1. Let \( \rho \) be a representation of a group \( G \). Show that \( \det \rho \) is a representation of \( G \). What is its degree?

2. Let \( \theta \) be a one-dimensional representation of a group \( G \) and \( \rho: G \to GL(V) \) another representation of \( G \). Show that \( \theta \otimes \rho: G \to GL(V) \) given by \( \theta \otimes \rho(g) = \theta(g) \cdot \rho(g) \) defines a representation of \( G \). If \( \rho \) is irreducible, must \( \theta \otimes \rho \) also be irreducible?

3. Suppose that \( N \) is a normal subgroup of a group \( G \). Given a representation of the quotient group \( G/N \) on a vector space \( V \), explain how to construct an associated representation of \( G \) on \( V \). Which representations of \( G \) arise in the way?

Recall that \( G' \) is defined to be the normal subgroup of \( G \) generated by elements of the form \( ghg^{-1}h^{-1} \) with \( g, h \in G \). Show that the 1-dimensional representations of \( G \) are precisely those that arise from 1-dimensional representations of \( G/G' \).

4. Let \( \rho: \mathbb{Z} \to GL_2(\mathbb{C}) \) be the representation defined by \( \rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Show that \( \rho \) is not completely reducible.

By a similar construction, show that if \( k \) is a field of characteristic \( p \) there is a two dimensional \( k \)-representation of \( C_p \) that is not completely reducible.

5. Let \( C_n \) be the cyclic group of order \( n \). Explicitly decompose the complex regular representation \( \mathbb{C}C_n \) as a direct sum of irreducible subrepresentations.

6. Let \( D_{10} \) be the dihedral group of order 10. Show that every irreducible \( \mathbb{C} \)-representation of \( D_{10} \) has degree 1 or 2. By describing them explicitly, show that there are precisely four such representations up to isomorphism. Show moreover that for each such representation it is possible to choose a basis so that all the representing matrices have real entries.

7. What are the irreducible real representations \( \rho: C_n \to GL(V) \) of a cyclic group of order \( n \)? Compute \( \text{Hom}_\mathbb{R}(V, V) \) in each case. How does the real regular representation \( \mathbb{R}C_n \) of \( C_n \) break up as a direct sum of irreducible representations?

8. Write down a presentation of the quaternion group \( Q_8 \) of order 8. Show that (up to isomorphism) there is only one irreducible complex representation of \( Q_8 \) of dimension at least two. Show that this representation cannot be realised over \( \mathbb{R} \) and deduce that that \( Q_8 \) is not isomorphic to a subgroup of \( GL_2(\mathbb{R}) \).

Find a four-dimensional irreducible real representation \( V \) of \( Q_8 \). Compute \( \text{Hom}_G(V, V) \) in this case.

9. Suppose that \( k \) is algebraically closed. Using Schur’s Lemma, show that if \( G \) is a finite group with trivial centre and \( H \) is a subgroup of \( G \) with non-trivial centre, then any faithful representation of \( G \) is reducible after restriction to \( H \). What happens for \( k = \mathbb{R} \)?

10. Let \((\rho, V)\) be an irreducible complex representation of a finite group \( G \). For each \( v \in V \), show that the \( \mathbb{C} \)-linear map \( CG \to V \) given by \( \delta_g \mapsto \rho(g)(v) \) is \( G \)-linear and deduce that \( V \) is isomorphic to a subrepresentation of \( CG \). What is \( \dim \text{Hom}_G(CG, V) \)?

11. Let \( G \) be the subgroup of the symmetric group \( S_6 \) generated by \((123), (456)\) and \((23)(56)\). Show that \( G \) has an index two subgroup of order 9 and four normal subgroups of order 3. By considering quotients show that \( G \) has two complex representations of degree 1, and four pairwise non-isomorphic irreducible complex representations of degree 2, none of which is faithful. Does \( G \) have a faithful irreducible complex representation?

12. Show that if \( \rho: G \to GL(V) \) is a representation of a finite group \( G \) on a real vector space \( V \) then there is a basis for \( V \) with respect to which the matrix representing \( \rho(g) \) is orthogonal for every \( g \in G \). Which finite groups have a faithful two-dimensional real representation?

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