1. Find all the characters of $S_5$ obtained by inducing irreducible representations of $S_4$. Use these to reconstruct the character table of $S_5$. Then repeat, replacing $S_4$ by the subgroup $\langle (12345), (2354) \rangle$ of $S_5$ of order 20.

2. Recall the character table of $D_{10}$ from sheet 2. Explain how to view $D_{10}$ as a subgroup of $A_5$ and then use induction from $D_{10}$ to $A_5$ to reconstruct the character table of $A_5$.

3. Let $H$ be a subgroup of a group $G$. Show that for every irreducible representation $(\rho, V)$ of $G$ there is an irreducible representation $(\rho', W)$ of $H$ such that $\rho$ is an irreducible component of $\text{Ind}^G_H W$.

Deduce that if $A$ is an abelian subgroup of $G$ then every irreducible representation of $G$ has dimension at most $|G/A|$.

4. Obtain the character table of the dihedral group $D_8$ by using induction from the cyclic group $C_4$; you will want to split into two cases according as $m$ is odd or even.

5. Prove that if $H$ is a subgroup of a group $G$, and $K$ is a subgroup of $H$, and $W$ is a representation of $K$ then $\text{Ind}^G_K W \cong \text{Ind}^H_K W$.

5. Calculate $S^2V$ and $\Lambda^2V$ for the two-dimensional irreducible representations of $D_8$ and of $Q_8$. Which has the trivial representation as a subrepresentation in each case?

7. Let $\rho: G \to \text{GL}(V)$ be a representation of $G$ of dimension $d$.
   
   (a) Compute $\dim S^nV$ and $\dim \Lambda^nV$ for all $n$.
   
   (b) Let $g \in G$ and $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\rho(g)$. What are the eigenvalues of $g$ on $S^nV$ and $\Lambda^nV$?
   
   (c) Let $f(t) = \det(tI - \rho(g))$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of $f$ and $\chi_{\Lambda^nV}$?
   
   (d) What is the relationship between $\chi_{S^nV}(g)$ and $f$? (Hint: start with case $d = 1$).

8. Let $G = S_n$ act naturally on the set $X = \{1, \ldots, n\}$. For each non-negative integer $r$, let $X_r$ be the set of all $r$-element subsets of $X$ equipped with the natural action of $G$, and $\pi_r$ be the character of the corresponding permutation representation. If $0 \leq l \leq k \leq n/2$, show that
   
   $$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that $\pi_r - \pi_{r-1}$ is a character of an irreducible representation for each $1 \leq r \leq n/2$. What happens for $r > n/2$?

9. Suppose $\rho: G \to \text{GL}(V)$ is an irreducible representation of $G$ with character $\chi$. By considering $V \otimes V$, $S^2V$ and $\Lambda^2V$ show that
   
   $$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \begin{cases} 0 & \text{if } \chi \text{ is not real-valued} \\ \pm 1 & \text{if } \chi \text{ is real valued}. \end{cases}$$

Deduce that if $|G|$ is odd then $G$ has only one real-valued irreducible character.

10. Suppose that $V$ is a faithful representation of a group $G$ such that $\chi_V$ takes $r$ distinct values. Show that each irreducible representation of $G$ is a summand of $V \otimes^n$ for some $n < r$.

   (Hint: Assume for contradiction that $\langle \chi_W, \chi_V \otimes^n \rangle = 0$ for some irreducible representation $W$.)


12. Show that if $V$ is an irreducible representation of a group $G$ then (up to rescaling) $V$ has only one $G$-invariant Hermitian inner product.

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