Groups: Example Sheet 1 of 4

1. Let G be any group. Show that the identity e is the unique solution of the equation $x^2 = x$ in G.

2. Let H_1 and H_2 be two subgroups of the group G. Show that the intersection $H_1 \cap H_2$ is a subgroup of G. Show that the union $H_1 \cup H_2$ is a subgroup of G if and only if one of the H_i contains the other.

3. Let $G = \{x \in \mathbb{R} : x \neq -1\}$, and let $x * y = x + y + xy$, where xy denotes the usual product of two real numbers. Show that $(G, *)$ is a group. What is the inverse 2^{-1} of 2 in this group? Solve the equation $2 * x * 5 = 6$.

4. Let G be a finite group. Show that every element of G has finite order. Show that there exists a positive integer n such that for all $g \in G$ we have $g^n = e$.

5. Show that the set G of complex numbers of the form $\exp(i\pi t)$ with t rational is a group under multiplication (with identity 1). Show that G is infinite, but that every element a of G has finite order.

6. Let G be a finite group and f a homomorphism from G to H. Let $a \in G$. Show that the order of $f(a)$ is finite and divides the order of a.

7. Let C_n be the cyclic group with n elements and D_{2n} the group of symmetries of the regular n-gon. If n is odd and $\theta\colon D_{2n} \to C_n$ is a homomorphism, show that $\theta(g) = e$ for all $g \in D_{2n}$. Can you find all homomorphisms $D_{2n} \to C_n$ if n is even? Find all homomorphisms $C_n \to C_m$.

8. Show that any subgroup of a cyclic group is cyclic.

9. Consider the Möbius maps $f(z) = e^{2\pi i/n} z$ and $g(z) = 1/z$. Show that the subgroup G of the Möbius group $\mathcal{M}$ generated by f and g is isomorphic to D_{2n}.

10. Express the Möbius transformation $f(z) = \frac{2z+3}{z-4}$ as the composition of maps of the form $z \mapsto az$, $z \mapsto z+b$ and $z \mapsto 1/z$. Hence show that f maps the circle $|z - 2i| = 2$ onto the circle $|8z + (6 + 11i)| = 11$.

11. Let G be the subgroup of Möbius transformations that map the set $\{0, 1, \infty\}$ to itself. What are the elements of G? Which standard group is isomorphic to G? What is the group of Möbius transformations that map the set $\{0, 2, \infty\}$ to itself.

12. (a) Is the Möbius group generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto z + b$? Why/why not?
    (b) Is the Möbius group generated by Möbius transformations of the form $z \mapsto az$ and $z \mapsto 1/z$? Why/why not?
    (c) Is the Möbius group generated by Möbius transformations of the form $z \mapsto z + b$ and $z \mapsto 1/z$? Why/why not?

13. Show that an invertible function $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ that preserves the cross-ratio, i.e. such that

$$[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)] \text{ for all distinct } z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty,$$

is a Möbius transformation.

14. Let G be a group in which every element other than the identity has order two. Show that G is abelian. *Show also that if G is finite, the order of G is a power of 2.

15. Let G be a group of even order. Show that G contains an element of order two.

16. Show that every isometry of $\mathbb{C}$ is either of the form $z \mapsto az + b$ or the form $z \mapsto a\bar{z} + b$ with $a, b \in \mathbb{C}$ and $|a| = 1$ in either case. *Describe the finite subgroups of the group of isometries of $\mathbb{C}$.