Groups: Example Sheet 3 of 4

1. Suppose that G is a finite group with a subgroup H. Show that if G contains twice as many elements as H then H is normal in G.

2. Show that any subgroup of D_{2n} consisting of rotations is normal.

3. Show that a subgroup H of a group G is normal if and only if it is a union of conjugacy classes in G.

4. Suppose that G is a group in which every subgroup is normal. Must G be abelian?

5. Suppose that H is a subgroup of C_n. What is C_n/H?

6. Show that $\mathbb{Q}/\mathbb{Z}$ is an infinite group in which every element has finite order.

7. Let K be a subgroup of a group G. Show that K is a normal subgroup if and only if it the kernel of some group homomorphism $\theta\colon G \to H$.

8. Let G be the set of all 3×3 matrices of the form
$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$
with $x, y, z \in \mathbb{R}$. Show that G is a subgroup of $GL_3(\mathbb{R})$. Let H be the subset of G consisting of those matrices with $x = y = 0$. Show that H is a normal subgroup of G. What known group is isomorphic to G/H?

9. Consider the subgroup Γ of the additive group $\mathbb{C}$ consisting of elements $m + in$ with $m, n \in \mathbb{Z}$. By considering the map
$$x + iy \mapsto (e^{2\pi ix}, e^{2\pi iy}),$$
show that $\mathbb{C}/\Gamma$ is isomorphic to the torus $S^1 \times S^1$.

10. Suppose $a, b \in \mathbb{Z}$ and consider $\theta\colon \mathbb{Z}^2 \to \mathbb{Z}; (x, y) \mapsto ax + by$. Show that θ is a group homomorphism and describe $\operatorname{Im}\theta$ and $\ker\theta$. What characterises the cosets of $\ker\theta$ in $\mathbb{Z}^2$?

11. Let G be a finite group and H a proper subgroup. Let k be the cardinality of the set of left cosets of H in G and suppose that $|G|$ does not divide $k!$. By considering the action of G on G/H, show that H contains a non-trivial normal subgroup of G. Deduce that a group of order 28 has a normal subgroup of order 7.

12. Show that if a group G of order 28 has a normal subgroup of order 4 then G is abelian.

13. Show that $\mathrm{Isom}(\mathbb{Z})$ acts on $\mathbb{Z}/n\mathbb{Z}$ via $f \cdot (a + n\mathbb{Z}) = f(a) + n\mathbb{Z}$. Deduce that D_{2n} is isomorphic to a quotient of $\mathrm{Isom}(\mathbb{Z})$. What is the corresponding normal subgroup of $\mathrm{Isom}(\mathbb{Z})$?

14. *For which natural numbers n is every pair of groups of order n isomorphic?