Linear Algebra: Example Sheet 4 of 4

1. The square matrices $A$ and $B$ over the field $F$ are congruent if $B = P^TAP$ for some invertible matrix $P$ over $F$. Which of the following symmetric matrices are congruent to the identity matrix over $\mathbb{R}$, and which over $\mathbb{C}$? (Which, if any, over $\mathbb{Q}$?) Try to get away with the minimum calculation.

\[
\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.
\]

2. Find the rank and signature of the following quadratic forms over $\mathbb{R}$.

\[x^2 + y^2 + z^2 - 2xz - 2yz, \quad x^2 + 2y^2 - 2z^2 - 4xy - 4yz, \quad 16xy - z^2, \quad 2xy + 2yz + 2zx.\]

If $A$ is the matrix of the first of these (say), find a non-singular matrix $P$ such that $P^TAP$ is diagonal with entries $\pm 1$.

3. (i) Show that the function $\psi(A, B) = \text{tr}(AB^T)$ is a symmetric positive definite bilinear form on the space $\text{Mat}_n(\mathbb{R})$ of all $n \times n$ real matrices. Deduce that $|\text{tr}(AB^T)| \leq \text{tr}(AA^T)^{1/2}\text{tr}(BB^T)^{1/2}$.

(ii) Show that the map $A \mapsto \text{tr}(A^2)$ is a quadratic form on $\text{Mat}_n(\mathbb{R})$. Find its rank and signature.

4. Let $\psi : V \times V \to \mathbb{C}$ be a Hermitian form on a complex vector space $V$.

(i) Find the rank and signature of $\psi$ in the case $V = \mathbb{C}^3$ and

\[\psi(x, x) = |x_1 + ix_2|^2 + |x_2 + ix_3|^2 + |x_3 + ix_1|^2 - |x_1 + x_2 + x_3|^2.\]

(ii) Show in general that if $n > 2$ then $\psi(u, v) = \frac{1}{n} \sum_{k=1}^n \zeta^{-k} \psi(u + \zeta^k v, u + \zeta^k v)$ where $\zeta = e^{2\pi i/n}$.

5. Show that the quadratic form $2(x^2+y^2+z^2+xy+yz+zx)$ is positive definite. Write down an orthonormal basis for the corresponding inner product on $\mathbb{R}^3$. Compute the basis of $\mathbb{R}^3$ obtained by applying the Gram-Schmidt process to the standard basis with respect to this inner product.

6. Let $W \leq V$ with $V$ an inner product space. An endomorphism $\pi$ of $V$ is called an idempotent if $\pi^2 = \pi$. Show that the orthogonal projection onto $W$ is a self-adjoint idempotent. Conversely show that any self-adjoint idempotent is orthogonal projection onto its image.

7. Let $S$ be an $n \times n$ real symmetric matrix with $S^k = I$ for some $k \geq 1$. Show that $S^2 = I$.

8. An endomorphism $\alpha$ of a finite dimensional inner product space $V$ is positive definite if it is self-adjoint and satisfies $\langle \alpha(x), x \rangle > 0$ for all non-zero $x \in V$.

(i) Prove that a positive definite endomorphism has a unique positive definite square root.

(ii) Let $\alpha$ be an invertible endomorphism of $V$ and $\alpha^*$ its adjoint. By considering $\alpha^*\alpha$, show that $\alpha$ can be factored as $\beta\gamma$ with $\beta$ unitary and $\gamma$ positive definite.

9. Let $V$ be a finite dimensional complex inner product space, and let $\alpha$ be an endomorphism on $V$. Assume that $\alpha$ is normal, that is, $\alpha$ commutes with its adjoint: $\alpha\alpha^* = \alpha^*\alpha$. Show that $\alpha$ and $\alpha^*$ have a common eigenvector $v$, and the corresponding eigenvalues are complex conjugates. Show that the subspace $\langle v \rangle$ is invariant under both $\alpha$ and $\alpha^*$. Deduce that there is an orthonormal basis of eigenvectors of $\alpha$.

10. Find a linear transformation which simultaneously reduces the pair of real quadratic forms

\[2x^2 + 3y^2 + 3z^2 - 2yz, \quad x^2 + 3y^2 + 3z^2 + 6xy + 2yz - 6xz\]

to the forms

\[X^2 + Y^2 + Z^2, \quad \lambda X^2 + \mu Y^2 + \nu Z^2\]

for some $\lambda, \mu, \nu \in \mathbb{R}$ (which should turn out in this example to be integers).

Does there exist a linear transformation which reduces the pair of real quadratic forms $x^2 - y^2$, $2xy$ simultaneously to diagonal forms?
11. Show that if $A$ is an $m \times n$ real matrix of rank $n$ then $A^T A$ is invertible. Find a corresponding result for complex matrices.

12. Let $P_n$ be the $(n+1)$-dimensional space of real polynomials of degree $\leq n$. Define

$$(f, g) = \int_{-1}^{+1} f(t)g(t)dt.$$  

Show that $(\ , \ )$ is an inner product on $P_n$ and that the endomorphism $\alpha : P_n \to P_n$ defined by

$$\alpha(f)(t) = (1 - t^2)f'''(t) - 2tf'(t)$$  

is self-adjoint. What are the eigenvalues of $\alpha$?

Let $s_k \in P_n$ be defined by $s_k(t) = \frac{d^k}{dt^k}(1 - t^2)^k$. Prove the following.

(i) For $i \neq j$, $(s_i, s_j) = 0$.

(ii) $s_0, \ldots, s_n$ forms a basis for $P_n$.

(iii) For all $1 \leq k \leq n$, $s_k$ spans the orthogonal complement of $P_{k-1}$ in $P_k$.

(iv) $s_k$ is an eigenvector of $\alpha$. (Give its eigenvalue.)

What is the relation between the $s_k$ and the result of applying Gram-Schmidt to the sequence $1, x, x^2, x^3$ and so on? (Calculate the first few terms?)

13. Let $f_1, \ldots, f_t, f_{t+1}, \ldots, f_{t+u}$ be linear functionals on the finite dimensional real vector space $V$. Show that $Q(x) = f_1(x)^2 + \cdots + f_t(x)^2 - f_{t+1}(x)^2 - \cdots - f_{t+u}(x)^2$ is a quadratic form on $V$. Suppose $Q$ has rank $p + q$ and signature $p - q$. Show that $p \leq t$ and $q \leq u$.

14. Let $a_1, a_2, \ldots, a_n$ be real numbers such that $a_1 + \cdots + a_n = 0$ and $a_1^2 + \cdots + a_n^2 = 1$. What is the maximum value of $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1$?

15. Suppose that $\alpha$ is an orthogonal endomorphism on the finite-dimensional real inner product space $V$. Prove that $V$ can be decomposed into a direct sum of mutually orthogonal $\alpha$-invariant subspaces of dimension 1 or 2. Determine the possible matrices of $\alpha$ with respect to orthonormal bases in the cases where $V$ has dimension 1 or dimension 2.