Commutative Algebra

Michaelmas 2010

Example Sheet 2

Recall that in this course all rings are commutative

1. Suppose that S is a m.c. subset of a ring A, and M is an A-module. Show that there is a natural isomorphism $A_S \otimes_A M \cong M_S$ of A_S-modules. Deduce that A_S is a flat A-module.

2. Suppose that $\{M_i | i \in I\}$ is a set of modules for a ring A. Show that $M = \bigoplus_{i \in I} M_i$ is a flat A-module if and only if each M_i is a flat A-module. Deduce that projective modules are flat.

3. Show that an A-module is flat if every finitely generated submodule is flat.

4. Show that flatness is a local properties of modules.

5. Suppose that A is a ring and $\phi\colon A^n \to A^m$ is an isomorphism of A-modules. Must $n = m$?

6. Let A be a ring. Suppose that M is an A-module and $f\colon M \to M$ is an A-module map.

   (i) Show by considering the modules $\ker f^n$ that if M is Noetherian as an A-module and f is surjective then f is an isomorphism

   (ii) Show that if M is Artinian as an A-module and f is injective then f is an isomorphism.

7. Let S be the subset of $\mathbb{Z}$ consisting of powers of some integer $n \geq 2$. Show that the $\mathbb{Z}$-module $\mathbb{Z}_S/\mathbb{Z}$ is Artinian but not Noetherian.

8. Show that $\mathbb{C}[X_1, X_2]$ has a $\mathbb{C}$-subalgebra that is not a Noetherian ring.

9. Suppose that A is a Noetherian ring and $A[[x]]$ is the ring of formal power series over A. Adapt the proof of Hilbert's Basis Theorem to show that $A[[x]]$ is a Noetherian ring.

10. Suppose that M is a Noetherian A-module. Show that $A/\operatorname{Ann}_A(M)$ is a Noetherian ring.

11. Suppose that A is any ring.

    Show that if the zero ideal may be written as a product of maximal ideals (possibly with repeats) then A is Artinian if and only if it is Noetherian.

    Deduce in general that A is Artinian if and only if it is Noetherian and every prime ideal is maximal.

12. Show that a ring A is Artinian if and only if it is a finite direct product of Artinian local rings.

13. Suppose $0 \to L \to M \to N \to 0$ is a short exact sequence of A-modules. Must we have

$$\mathrm{Ass}(M) = \mathrm{Ass}(L) \cup \mathrm{Ass}(N)?$$

14. Suppose that A is a ring, and M and N are finitely generated A-modules. Use Nakayama's lemma to show that $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$.

15. Let A be a ring. Show that the following are equivalent:

    (i) Every prime ideal in A is an interesection of maximal ideals;

    (ii) For every ideal I in A, $N(A/I) = \mathrm{Jac}(A/I)$;

    (iii) Every prime ideal in A which is not maximal is the intersection of the prime ideals strictly containing it.

16. Let A be a ring. Show that if M is a flat A-module and $\sum_{i=1}^{n} a_i x_i = 0$ with a_i in A and $x_i \in M$ then there are elements b_{ij} in A and y_j in M (with $1 \leq j \leq m$, say) such that $x_i = \sum_j b_{ij} y_j$ for each $1 \leq i \leq n$ and $\sum_i a_i b_{ij} = 0$ for each j.

17. Suppose that A is a Noetherian local ring show that a finitely generated flat module is free. By showing that being a finitely generated projective module over a Noetherian ring is a local property, deduce, in general that if A is a Noetherian ring, then a finitely generated flat module is projective.

18. Suppose that A is a ring such that A_P is Noetherian for every prime ideal P in A. Must A be Noetherian?

19. Let A be a Noetherian ring, $S \subset A$ a multiplicatively closed subset of A and let M and N be A-modules. Show that if M is finitely generated then there is a canonical isomorphism

$$\mathrm{Hom}_A(M, N)_S \cong \mathrm{Hom}_{A_S}(M_S, N_S).$$

    Can you weaken the hypotheses on the pair (A, M)?

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