Throughout this note we fix a symmetric monoidal category $(\mathcal{C}, \otimes, \tau)$ where τ is the symmetry functor with $\tau_{AB} : A \otimes B \to B \otimes A$.

A bicommutative bialgebra object in $\mathcal{C}$ is an object A, with functors $\mu_A : A \otimes A \to A$, $\eta_A : I \to A$, $\Delta_A : A \to A \otimes A$, and $\epsilon_A : A \to I$ called multiplication, unit, comultplication and counit respectively, satisfying certain axioms.

Given two bicommutative bialgebra objects A and B in $\mathcal{C}$, we may give $A \otimes B$ the structure of a bicommutative bialgebra: $\mu_{A\otimes B} := (\mu_A \otimes \mu_B)(1 \otimes \tau_{BA} \otimes 1)$, $\eta_{A\otimes B} = \eta_A \otimes \eta_B$, $\Delta_{A\otimes B} = (1 \otimes \tau_{AB} \otimes 1)(\Delta_A \otimes \Delta_B)$, and $\epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B$. In this way we may make $A^{\otimes m}$ into a bicommutative bialgebra for each $m \geq 2$.

As a point of notation, we will write μ_A^n for the functor from $A^{\otimes n} \to A$, given inductively by $\mu_A^0 = \eta_A$, and $\mu_A^{n+1} = \mu(1 \otimes \mu_A^n)$, and Δ_A^n for the functor from $A \to A^{\otimes n}$ given inductively by $\Delta_A^0 = \epsilon_A$ and $\Delta_A^{n+1} = \Delta_A(1 \otimes \Delta_A^n)$. Notice every μ_A^n and Δ_A^n is bialgebra map.

Lemma. *The monoidal subcategory of $\mathcal{C}$ whose objects are bicommutative bialgebra objects in $\mathcal{C}$ and whose morphisms are bialgebra bihomomorphisms may be enriched over commutative monoids.*

From now on we will refer to this enriched category as Bialg. Notice that the enriched structure makes $\mathrm{Hom}_{\mathrm{Bialg}}(A, A)$ is a rig with identity id_A whenever A is an object in Bialg. We will just write $\mathrm{End}(A)$ for this rig.

Proof. We first need to explain how to define an addition on the Hom sets. Suppose A and B are bicommutative bialgebras, and suppose that f and g are two morphisms from A to B. We define

$$f + g := \mu_B(f \otimes g)\Delta_A$$

For each Hom set the axioms for a commutative monoid now follow easily, but for completeness: suppose f, g and h are in $\mathrm{Hom}(A, B)$

$$\begin{aligned}(f+g)+h &= \mu_B((\mu_B(f\otimes g)\Delta_A)\otimes h)\Delta_A \\ &= \mu_B(\mu_B\otimes 1)(f\otimes g\otimes h)\Delta_A(1\otimes\Delta_A) \\ &= \mu_B(1\otimes\mu_B)(f\otimes g\otimes h)(1\otimes\Delta_A)\Delta_A \\ &= f+(g+h)\end{aligned}$$

where the third equality follows from the coassociativity of A, and associativity of B.

The zero map from A to B is the composite of the counit ϵ_A and the unit η_B, and

$$0 + f = \mu_B(\eta_B\epsilon_A \otimes f)\Delta_A = \mu_B(\eta_B \otimes 1)(1 \otimes f)(\epsilon_A \otimes 1)\Delta_A = f$$

where the last equality follows from the axioms for unit and counit.

The symmetry of $+$ follows from the commutativity of B, and cocommutativity of A: if τ_A is the symmetry map $A \otimes A \to A \otimes A$ then

$$f+g = \mu_B(f\otimes g)\Delta_A = \mu_B\tau_B(f\otimes g)\Delta_A = \mu_B(g\otimes f)\tau_A\Delta_A = \mu_B(g\otimes f)\Delta_A = g+f$$

To complete the proof we need to check that the composition of morphisms gives a monoid map $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$: suppose f_1 and f_2 are in $\mathrm{Hom}(B, C)$ and g_1 and g_2 are in $\mathrm{Hom}(A, B)$ then

$$f_1(g_1+g_2) = f_1\mu_B(g_1\otimes g_2)\Delta_A = \mu_C(f_1\otimes f_1)(g_1\otimes g_2)\Delta_A = \mu_C(f_1g_1\otimes f_1h_1)\Delta_A = f_1g_1+f_1h_1$$

and

$$(f_1+f_2)g_1 = \mu_C(f_1\otimes f_2)\Delta_B g_1 = \mu_C(f_1\otimes f_2)(g_1\otimes g_1)\Delta_A = \mu_C(f_1 g_1\otimes f_2 g_1)\Delta_A = f_1 g_1+f_2 g_1.$$

□

Our goal now is the following theorem:

Theorem. *If R is a rig then* $\mathrm{Mat}(R)$ *is just the PROP for bicommutative bialgebras A equipped a map of rigs $R \to \mathrm{End}(A)$.*

We prove this theorem with three lemmas. Firstly we show

Lemma. *If A is a bicommutative bialgebra and $\phi : R \to \mathrm{End}(A)$ a map of rigs, then there is a strict monoidal functor F_A enriched over commutative moniods from* $\mathrm{Mat}(R)$ *to* Bialg *such that $F_A(1) = A$ and F_A is just ϕ on* $\mathrm{Hom}(1,1)$.

Proof. Because F_A is a strict monoidal functor with $F_A(1) = A$, $F_A(n)$ is necessarily $A^{\otimes n}$ for every n.

We begin by defining F_A on $\mathrm{Hom}(n,1)$ for each n. If (r_i) is an $(1\times n)$ matrix with entries in R we set $F_A(r_i) = \mu_A^n(\phi(r_1)\otimes\cdots\otimes\phi(r_n))$. Notice that in particular $F_A(r : 1 \to 1)$ is just $\phi(r)$ as required. Also notice $F_A(0 : 0 \to 1) = \eta_A$ and $F_A(11 : 2 \to 1)$ is just μ_A. We need all these maps $F_A(r_i)$ to be bialgebra maps, but this is true because they are defined as a composite of bialgebra maps.

Now suppose that (r_{ij}) is any $(m \times n)$ matrix with entries in R. We define $F_A(r_{ij}) = (F_A(r_{i1})\otimes\cdots\otimes F_A(r_{im}))\Delta^m_{A^{\otimes n}}$. This time it may be easily seen that $F_A(0 : 1 \to 0) = \epsilon_A$ and $F_A((11)^t : 1 \to 2) = \Delta_A$. As before all these maps $F_A(r_{ij})$ are bialgebra maps because they are a composite of such.

We need to check that F as defined is an enriched functor. First, we check that $F_A(r_{ij} + s_{ij}) = F_A(r_{ij}) + F_A(s_{ij})$ for every pair of R-valued $(m \times n)$ matrices (r_{ij}) and (s_{ij}). As before we begin by considering the case $m = 1$ suppressing the second index as we may:

$$\begin{aligned}
F_A(r_i + s_i) &= \mu_A^n(\phi(r_1 + s_1)\otimes\cdots\otimes\phi(r_n + s_n))\\
&= \mu_A^n((\phi(r_1) + \phi(s_1))\otimes\cdots(\phi(r_n) + \phi(s_n)))\\
&= \mu_A^n((\mu_A(\phi(r_1)\otimes\phi(s_1))\Delta_A)\otimes\cdots\otimes(\mu_A(\phi(r_n)\otimes\phi(s_n))\Delta_A))\\
&= \mu_A^n(\mu_A\otimes\cdots\otimes\mu_A)(\phi(r_1)\otimes\phi(s_1)\otimes\cdots\otimes\phi(r_n)\otimes\phi(s_n))(\Delta_A\otimes\cdots\otimes\Delta_A)\\
&= \mu_A^n(\mu_{A^{\otimes n}}(\phi(r_1)\otimes\cdots\otimes\phi(r_n)\otimes\phi(s_1)\otimes\cdots\otimes\phi(s_n))\Delta_{A^{\otimes n}}\\
&= \mu_A(\mu_A^n(\phi(r_1)\otimes\cdots\otimes\phi(r_n))\otimes\mu_A^n(\phi(s_1)\otimes\cdots\otimes\phi(s_n)))\Delta_{A^{\otimes n}}\\
&= F_A(r_i) + F_A(s_i)
\end{aligned}$$

Now we consider the general case:

$$\begin{aligned}
F_A(r_{ij} + s_{ij}) &= (F_A(r_{i1} + s_i 1)\otimes\cdots\otimes F_A(r_{im} + s_{im}))\Delta^m_{A^{\otimes n}}\\
&= ((F_A(r_{i1}) + F_A(s_{i1}))\otimes\cdots\otimes(F_A(r_{im}) + F_A(s_{im})))\Delta^m_{A^{\otimes n}}\\
&= (\mu_A(F_A(r_{i1})\otimes F_A(s_{i1}))\Delta_{A^{\otimes n}})\otimes\cdots\otimes(\mu_A(F_A(r_{im})\otimes F_A(s_{im})\Delta_{A^{\otimes n}}))\Delta^m_{A^{\otimes n}}\\
&= \mu_{A^{\otimes m}}((F_A(r_{i1})\otimes\cdots\otimes F_A(r_{im}))\Delta^m_{A^{\otimes n}})\otimes(F_A(s_{i1})\otimes\cdots\otimes F_A(s_{im}))\Delta^m_{A^{\otimes n}})\Delta_{A^{\otimes n}}\\
&= \mu_{A^{\otimes m}}(F_A(r_{ij})\otimes F_A(s_{ij}))\Delta_{A^{\otimes n}}\\
&= F_A(r_{ij}) + F_A(s_{ij})
\end{aligned}$$

We can now complete the proof by showing that F_A is a functor i.e. that F_A preserves composition. Because we have already checked that F_A preserves the enriched structure it suffices to check this on matrices with precisely one non-zero entry since these generate all matrices under $+$.

But if we have a $(1 \times n)$ row matrix with all entries zero except possibly the ith entry which takes value r, it is easy to check that F_A sends it to the map $\epsilon_{A^{\otimes i-1}} \otimes \phi(r) \otimes \epsilon_{A^{\otimes n-i}}$. Then we see that F_A sends a general $(m \times n)$ matrix with all entries 0 except the ijth which takes value r to $\eta_{A^{\otimes j-1}} \otimes (\epsilon_{A^{\otimes i-1}} \otimes \phi(r) \otimes \epsilon_{A^{\otimes n-i}}) \otimes \eta_{A^{\otimes n-j}}$. Now if we take two matrices of this form that compose it is easy to see that F_A preserves their composition. □

Secondly,

Lemma. *If R is a rig then an algebra over the PROP* $\mathrm{Mat}(R)$ *is a bicommutative bialgebra with a map of rigs $R \to \mathrm{End}(A)$.*

Proof. Suppose that $F : \mathrm{Mat}(R) \to \mathcal{C}$ is a strict monoidal functor. We set $A = F(1)$, $\mu = F((11))$, and $\eta = F(0 : 0 \to 1)$. This makes (A, μ, η) into a commutative monoid:

Associativity follows from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 0\end{pmatrix},$$

commutativity from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix},$$

and the unit axiom from

$$\begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}1\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}.$$

The coalgebra structure is given by comultiplication $\Delta = F((11)^t)$ and $\epsilon = (0 : 1 \to 0)$ that this does define a coalgebra follows from what has gone before and the fact that the transpose map from $\mathrm{Mat}(R)$ to itself is a contravariant functor that is self-inverse.

Next we must check that the algebra and coalgebra structures are compatible. It suffices to check that Δ is an algebra map and the counit respects the algebra structure. The first follows from the matrix equations

$$\begin{pmatrix}1\\ 1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\\ 0 & 1\\ 0 & 1\end{pmatrix},$$

and the latter from the fact that 0 is a terminal object in $Mat(R)$. □

Finally we need to show

Lemma. *If R is a rig then a morphism between algebras F and G over $Mat(R)$ in $\mathcal{C}$ is just a bialgebra map between $F(1)$ and $G(1)$ that commutes with the maps $R \to \mathrm{End}(F(1))$ and $R \to \mathrm{End}(G(1))$.*

Proof. Let's write A for $F(1)$ and B for $G(1)$

A natural transformation from F to G is a map $\theta : A \to B$ in $\mathcal{C}$ such that if X is an $(m \times n)$ matrix with coeffecients in R then $\theta^{\otimes n} F(X) = G(X)\theta^{\otimes m}$.

This condition for (1×1)-matrices says precisely that θ commutes with the maps $R \to \mathrm{End}(A)$ and $R \to \mathrm{End}(B)$. Then the condition for the matrix (11) implies that θ is an algebra map, and for $(11)^t$ that it is an coalgebra map.

It now remains to show that a bialgebra map θ of the given form defines a natural transformation. Because F and G are enriched functors and θ^n is a bialgebra map $A^{\otimes n} \to B^{\otimes n}$ for every n it suffices to check the equation $\theta^{\otimes n} F(X) = G(X)\theta^{\otimes m}$ for $(m \times n)$ matrices with only one non-zero entry. This is a straightforward check. $\square$