1. Bicommutative bialgebras

Throughout this note we fix a symmetric monoidal category $(\mathcal{C}, \otimes, \tau)$ where τ is the symmetry functor with $\tau_{AB} : A \otimes B \to B \otimes A$.

A bicommutative bialgebra in $\mathcal{C}$ is an object A in $\mathcal{C}$, with functors $\mu_A : A \otimes A \to A$, $\eta_A : I \to A$, $\Delta_A : A \to A \otimes A$, and $\epsilon_A : A \to I$ called multiplication, unit, comultplication and counit respectively, satisfying certain axioms.

Given two bicommutative bialgebra objects A and B in $\mathcal{C}$, we may give $A \otimes B$ the structure of a bicommutative bialgebra: $\mu_{A\otimes B} := (\mu_A \otimes \mu_B)(1 \otimes \tau_{BA} \otimes 1)$, $\eta_{A\otimes B} = \eta_A \otimes \eta_B$, $\Delta_{A\otimes B} = (1 \otimes \tau_{AB} \otimes 1)(\Delta_A \otimes \Delta_B)$, and $\epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B$. In this way we may make $A^{\otimes m}$ into a bicommutative bialgebra for each $m \geq 2$.

As a point of notation, we will write μ_A^n for the functor from $A^{\otimes n} \to A$, given inductively by $\mu_A^0 = \eta_A$, and $\mu_A^{n+1} = \mu(1 \otimes \mu_A^n)$, and Δ_A^n for the functor from $A \to A^{\otimes n}$ given inductively by $\Delta_A^0 = \epsilon_A$ and $\Delta_A^{n+1} = \Delta_A(1 \otimes \Delta_A^n)$. Notice every μ_A^n and Δ_A^n is a bialgebra map.

Lemma 1. *The monoidal subcategory of $\mathcal{C}$ whose objects are bicommutative bialgebra objects in $\mathcal{C}$ and whose morphisms are bialgebra bihomomorphisms may be enriched over commutative monoids.*

Proof. We first need to explain how to define an addition on the Hom sets. Suppose A and B are bicommutative bialgebras, and suppose that f and g are two morphisms from A to B. We define

$$f + g := \mu_B(f \otimes g)\Delta_A$$

For each Hom set the axioms for a commutative monoid now follow easily, but for completeness: suppose f, g and h are in $\mathrm{Hom}(A, B)$

$$\begin{aligned} (f+g)+h &= \mu_B((\mu_B(f \otimes g)\Delta_A) \otimes h)\Delta_A \\ &= \mu_B(\mu_B \otimes 1)(f \otimes g \otimes h)\Delta_A(1 \otimes \Delta_A) \\ &= \mu_B(1 \otimes \mu_B)(f \otimes g \otimes h)(1 \otimes \Delta_A)\Delta_A \\ &= f + (g+h) \end{aligned}$$

where the third equality follows from the coassociativity of A, and associativity of B.

The zero map from A to B is the composite of the counit ϵ_A and the unit η_B, and

$$0 + f = \mu_B(\eta_B\epsilon_A \otimes f)\Delta_A = \mu_B(\eta_B \otimes 1)(1 \otimes f)(\epsilon_A \otimes 1)\Delta_A = f$$

where the last equality follows from the axioms for unit and counit.

The symmetry of $+$ follows from the commutativity of B, and cocommutativity of A: if τ_A is the symmetry map $A \otimes A \to A \otimes A$ then

$$f+g = \mu_B(f\otimes g)\Delta_A = \mu_B\tau_B(f\otimes g)\Delta_A = \mu_B(g\otimes f)\tau_A\Delta_A = \mu_B(g\otimes f)\Delta_A = g+f$$

To complete the proof we need to check that the composition of morphisms gives a monoid map $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$: suppose f_1 and f_2 are in $\mathrm{Hom}(B, C)$ and g_1 and g_2 are in $\mathrm{Hom}(A, B)$ then

$$f_1(g_1+g_2) = f_1\mu_B(g_1\otimes g_2)\Delta_A = \mu_C(f_1\otimes f_1)(g_1\otimes g_2)\Delta_A = \mu_C(f_1g_1\otimes f_1h_1)\Delta_A = f_1g_1+f_1h_1$$

and

$$(f_1+f_2)g_1 = \mu_C(f_1\otimes f_2)\Delta_Bg_1 = \mu_C(f_1\otimes f_2)(g_1\otimes g_1)\Delta_A = \mu_C(f_1g_1\otimes f_2g_1)\Delta_A = f_1g_1+f_2g_1.$$

□

From now on we will refer to this enriched category as Bialg. Notice that the enriched structure makes $\mathrm{Hom}_{\mathrm{Bialg}}(A,A)$ is a rig with identity id_A whenever A is an object in Bialg. We will just write $\mathrm{End}(A)$ for this rig.

If R is a rig, we'll write Bialg^R for the category whose objects are pairs (A,ϕ_A) with A an object in Bialg and ϕ_A a morphism of rigs $R \to \mathrm{End}(A)$, and such that $\mathrm{Hom}_{\mathrm{Bialg}^R}((A,\phi_A),(B,\phi_B))$ is the set of morphisms f in $\mathrm{Hom}_{\mathrm{Bialg}}(A,B)$ such that $f\phi_A = \phi_B f$. Since there is a unique map of rigs $\mathbb{N} \to \mathrm{End}(A)$ for each $A \in$ Bialg that sends 1 to id_A, $\mathrm{Bialg}^{\mathbb{N}}$ is just Bialg. In general Bialg^R inherits a monoidal structure from Bialg: $(A,\phi_A)\otimes(B,\phi_B) = (A\otimes B, \phi_A \otimes \phi_B)$.

2. The PROP Mat(R)

Suppose that R is a rig. Recall that the PROP $\mathrm{Mat}(R)$ is the monoidal category enriched over commutative monoids whose objects are the natural numbers, with tensor product given by addition, and whose morphisms $\mathrm{Hom}(m,n)$ are $(m \times n)$ matrices with coefficients in R. Composition is given by usual matrix mulitplication and the addition providing the enriched structure is given usual matrix addition. If f and g are two matrices then $f \otimes g$ is just given by the block matrix $\begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}$

It is easy to check that transposing matrices gives an self-inverse contravariant enriched monoidal functor from $\mathrm{Mat}(R)$ to itself.

Notice that 0 is both an initial and terminal object in this category, as there is precisely one matrix with n rows and no columns and its transpose is the unique matrix with n columns and no rows. We'll write 0^n for the former and 0_n for the latter. We have identities such as $0_n = (0_1)^{\otimes n}$. The $m \times n$ matrix all of whose entries are 0, which we'll write 0^m_n, is just $0^m \otimes 0_n$.

We will write $E^{mn}_{ij}(r)$ for the $m \times n$-matrix whose only entries are zero except for an r in the ijth place. Alternatively we may describe this by the equation

$$E^{mn}_{ij}(r) = 0^{i-1}_{j-1} \otimes r \otimes 0^{m-i}_{n-j},$$

where r is the 1×1 matrix whose only entry is r. These matrices generate all the morphisms in $\mathrm{Mat}(R)$ using the enriched structure.

3. Algebras over Mat(R)

Our goal now is the following theorem:

Theorem. *If R is a rig then* $\mathrm{Mat}(R)$ *is the PROP for bicommutative bialgebras A equipped a map of rigs $R \to \mathrm{End}(A)$.*

We prove this theorem by constructing mutually inverse functors between the relevant categories.

Firstly we show

Lemma 2. *If A is a bicommutative bialgebra and $\phi : R \to \mathrm{End}(A)$ a map of rigs, then there is a unique strict monoidal functor F_A enriched over commutative moniods from* $\mathrm{Mat}(R)$ *to* Bialg *such that*

(1) $A = F_A(1)$
(2) $\mu_A = F_A((11) : 2 \to 1)$
(3) $\eta_A = F_A(0 : 0 \to 1)$
(4) $\Delta_A = F_A((11)^t : 1 \to 2)$
(5) $\epsilon_A = F_A(0 : 1 \to 0)$

(6) $\phi(r) = F_A(r : 1 \to 1)$ *for each* $r \in R$.

Proof. Because F_A is a strict monoidal functor with $F_A(1) = A$, $F_A(n)$ is necessarily $A^{\otimes n}$ for every n.

We begin by defining F_A on $\mathrm{Hom}(n, 1)$ for each n. If (r_i) is an $(1 \times n)$ matrix with entries in R we set $F_A(r_i) = \mu_A^n(\phi(r_1) \otimes \cdots \otimes \phi(r_n))$. Notice that in particular $F_A(r : 1 \to 1)$ is just $\phi(r)$, $F_A(0 : 0 \to 1) = \eta_A$ and $F_A(11 : 2 \to 1)$ is just μ_A all as required. We need the maps $F_A(r_i)$ to be bialgebra maps, but this is true because they are defined by composing bialgebra maps.

Now suppose that (r_{ij}) is any $(m \times n)$ matrix with entries in R. We define $F_A(r_{ij}) = (F_A(r_{i1}) \otimes \cdots \otimes F_A(r_{im}))\Delta^m_{A^{\otimes n}}$. This time it may be easily seen that $F_A(0 : 1 \to 0) = \epsilon_A$ and $F_A((11)^t : 1 \to 2) = \Delta_A$ as required. As before all these maps $F_A(r_{ij})$ are bialgebra maps because they are a composite of such.

We need to check that F as defined is an enriched functor. First, we check that $F_A(r_{ij} + s_{ij}) = F_A(r_{ij}) + F_A(s_{ij})$ for every pair of R-valued $(m \times n)$ matrices (r_{ij}) and (s_{ij}). As before we begin by considering the case $m = 1$, suppressing the second index as we may:

$$\begin{aligned}
F_A(r_i + s_i) &= \mu_A^n(\phi(r_1 + s_1) \otimes \cdots \otimes \phi(r_n + s_n)) \\
&= \mu_A^n((\phi(r_1) + \phi(s_1)) \otimes \cdots (\phi(r_n) + \phi(s_n))) \\
&= \mu_A^n((\mu_A(\phi(r_1) \otimes \phi(s_1))\Delta_A) \otimes \cdots \otimes (\mu_A(\phi(r_n) \otimes \phi(s_n))\Delta_A)) \\
&= \mu_A^n \mu_A^{\otimes n}(\phi(r_1) \otimes \phi(s_1) \otimes \cdots \otimes \phi(r_n) \otimes \phi(s_n))\Delta_A^{\otimes n} \\
&= \mu_A^n(\mu_{A^{\otimes n}}(\phi(r_1) \otimes \cdots \otimes \phi(r_n) \otimes \phi(s_1) \otimes \cdots \otimes \phi(s_n))\Delta_{A^{\otimes n}} \\
&= \mu_A(\mu_A^n(\phi(r_1) \otimes \cdots \otimes \phi(r_n)) \otimes \mu_A^n(\phi(s_1) \otimes \cdots \otimes \phi(s_n)))\Delta_{A^{\otimes n}} \\
&= F_A(r_i) + F_A(s_i)
\end{aligned}$$

Now we consider the general case:

$$\begin{aligned}
F_A(r_{ij} + s_{ij}) &= (F_A(r_{i1} + s_i 1) \otimes \cdots \otimes F_A(r_{im} + s_{im}))\Delta^m_{A^{\otimes n}} \\
&= ((F_A(r_{i1}) + F_A(s_{i1})) \otimes \cdots \otimes (F_A(r_{im}) + F_A(s_{im})))\Delta^m_{A^{\otimes n}} \\
&= \left(\bigotimes_{k=1}^{m} \mu_A(F_A(r_{ik}) \otimes F_A(s_{ik}))\Delta_{A^{\otimes n}}\right)\Delta^m_{A^{\otimes n}} \\
&= \mu_{A^{\otimes m}}\left(\left(\bigotimes_{k=1}^{m} F_A(r_{ik})\right)\Delta^m_{A^{\otimes n}} \otimes \left(\bigotimes_{k=1}^{m} F_A(s_{ik})\right)\Delta^m_{A^{\otimes n}}\right)\Delta_{A^{\otimes n}} \\
&= \mu_{A^{\otimes m}}(F_A(r_{ij}) \otimes F_A(s_{ij}))\Delta_{A^{\otimes n}} \\
&= F_A(r_{ij}) + F_A(s_{ij})
\end{aligned}$$

We can now complete the existence part of the proof by showing that F_A is a functor i.e. that F_A preserves composition. Because we have already checked that F_A preserves the enriched structure it suffices to check this on matrices of the form $E^{mn}_{ij}(r)$ since these generate all matrices under $+$.

But it is easy to check that $F_A(E^{1n}_{1i}(r))$ is the map $\epsilon_{A^{\otimes i-1}} \otimes \phi(r) \otimes \epsilon_{A^{\otimes n-i}}$. Then we see that

$$F_A(E^{mn}_{ij}(r)) = \eta_{A^{\otimes j-1}} \otimes \epsilon_{A^{\otimes i-1}} \otimes \phi(r) \otimes \epsilon_{A^{\otimes m-i}} \otimes \eta_{A^{\otimes n-j}}.$$

Now if we take two matrices of this form that compose it is easy to see that F_A preserves their composition.

For uniqueness, we have already observed that F_A is uniquely determined on objects by the conditions given. Because we are insisting F_A be enriched over commutative monoids, to show F_A is uniquely determined on morphims it suffices to check that this is so for matrices with at most one non-zero entry. But the monoidal structure forces $F_A(E_{ij}^{mn}(r)$ to be the same as

$$F_A(0_{j-1}^{i-1}) \otimes F_A(r) \otimes F_A(0_{n-j}^{m-i}).$$

But $F_A(0_n^m)$ is required to be $\epsilon_{A^{\otimes n}} \otimes \eta_{A^{\otimes m}}$. □

It is now easy to prove the following propostion

Proposition. *Suppose that R is a rig. There is a functor $\mathbf{F}$ from Bialg^R to the category of algebras over the PROP $\mathrm{Mat}(R)$ such that*

(1) $A = \mathbf{F}((A, \phi_A))(1)$
(2) $\mu_A = \mathbf{F}((A, \phi_A))((11) : 2 \to 1)$
(3) $\eta_A = \mathbf{F}((A, \phi_A))(0 : 0 \to 1)$
(4) $\Delta_A = \mathbf{F}((A, \phi_A))((11)^t : 1 \to 2)$
(5) $\epsilon_A = \mathbf{F}((A, \phi_A))(0 : 1 \to 0)$
(6) $\phi_A(r) = \mathbf{F}((A, \phi_A))(r : 1 \to 1)$ *for each* $r \in R$.

Proof. By lemma 2 there is a unique way to define $\mathbf{F}$ on objects subject to the given conditions.

Suppose that $f : (A, \phi_A) \to (B, \phi_B)$ is a morphism in Bialg^R. We must define a natural transformation $\mathbf{F}(f)$ from $\mathbb{F}(A, \phi_A)$ to $\mathbf{F}(B, \phi_B)$. To this end, let $\mathbf{F}(f)(n) = f^{\otimes n}$ for each n. We need to show that $\mathbf{F}(f)(m)\mathbf{F}(A)(X) = \mathbf{F}(B)(X)\mathbf{F}(f)(n)$ for each $m \otimes n$ matrix X with coefficients in R. Because of the enriched structure on the functors $\mathbf{F}(A, \phi_A)$ and $\mathbf{F}(B, \phi_B)$, it suffices to check this for matrices with precisely one non-zero entry. For (1×1)-matrices this is just the fact that $f\phi_A = \phi_B f$. It is also true when m or n is 0 since $\mathbf{F}((A, \phi_A))(0)$ is both an intial and terminal object. Since each matrix with one non-zero entry is a tensor product of matrices of these types we are done. □

Next we want,

Lemma 3. *If R is a rig and $F : \mathrm{Mat}(R) \to \mathcal{C}$ is an algebra over the PROP $\mathrm{Mat}(R)$ then $(A = F(1), \phi_A)$ is in Bialg^R with $\mu_A = F((11) : 2 \to 1))$, $\eta_A = F(0 : 0 \to 1)$, $\Delta_A = F((11)^t : 1 \to 2)$, $\epsilon_A = F(0 : 1 \to 0$, and $\phi_A(r) = F(r : 1 \to 1)$. Finally, F is necessarily an enriched functor.*

Proof. First we show This makes (A, μ_A, η_A) is a commutative monoid:

Associativity follows from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\end{pmatrix},$$

commutativity from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix},$$

and the unit axiom from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

That $(A, \Delta_A, \epsilon_A)$ does form a cocommutative coalgebra follows from what has gone before and the fact observed previously that the transpose map from $\mathrm{Mat}(R)$ to itself is a contravariant functor that is self-inverse.

Next we must check that the algebra and coalgebra structures are compatible. It suffices to check that Δ_A is an algebra map and the counit respects the algebra structure. The first follows from the matrix equations

$$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 & 0\\0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 0\\0 & 1\\0 & 1\end{pmatrix},$$

and the latter from the fact that 0 is a terminal object in $\mathrm{Mat}(R)$. To see that ϕ_A is a map of rigs we notice

$$\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}r & 0\\0 & r'\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix}.$$

□

To finish, we need to show that F is an enriched functor: suppose X and Y are $m \times n$ matrices in $\mathrm{Mat}(R)$. We want $F(X+Y) = \mu_{A^{\otimes m}}(F(X) \otimes F(Y))\Delta_{A^{\otimes n}}$. This follows from the matrix equations

$$\begin{pmatrix}I_m & I_m\end{pmatrix}\begin{pmatrix}X & 0\\0 & Y\end{pmatrix}\begin{pmatrix}I_n\\I_n\end{pmatrix}$$

where I_n is the $(n \times n)$ identity matrix.

Proposition. *Suppose that R is a rig. There is a functor* $\mathbf{G}$ *from the category of algebras over the PROP* $\mathrm{Mat}(R)$ *to* Bialg^R *such that*

(1) $\mathbf{G}(F : \mathrm{Mat}(R) \to \mathcal{C}) = (F(1), F(\mathrm{Hom}_{\mathrm{Mat}(R)}(1,1)))$
(2) $\mu_{\mathbf{G}(F)} = \mathbf{G}(F)((11) : 2 \to 1)$
(3) $\eta_{\mathbf{G}(F)} = \mathbf{G}(F)(0 : 0 \to 1)$
(4) $\Delta_{\mathbf{G}(F)} = \mathbf{G}(F)((11)^t : 1 \to 2)$
(5) $\epsilon_{\mathbf{G}(F)} = \mathbf{G}(F)(0 : 1 \to 0)$

Proof. Lemma 3 shows that our definition makes sense on objects. We need to define $\mathbf{G}$ on morphisms. Suppose F and G are two algebras over $\mathrm{Mat}\, R$, and Θ is a natural transformation from F to G. Then $\Theta(1)$ is a map f from $F(1)$ to $G(1)$ such that if X is an $(m \times n)$ matrix with coeffecients in R then $f^{\otimes n}F(X) = G(X)f^{\otimes m}$.

This condition for (1×1)-matrices says precisely that f commutes with the maps $R \to \mathrm{End}(A)$ and $R \to \mathrm{End}(B)$. Then the condition for the matrix (11) implies that f is an algebra map, and for $(11)^t$ that it is an coalgebra map. □

We are now ready to complete the proof of our theorem:

Proof. We have constructed a functor $\mathbf{F}$ from Bialg^R to algebras over $\mathrm{Mat}\, R$ and a functor $\mathbf{G}$ in the opposite direction. It is clear from the construction that these functors are mutual inverses, that is they define an isomorphism of categories. □

Corollary. *The category of algebras over* $\mathrm{Mat}(\mathbb{N})$ *is the category of bicommutative bialgebras.*

□

Corollary. *The category of algebras over* $\mathrm{Mat}(\mathbb{Z})$ *is the category of bicommutative Hopf algebras.*

Proof. It follows from the theorem that we must show that a bicommutive Hopf algebra is just a bicommutative bialgebra A equipped with a map of rigs $\phi_A : \mathbb{Z} \to A$.

It is easy to see that such a bialgebra must be a bicommutative Hopf algebra with antipode $\phi(-1)$ since the condition $\phi(0) = \phi(1) + \phi(-1)$ translates as

$$\eta_A \epsilon_A = \mu_A(\mathrm{id}_A \otimes \phi(-1))\Delta_A$$

which is half of the axiom for the antipode. The other half is just encoded by $\phi(0) = \phi(-1) + \phi(1)$.

To finish, we must see that if A is a bicommutative Hopf algebra with antipode S then $\phi(-1) = S$ extends uniquely to a map of rigs $\mathbb{Z} \to \mathrm{End}(A)$. It is clear that we must define $\phi(0) = 0$, $\phi(n) = \mathrm{id}_A + \phi(n-1)$ and $\phi(-n) = S + \phi(-n+1)$ inductively for each postive integer n. This will define an additive homomorphism of commutative monoids, since we have already checked that $S + \mathrm{id}_A = 0$ is the antipode axiom.

That ϕ is a homomorphism of rigs now follows from the distributive law for rigs and the fact the $S^2 = \mathrm{id}_A$ since A is a bicommutative Hopf algebra. $\square$