so
\[
\varphi_M = \zeta_{\text{gr}} \cdot M|_{t=1},
\]
thought of as \( F \)-valued functions on \( \mathcal{G}_{\text{reg}} \). Applying the explicit formula for \( \rho_N \)
given in Theorem 4.5 shows that
\[
\rho_N[M](g) = \varphi_M(g) \cdot \det(1 - \text{Ad}(g^{-1}))
\]
for any $g \in G_{\text{reg}}$. We therefore have a commutative diagram

$$
\begin{array}{ccc}
FK_0(F_0) & \xrightarrow{\varphi} & C(G_{\text{reg}}; F)^{G \times \mathfrak{G}_k} \\
\downarrow{\alpha_0} & & \downarrow{\eta} \\
FK_0(F_d) & \xrightarrow{\rho_N} & C(G_{\text{reg}}; F)^{G \times \mathfrak{G}_k}
\end{array}
$$

where $\eta$ is multiplication by the locally constant function

$$
\Psi|_{t=1} : g \mapsto \det(1 - \text{Ad}(g^{-1})).
$$

Using Lemma 8.5 we see that $\Psi|_{t=1}(g) \neq 0$ if and only if $\dim C_G(g) = 0$. It follows that the image of $\eta$ is precisely $C(S_0; F)^{G \times \mathfrak{G}_k}$. As the maps $\varphi$ and $\rho_N$ are isomorphisms by Theorem 3.4 and Corollary 8.2,

$$
\text{rk} \alpha_0 = \text{rk} \eta = |\{G \times \mathfrak{G}_k\} \setminus S_0|
$$

as required.

We now start preparing for the proof of Theorem 11.3 which says that the upper bound of Theorem B is always attained if the group $G$ is virtually abelian.

### 11. Induction of modules

#### 11.1. Dimensions

In what follows we fix a closed subgroup $H$ of $G$ of dimension $e$. Recall [9, Lemma 4.5] that $kG$ is a flat $kH$-module. Therefore the induction functor

$$
\text{Ind}_H^G : \mathcal{M}(kH) \to \mathcal{M}(kG)
$$

which sends $M \in \mathcal{M}(kH)$ to $M \otimes_k kG \in \mathcal{M}(kG)$ is exact and induces a map

$$
\text{Ind}_H^G : \mathcal{G}_0(kH) \to \mathcal{G}_0(kG).
$$

We can obtain precise information about the dimension of an induced module.

**Lemma.** Let $M \in \mathcal{M}(kH)$. Then

$$
\text{d}(M \otimes_k kH) = \text{d}(M) + d - e.
$$

**Proof.** Recall that the grade $j(X)$ of a finitely generated $R$-module $X$ over an Auslander-Gorenstein ring $R$ is defined by the formula

$$
j(X) = \min\{j : \text{Ext}_R^j(X, R) \neq 0\}.
$$

The canonical dimension of $X$ is the non-negative integer $\text{id}(R) - j(X)$ where $\text{id}(R)$ is the injective dimension of $R$.

Now, choosing a free resolution of $M$ and using the fact that $kG$ is a flat $kH$-module, we see that

$$
kG \otimes_k kH \text{ Ext}_{kH}^j(M, kH) \cong \text{ Ext}_{kG}^j(M \otimes_k kH, kG)
$$

as left $kG$-modules, for any $j \geq 0$. In fact, $kG$ is a faithfully flat (left) $kH$-module by [2, Lemma 5.1], which implies that $kG \otimes_k A = 0$ if and only if $A = 0$ for any finitely generated left $kH$-module $A$.

Hence $j(M) = j(M \otimes_k kG)$ and the result follows because $\text{id}(kG) = \dim G = d$ and $\text{id}(kH) = \dim H = e$. 

\[□\]
11.2. Proposition. Suppose that the group \( H \cap N \) is uniform. Then there exists a map \( \iota_N : C(H_{\text{reg}}; F)^{H \times G_k} \to C(G_{\text{reg}}; F)^{G \times G_k} \) such that the following diagram commutes:

\[
\begin{array}{c}
FK_0(\mathcal{F}_{d-e}(G)) \xrightarrow{\alpha_{d-e}} FK_0(\mathcal{F}_d(G)) \xrightarrow{\rho_N} C(G_{\text{reg}}; F)^{G \times G_k} \\
\Ind_{H}^{G} \downarrow \quad \Ind_{H}^{G} \downarrow \quad \iota_N \\
FK_0(\mathcal{F}_{e}(H)) \xrightarrow{\rho_{H \cap N}} FK_0(\mathcal{F}_{e}(H)) \xrightarrow{\iota_N} C(H_{\text{reg}}; F)^{H \times G_k}.
\end{array}
\]

Proof. We assume that \( H \cap N \) is uniform only to make sure that the map \( \rho_{H \cap N} \) makes sense. We construct this diagram in several steps — note that the left-hand square makes sense by Lemma 11.1.

Let \( \overline{H} := HN/N \cong H/(H \cap N) \), let \( \overline{G} := G/N \) and define a map

\[\Ind_{\overline{H}}^{\overline{G}} : C(\overline{H}_{\text{reg}}; F)^{\overline{H} \times G_k} \to C(\overline{G}_{\text{reg}}; F)^{\overline{G} \times G_k}\]

as follows:

\[\Ind_{\overline{H}}^{\overline{G}}(f)(g) = \frac{1}{|\overline{H}|} \sum_{x \in \overline{G}} f(xgx^{-1})\]

for any \( f \in C(\overline{H}_{\text{reg}}; F)^{\overline{H} \times G_k} \) and \( g \in \overline{G}_{\text{reg}} \), with the understanding that \( f(u) = 0 \) if \( u \notin \overline{H} \). Consider the following diagram:

\[
\begin{array}{ccc}
FK_0(\mathcal{F}_d(G)) & \xrightarrow{\theta_N} & FK_0(\mathcal{F}_e(G)) \\
\Ind_{H}^{G} \downarrow & & \Ind_{H}^{G} \downarrow \\
FK_0(\mathcal{F}_e(H)) & \xrightarrow{\rho_{H \cap N}} & FK_0(\mathcal{F}_e(H))
\end{array}
\]

The middle square commutes by functoriality of \( \mathcal{G}_0 \) and the right-hand square commutes by the well-known formula [22, Theorem 12 and Exercise 18.2] for the character of an induced representation. So the whole diagram commutes.

Finally, using the isomorphisms \( \pi_N \) and \( \pi_{\overline{H} \cap N} \) which feature in (3.8) we can define the required map \( \iota_N \) to be the map which makes the following diagram commute:

\[
\begin{array}{ccc}
C(\overline{G}_{\text{reg}}; F)^{\overline{G} \times G_k} & \xrightarrow{\pi_N} & C(G_{\text{reg}}; F)^{G \times G_k} \\
\Ind_{\overline{G}}^{\overline{G}} \downarrow & & \iota_N \downarrow \\
C(\overline{H}_{\text{reg}}; F)^{\overline{H} \times G_k} & \xrightarrow{\pi_{\overline{H} \cap N}} & C(H_{\text{reg}}; F)^{H \times G_k}.
\end{array}
\]

The commutative diagram appearing in (3.8) now shows that \( \pi_N \circ \varphi = \varphi \circ \lambda_N \) and \( \pi_{\overline{H} \cap N} \circ \varphi = \varphi \circ \lambda_{\overline{H} \cap N} \) and the result follows by pasting the above diagrams together. \( \square \)

11.3. The case when \( G \) is virtually abelian. We can now apply the theory developed above. If \( g \in G_{\text{reg}} \) let \( \delta_g : G_{\text{reg}} \to F \) be the function which takes the value 1 on the \( G \times G_k \)-orbit of \( g \) and is zero elsewhere. Note that \( \delta_g \) is locally constant by Corollary 3.6.
Theorem. Suppose that \( G \) is virtually abelian. Then the upper bound of Theorem B is always attained:

\[
\text{rk} \alpha_i = |(G \times G_k) \setminus S_i|
\]

for all \( i = 0, \ldots, d \).

Proof. Fix the integer \( i \) and fix \( g \in G_{\text{reg}} \) such that \( \dim C_G(g) \leq i \). As \( \text{rk} \alpha_i = \text{rk}(\rho_N \circ \alpha_i) \) by Corollary 8.2, it will be sufficient to show that

\[
\delta_g \in \text{Im}(\rho_N \circ \alpha_i).
\]

Because we are assuming that \( G \) is virtually abelian, the uniform pro-\( p \) subgroup \( N \) is abelian. Thus \( N \) is a free \( \mathbb{Z}_p \)-module of rank \( d = \dim G \). By considering the conjugation action of \( g \) on \( N \) we see that the submodule of fixed points

\[
C_N(g) = \{ x \in N : gx = xg \} = N \cap C_G(g)
\]

has a unique \( g \)-invariant \( \mathbb{Z}_p \)-module complement in \( N \) which we will denote by \( L \).

Let \( H \) be the closed subgroup of \( G \) generated by \( L \) and \( g \). Because \( g \) normalises \( L \), \( H \) is isomorphic to a semi-direct product of \( L \) with the finite group \( \langle g \rangle \):

\[
H = L \rtimes \langle g \rangle.
\]

Let \( \epsilon_g : H_{\text{reg}} \to F \) be the locally constant function which is 1 on the \( H \times G_k \)-orbit of \( g \) inside \( H_{\text{reg}} \) and zero elsewhere. Note that \( \epsilon_g \) is constant on the cosets of the open uniform subgroup \( H \cap N = L \) of \( H \).

By construction, \( g \) acts without nontrivial fixed points on \( L \) by conjugation. So if \( \beta_g \) denotes this action, then

\[
D_g := \det(1 - \beta_g) \in F
\]

is a nonzero constant. In view of the commutative diagram which appeared in the proof of Proposition 10.7 the element \( X := \varphi^{-1}(\epsilon_g) \in F K_0(F_0(H)) \) satisfies

\[
(\rho_{H \cap N} \circ \alpha_0)(X) = D_g \epsilon_g.
\]

Using the definition of the map \( \iota_N : C(H_{\text{reg}}; F)^{H \times G_k} \to C(G_{\text{reg}}; F)^{G \times G_k} \) of Proposition 11.2 we may calculate that there exists a nonzero constant \( A_g \in F \) such that for all \( y \in G_{\text{reg}} \) we have

\[
\iota_N(\epsilon_g)(y) = A_g \cdot \delta_g(y).
\]

We can now apply Proposition 11.2 and obtain

\[
(\rho_N \circ \alpha_{d-e}) \left( \text{Ind}_{H}^{G} X \right) = \iota_N(\rho_{H \cap N}(\alpha_0(X))) = \iota_N(D_g \epsilon_g) = D_g A_g \delta_g.
\]

But \( d - e = \dim G - \dim H = \dim C_G(g) \leq i \) and \( D_g A_g \neq 0 \) so

\[
\delta_g \in \text{Im}(\rho_N \circ \alpha_{d-e}) \subseteq \text{Im}(\rho_N \circ \alpha_i)
\]

as required. \( \square \)
12. An example

12.1. Central torsion modules. Suppose that $z$ is a central element of $G$ contained in $N$. Write $Z$ for the closed central subgroup of $G$ generated by $z$.

Because $N$ is torsion-free by [13, Theorem 4.5], $Z$ is isomorphic to $\mathbb{Z}_p$. Hence $kZ$ is isomorphic to the power series ring $k[[z-1]]$ and its maximal ideal is generated by $z-1$. Because $kG$ is a flat $kZ$-module, it follows that $z-1$ generates the kernel of the map $kG \to k[[G/Z]]$ as an ideal in $kG$.

Lemma. Let $M \in \mathcal{M}(kG)$ and suppose that $M.(z-1) = 0$ so that $M$ is also a right $k[[G/Z]]$-module. Then as right $k[[G/Z]]$-modules, we have

$$\text{Tor}^kG_n(M, k[[G/Z]]) \cong \left\{ \begin{array}{ll} M & \text{if } n = 0 \text{ or } n = 1 \\ 0 & \text{otherwise.} \end{array} \right.$$ 

Proof. Because $z-1$ is not a zero-divisor in $kG$, 

$$0 \to kG \to k[[G/Z]] \to 0$$

is a free resolution of $k[[G/Z]]$ as a left $kG$-module. Hence the complex

$$0 \to M \to 0$$

computes the required modules $\text{Tor}^kG_n(M, k[[G/Z]])$. The result follows because $M.(z-1) = 0$. □

12.2. Proposition. Let $M \in \mathcal{M}(kG)$ and suppose that $M.(z-1)^a = 0$ for some $a \geq 1$. Then $\theta_N[M] = 0$.

Proof. Suppose first that $a = 1$ so that $M$ is killed by $z-1$. We recall the base-change spectral sequence for Tor [27, Theorem 5.6.6] associated with a ring map $f : R \to S$:

$$E^2_{ij} = \text{Tor}^R_{i}(A,S,B) \implies \text{Tor}^S_{i+j}(A,B).$$

This is a first quadrant convergent homological spectral sequence. We apply it to the map $R := kG \to k[[G/Z]] := S$ with $A := M$ and $B := kG = k|G/N|$ — note that $B$ is a left $k[[G/Z]]$-module because we are assuming that $Z \leq N$.

By Lemma 12.1 this spectral sequence is concentrated in rows $j = 0$ and $j = 1$, so we may apply [27, Exercise 5.2.2] and obtain a long exact sequence

$$\cdots \to \text{Tor}^{kG}_{n+1}(M, k\overline{G}) \to \text{Tor}^{k[[G/Z]]}_{n+1}(M, k\overline{G}) \to \text{Tor}^{k[[G/Z]]}_{n}(M, k\overline{G}) \to \text{Tor}^{kG}_{n}(M, k\overline{G}) \to \text{Tor}^{k[[G/Z]]}_{n}(M, k\overline{G}) \to \text{Tor}^{kG}_{n-2}(M, k\overline{G}) \to \cdots .$$

We can now apply Lemma 2.2 and deduce that

$$\theta_N[M] = \sum_{n=0}^{d} (-1)^n [\text{Tor}^{kG}_n(M, k\overline{G})] = 0 \in G_0(k\overline{G})$$

as required. The general case follows quickly by an induction on $a$. □

Corollary. Let $M$ be as above. Then its Euler characteristic is trivial:

$$\chi(G, M) = 1.$$ 

Proof. This follows from Definition 8.4. □

Using this result we now show that the upper bound for $\text{rk} \alpha_i$ given in Theorem B is not always attained.
12.3. Example. Let $p$ be odd and let $N$ be a clean Heisenberg pro-$p$ group \cite[Definition 4.2]{26} of dimension $2r+1$. By definition, $N$ has a topological generating set $\{x_1, \ldots, x_r, y_1, \ldots, y_r, z\}$ and relations $[x_i, y_i] = z^p$ for each $i = 1, \ldots, r$, all other commutators being trivial. Note that $N$ is uniform.

The presentation for $N$ given above makes it possible to define an automorphism $\gamma$ of $N$ which fixes $z$ and sends all the other generators to their inverses:

$$
\gamma(x_i) = x_i^{-1}, \gamma(y_i) = y_i^{-1}, \gamma(z) = z.
$$

Now let $G$ be the semidirect product of $N$ with a cyclic group $\langle g \rangle$ of order 2, where the conjugation action of $g$ on $N$ is given by the automorphism $\gamma$. Thus the $p'$-part of $|G/N|$ is equal to 2 so the Galois group $G_k$ defined in (3.1) is trivial for any finite field $k$ of characteristic $p$.

Now $(G/N)_{\text{reg}} = G/N$ has two $G/N$-conjugacy classes, so by Proposition 3.6 there are just two $G \times G_k$-orbits on $G_{\text{reg}}$, represented by the elements 1 and $g$.

By construction, the endomorphism $\gamma = \text{Ad}(g)$ has eigenvalue $-1$ with multiplicity $2r$ and eigenvalue 1 with multiplicity 1:

$$
\Psi(g) = (1 + t)^{2r}(1 - t),
$$

so $\dim C_G(g) = 1$ and of course $\dim C_G(1) = 2r + 1$.

Hence in the notation of (9.3)

$$
|(G \times G_k) \backslash S_i| = \begin{cases} 
0 & \text{if } i = 0, \\
1 & \text{if } 1 \leq i \leq 2r, \\
2 & \text{if } i = 2r + 1.
\end{cases}
$$

Now if $M$ is a finitely generated $kG$-module with $d(M) \leq r$ then $M$ is killed by some power of $z - 1$ by \cite[Theorem B]{26}. It follows from Proposition 12.2 that $\theta_N[M] = 0$, so $\text{rk}\alpha_i = 0$ for all $i \leq r$ — thus the upper bound given in Theorem B is not attained for all values of $i$ between $i = 1$ and $i = r$.

References

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Christ's College, Cambridge; DPMMS, University of Cambridge
E-mail address: K.Ardakov@dpmms.cam.ac.uk; S.J.Wadsley@dpmms.cam.ac.uk